IDEMPOTENT RINGS WHICH ARE EQUIVALENT TO RINGS WITH IDENTITY

By

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Let $A$ be a ring such that $A=A^3$, but which does not necessarily have an identity element. In studying properties of the ring $A$ through properties of its modules, it is pointless to consider the category $A$-MOD of all the left $A$-modules: for instance, every abelian group —with trivial multiplication— is in $A$-MOD. The natural choice for an interesting category of left $A$-modules seems to be the following: if a left $A$-module $\Delta M$ is unital when $AM=M$, and is $A$-torsionfree when the annihilator $\gamma_M(A)$ is zero, then $A$-mod will be the full subcategory of $A$-MOD whose objects are the unital and $A$-torsionfree left $A$-modules. The category $A$-mod appears in a number of papers (for instance, [7-9]) and when $A$ has local units [1, 2] or is a left $s$-unital ring [6, 12], then the objects of $A$-mod are the unital left $A$-modules. $A$-mod is a Grothendieck category and we study here the question of finding necessary and sufficient conditions on the ring $A$ for $A$-mod to be equivalent to a category $R$-mod of modules over a ring with 1. This was already considered for rings with local units in [1], [2] or [3], and for left $s$-unital rings in [6]. Our situation is therefore more general.

In this paper, all rings will be associative rings, but we do not assume that they have an identity. A ring $A$ has local units [2] when for every finite family $a_1, \cdots, a_n$ of elements of $A$ there is an idempotent $e \in A$ such that $ea_j = a_j e$ for all $j=1, \cdots, n$. A left $A$-module $M$ is said to be unital if $M$ has a spanning set (that is, if $AM=M$); and $M$ has a finite spanning set when $M=\Sigma Ax_i$ for a finite family of elements $x_1, \cdots, x_n$ of $M$. The module $\Delta M$ will be called $A$-torsionfree when $\gamma_M(A)=0$. A ring $A$ is said to be left nondegenerate if the left module $\Delta A$ is $A$-torsionfree, and $A$ is nondegenerate when it is both left and right nondegenerate (see [10, p. 88]). Clearly, a ring with local units is nondegenerate. The ring $A$ will be called (left) $s$-unital [12] in case for each $a \in A$ (equivalently, for every finite family $a_1, \cdots, a_n$ of elements

Received March 25, 1991, Revised November 1, 1991.

With partial support from the D.G.I.C.Y.T. of Spain (PB 87-0703)
of $A$) there is some $u \in A$ such that $ua=a$ (respectively, $ua_i=a_i$, for all $i$): see [12, Theorem 1]. Any left $s$-unital ring is idempotent and left nondegenerate.

We will say that a ring $A$ is generated by the element $a \in A$ in case $A=A^aA$. The above mentioned results of Abrams and Ánh-Márki [1], [2], and Komatsu [6] may be stated as follows: if $A$ has local units, then $A$-mod is equivalent to a category of modules over a ring with 1 if and only if $A$ is generated by an idempotent $e$ [2, Proposition 3.5]; if $A$ is left $e$-unital and $A$-mod is equivalent to the category of left modules over a ring with 1, then $A$ is generated by some element $a$ [6, Proposition 4.7].

In the sequel, we will be dealing with left modules, and so we follow the convention of denoting the composition $g \cdot f$ of two module homomorphisms as the product $fg$. On the other hand, if $R$ is a ring with 1, $R^A$ is a left $R$-module and $E=\text{End}(\mu M)$ is its endomorphism ring, then we will denote by $E^E=\text{End}(\mu M)$ the following subring -in general, without identity- of $E$: $E^E=\{f \in E \mid f : M \rightarrow M$ factors through a finitely generated free module$\}$.

We now state and prove the following result.

**Theorem.** Let $A$ be an idempotent ring. Then the category $A$-mod is equivalent to the category $R$-mod of left modules over a ring $R$ with 1 if and only if there is some integer $n \geq 1$ such that the matrix ring $M_n(A)$ is generated by an idempotent.

**Proof.** We divide the proof in several steps.

**Step 1.** For any idempotent ring $A$, let us put $\text{ann}(A)=\{x \in A \mid AxA=0\}$ and $A':=A/\text{ann}(A)$. Then $A'$ is a nondegenerate idempotent ring and $A$-mod and $A'$-mod are equivalent categories.

The fact that $A'$ is nondegenerate is easy to verify. On the other hand, if $\varepsilon : A \rightarrow A'$ is the canonical projection, then one may see that the restriction of scalars functor $\varepsilon_*$ gives indeed a functor from $A'$-mod to $A$-mod. Now, if $\hat{A}$ belongs to $A$-mod and $a \in \text{ann}(A)$, then $AaM=AAaM=0$, so that $aM \subseteq \varepsilon_*(A)$, and $aM=0$, because $M$ is $A$-torsionfree. As a consequence, there is a functor $F : A$-mod $\rightarrow A'$-mod which views each $\hat{A}$ of $A$-mod as a left $A'$-module. Then $F$ and $\varepsilon_*$ are inverse equivalences and hence $A$-mod and $A'$-mod are equivalent categories.

**Step 2.** For each $n \geq 1$, let $\Delta$ be the matrix ring $M_n(A)$. Then $A$-mod and $\Delta$-mod are also equivalent categories.

To see this, consider the bimodules $\varepsilon(A^*)_\Delta$ and $\varepsilon(A^*)_A$, and the natural mappings $\Phi : A^* \otimes_A A^* \rightarrow \Delta$, $\Psi : A^* \otimes_A A^* \rightarrow A$. It is clear that they are bimodule homomorphisms which give a Morita context between $A$ and $\Delta$ (if we represent
elements in $\mathcal{A}(A^n)_A$ in row form, and elements of $\mathcal{A}(A^n)_A$ in column form, then $\phi$ and $\psi$ are induced by products of matrices). Also, the fact that $A$ is idempotent allows us to deduce that $\phi$ and $\psi$ are surjective. Then, by [7, Theorem], $A$-mod and $\Delta$-mod are equivalent categories.

Step 3. We prove now the sufficiency of the condition of the Theorem. Assume that $\Delta = M_n(A)$ is generated by an idempotent. By step 1, $\Delta$ is equivalent to $\Delta' = \Delta/\text{ann}(\Delta)$. But $\Delta = \Delta e \Delta$ for the idempotent $e$ implies that $\Delta' = \Delta' e' \Delta'$ for the idempotent $e' = e + \text{ann}(\Delta)$; so, we can assume that $\Delta$ is a nondegenerate ring. Then $\Delta$ belongs to the category $\Delta$-mod and is a generator of this category. But $\phi(\Delta e)$ generates $\Delta$, so that it is also a generator of $\Delta$-mod. $\Delta e$, being finitely spanned, is clearly a finitely generated object of $\Delta$-mod [11, p. 121]. Finally, let $p : Y \to X$ be an epimorphism in $\Delta$-mod, and put $U = \text{Im} \, p$, $V = X/U, W = V/\text{ker}(A)$. Then $W$ belongs to $\Delta$-mod and hence the canonical projection from $X$ to $W$ must be 0; thus, $V = 0$ and $X = U$, so that $p$ is a surjective morphism. If $f : \Delta e \to X$ is now a homomorphism, then $f(e) = ea$ for some $a \in X$, and $\alpha(e) := ea$, with $y$ such that $p(y) = ea$, gives a morphism $\alpha$ with $f = \alpha \cdot \phi$. This shows that $\Delta e$ is projective. It follows that $\Delta$-mod is equivalent to the category of left modules over the ring $\text{End}_{\phi}(\Delta e) \cong e \Delta e$. By step 2, $A$ is equivalent to a ring with 1.

Step 4. Let us now suppose that $A$ is an idempotent and left nondegenerate ring and that there is an equivalence $F : A$-mod $\to$ $R$-mod, $R$ being a ring with 1. We are to show that $M_n(A)$ is generated by an idempotent, for some $n \geq 1$.

By [4, Theorem 2.4], there exists a generator $\rho M$ of $R$-mod with the property that, if $E = \text{End}(\rho M)$, and $E_\rho = f \text{End}(\rho M)$, then $A$ is isomorphic to some right ideal $T$ of $E_\rho$, such that $E_\rho T = E_\rho$.

We now point out that we can further assume that there is an epimorphism of left $R$-modules $\pi : M \to R$. Indeed, this is true for some $M^*$, and we put $S := \text{End}(\rho M^*), S_\rho := f \text{End}(\rho M^*)$, so that there is an isomorphism $S \cong M_k(E)$. We assert that, in this isomorphism, $S_\rho \cong M_k(E_\rho)$; in fact, the inclusion $S_\rho \subseteq M_k(E_\rho)$ is obvious, and the inclusion $M_k(E_\rho) \subseteq S_\rho$ depends on the easily verified fact that morphisms $M^* \to M$ or $M \to M^*$ factor through free modules of finite type whenever they are induced by endomorphisms of $\rho M$ belonging to $E_\rho$. By substituting $M^*$, $S$ and $S_\rho$ for $M, E$ and $E_\rho$, we have that the matrix ring $M_k(A)$ is still (isomorphic to) a right ideal of $S_\rho$ in such a way that -assuming the obvious identification - $S_\rho \cdot M_k(A) = S_\rho$. So, by replacing $A$ by $M_k(A)$ if necessary (note that $M_k(A)$ is again idempotent and left nondegenerate), we may indeed assume that $\pi : M \to R$ is an epimorphism.

Let $x \in M$ be such that $\pi(x) = 1$. Since $E_\rho A = E_\rho$ and $\sum_{\sigma \in E_\rho} \text{Im} \, \sigma = M$ we
deduce that $\Sigma_{x \in \mathcal{A}} \text{Im } a = M$. Therefore there exists a homomorphism $\alpha : M^n \to M$ such that $x \in \text{Im } a$; and each component $\alpha_j := \mu_j \cdot a$, with $\mu_j : M \to M^n$ being the canonical inclusion, satisfies $\alpha_j \in \mathcal{A}$. So we have that $\alpha \cdot \pi : M^n \to R$ is an epimorphism and hence there is $g : R \to M^n$ with $g\alpha \pi = 1_R$ and $\alpha \pi g = e$ an idempotent in the ring $\text{End} (\mathfrak{p} M^n) \cong M_n(E)$. Moreover, each of the components of $e$, when considered as a matrix, consists of $\mu_j \alpha \pi g p_x \in E j E \subseteq A$ (where the $p_x$ are the canonical projections $M^n \to M$). This means that $e \in M_n(A)$.

As before, we may put $S := \text{End} (\mathfrak{p} M^n) \cong M_n(E)$, $S_0 := f \text{End} (\mathfrak{p} M^n) \cong M_n(E_0)$ so that $M_n(A)$ is an idempotent right ideal in $S_0$ which satisfies $S_0 M_n(A) = S_0$. Thus, $e$ is an idempotent element in $M_n(A) \subseteq S_0$ and is an endomorphism of $M^n$ such that $\text{Im } e$ is a direct summand of $M^n$ isomorphic to $R$. Consequently, $\text{Im } e$ generates $M^n$ and hence, if we let $t$ range over all the elements in $e S_0$, we have $\Sigma_1 \text{Im } f = M^n$. This shows that $e S_0$ is a right ideal of $S$ which satisfies $M^n \cdot (e S_0) = M^n$. If we apply now [5, Proposition 2.5], we see that this implies $S_0 e S_0 = S_0$.

Since $A = A^4$, $M_n(A) \cdot S_0 = M_n(A)$ and so we have: $M_n(A) \cdot e \cdot M_n(A) = M_n(A) \cdot S_0 e \cdot S_0 = M_n(A) \cdot S_0 = M_n(A)$. This proves that $M_n(A)$ is generated by an idempotent element.

Step 5. Now we complete the proof of the Theorem. Let $A$ be an idempotent ring (but not necessarily left nondegenerate), and assume that there is an equivalence of categories between $A$-$\text{mod}$ and $R$-$\text{mod}$ for $R$ a ring with 1. Put $\mathcal{A}(A) := \{ a \in A | A a = 0 \}$, and $A^* := A / \mathcal{A}(A)$. In a way analogous to that of Step 1, we may show that $A$ and $A^*$ are equivalent rings, so that we can deduce from stea 4, that for some $n \geq 1$, the matrix ring $M_n(A^*)$ is generated by an idempotent. Thus, all that is left to show is that this property can be lifted from $M_n(A^*)$ to $M_n(A)$. But we have that $M_n(A^*) = M_n(A / \mathcal{A}(A)) \cong (M_n(A)) / (M_n(\mathcal{A}(A)))$, and this last quotient is nothing else than $M_n(A) / \mathcal{A}(A) M_n(A)$, that is, $(M_n(A))^*$. Therefore, it will suffice to prove that if a ring of the form $A^* = A / \mathcal{A}(A)$ is generated by an idempotent, then so is the ring $A$.

So, let us assume that $A^* = A^* \cdot e \cdot A^*$ for some idempotent $e$. There is $u \in A$ with $u + \mathcal{A}(A) = e$, and then $u^2 - u \in \mathcal{A}(A)$, from which we see that $u^2 = u^2 = u^4$. Therefore, $w = u^2$ is an idempotent of $A$ such that $w + \mathcal{A}(A) = e$. Now, let $a$, $b \in A$; by hypothesis, $b + \mathcal{A}(A) = \sum a_j \cdot e \cdot b_j$ in the ring $A^*$, so that $b - \sum a_j \cdot w \cdot b_j \in \mathcal{A}(A)$, for some $a_j$ and $b_j$ in $A$. Then $ab = \sum a_j \cdot wb_j$ and $ab \in AwA$. But since $A$ is idempotent, we have finally that $A = AwA$ and $A$ is generated by an idempotent.

Remarks. 1) It follows from the Theorem that an idempotent ring $A$
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which is equivalent to a ring with 1 must be finitely generated as a bimodule over \( A \): the coordinates of the idempotent matrix \( e \) in the adequate \( M_e(A) \) give the family of generators. When \( A \) is left s-unital this gives as a consequence the already mentioned result of Komatsu [6, Proposition 4.7]. If \( A \) has local units, we get [2, Proposition 3.5].

2) However, the condition that \( A \) be finitely generated as a bimodule over itself is not sufficient for \( A \) to be equivalent to a ring with 1. To see this, take a ring \( A \) such that \( A = A^4 \), \( A \) is finitely generated as an \( A - A \)-bimodule, is nondegenerate and coincides with its Jacobson radical (Sasiada's example [10, p. 314] of a simple radical ring fulfills these requirements). It is not difficult to show that the Jacobson radical of such a ring is the intersection of all the subobjects of \( A \) in \( A \)-mod which give a simple quotient of \( A \) in \( A \)-mod, so that \( A \) has no simple quotients in \( A \)-mod. Suppose that the category \( A \)-mod were equivalent to \( R \)-mod for \( R \) a ring with 1. Then, if \( _RM \) corresponds to \( A \) in this equivalence, we would have that \( _RM \) is a generator of \( R \)-mod without simple quotients. But this is absurd, since \( R \) is isomorphic to a summand of some \( M^4 \).

3) It may happen that \( A \) be an idempotent ring such that \( A \)-mod is equivalent to a category \( R \)-mod for a ring \( R \) with 1 but, nevertheless, \( A \) is not generated by an idempotent. For instance, let \( R \) be a simple domain which is not a division ring and let \( I \) be a right ideal of \( R \) such that \( I \neq 0, I \neq R \). Then \( RI = R, I = IR = I^2 \) and \( I \) is a faithful right ideal of \( R \), so that we can view \( I \) as a left nondegenerate and idempotent ring contained in \( R = \text{End}(R) \). By [4, Theorem 2.4], we see that \( I \)-mod is equivalent to the category \( R \)-mod. But \( I \) contains no idempotent other than 0, so that \( I \) is not generated by an idempotent.

References


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