COMPLETE SPACE-LIKE SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR OF AN INDEFINITE SPACE FORM

By

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1. Introduction.

Let \( M^p_{n+p}(c) \) be an \((n+p)\)-dimensional connected indefinite Riemannian manifold of index \( p \) and of constant curvature \( c \), which is called an indefinite space form of index \( p \). According to \( c>0 \), \( c=0 \) or \( c<0 \) it is denoted by \( S^p_{n+p}(c) \), \( R^p_{n+p} \) or \( H^p_{n+p}(c) \). A submanifold \( M \) of an indefinite space form \( M^p_{n+p}(c) \) is said to be space-like if the induced metric on \( M \) from that of the ambient space is positive definite. It is pointed out by some physicians that space-like hypersurfaces with constant mean curvature of arbitrary spacetimes get interested in relativity theory and an entire space-like hypersurface with constant mean curvature of an indefinite space form are studied by many authors (for examples: [1], [2], [3], [4], [7], [12] and so on).

Now, for a complete space-like submanifold \( M \) with parallel mean curvature vector of \( S^p_{n+p}(c) \), it is also seen by the first author [5] that \( M \) is totally umbilic if \( n=2 \) and \( h^2 \leq 4c \) or if \( n>2 \) and \( h^2 < 4(n-1)c \), where \( H \) denotes the mean curvature, i.e., the norm of the mean curvature vector and \( h = nH \). On the other hand, the first author and Nakagawa [6] investigated the total umbilicness of such hypersurfaces from the different point of view. They proved that the squared norm \( S \) of the second fundamental form of \( M \) is bounded from above by \( S_>(1) \) and if \( \sup S < S_>(1) \) and \( H^2 \leq c \), then \( M \) is totally umbilic, where

\[
S_>(p) = -pnc + \frac{nh^2 \pm (n-2)\sqrt{h^4 - 4(n-1)ch^2}}{2(n-1)}.
\]

In this paper, we research the similar problem to the above property for the complete space-like submanifolds with parallel mean curvature vector of an indefinite space form. That is, we prove the following

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THEOREM 1. Let $M$ be an $n$-dimensional complete space-like submanifold with parallel mean curvature vector of an indefinite space form $M^{n+p}_p(c)$. If the one of the following conditions is satisfied:

1. $c \leq 0$,
2. $c > 0$ and $n^2H^2 \geq 4(n-1)c$,

then

(1.1) \[ S \leq S_+(p) + K(p), \]

where $K(p)$ is a constant defined by

\[ K(p) = (p-1)H \{ nH + \sqrt{n(n-1)}\{ S_+(1) - nH^2 \} \}. \]

THEOREM 2. The hyperbolic cylinder $H^1(c_1) \times R^{n-1}_{1+1}$ in $R^{n+1}_{1+1}$ is the only complete connected space-like $n$-dimensional submanifolds with parallel mean curvature vector of $R^{n+p}_p$ satisfying $S = S_+(p) + K(p)$.

THEOREM 3. The hyperbolic cylinder $H^1(c_1) \times H^{n-1}(c_2)$ of $H^{n+1}_{1+1}(c)$ and the maximal submanifolds $H^{n_1}(c_1) \times \cdots \times H^{n_{p+1}}(c_{p+1})$ of $H^{n+p}_p(c)$ are the only complete connected space-like $n$-dimensional submanifolds with parallel mean curvature vector satisfying $S = S_+(p) + K(p)$, where $c_r = (n/n_r)c$ and $\sum_{r=1}^{p+1} n_r = n$ in the latter case.

2. Standard models.

This section is concerned with some standard models of complete space-like submanifolds with parallel mean curvature vector of an indefinite space form $M^{n+p}_p(c)$, $c \leq 0$. In particular, we only consider non-totally umbilic cases. Moreover, the squared norms of the second fundamental forms of such standard models are calculated. Without loss of generality, an $(n+p)$-dimensional indefinite Euclidean space $R^{n+p}_p$ of index $p \geq 1$ can be first regarded as a product manifold of

$R^{n_1+1}_{1+1} \times \cdots \times R^{n_{p+1}}_{1+1} \times R^n$,

where $\sum_{r=1}^{p+1} n_r + m = n$. With respect to the standard orthonormal basis of $R^{n+p}_p$ a class of space-like submanifolds

$H^{n_1}(c_1) \times \cdots \times H^{n_p}(c_p) \times R^n$

of $R^{n+p}_p$ is defined as the Pythagorean product

$H^{n_1}(c_1) \times \cdots \times H^{n_p}(c_p) \times R^n$

$\{ (x_1, \cdots, x_{p+1}) \in R^{n+p}_p | x_r^2 = -\frac{1}{c_r} > 0 \}$. 
where \( r=1, \ldots, p \) and \(| |\) denotes the norm defined by the product on the Minkowski space \( R^{p+1}_i \) which is given by \( \langle x, x \rangle = -(x_0)^2 + \sum_{i=1}^{p} (x_i)^2 \). The mean curvature vector \( h \) of \( M \) is given by

\[
(2.1) \quad h = -\frac{1}{n} \sum_{r=1}^{p} n_r c_r x_r,
\]

at \((x_1, \ldots, x_{p+1}) \in M\), which is parallel in the normal bundle of \( M \). The number \( S_r(1) \) and the squared norm \( S \) of the second fundamental form are given by

\[
(2.2) \quad S_r(1) = n^2 H^2 = -\sum_{r=1}^{p} n_r^2 c_r, \quad S = -\sum_{r=1}^{p} n_r c_r.
\]

Then we get

\[
S_r(p) + K(p) = pn^2 H^2 = -p \sum_{r=1}^{p} n_r^2 c_r \geq S,
\]

where the equality holds if and only if \( p=1 \) and \( n_1 = 1 \).

Next we consider an \( n \)-dimensional space-like submanifold of \( H^{p+1}(c), \ p \geq 1 \). Without loss of generality, an \((n+p+1)\)-dimensional indefinite Euclidean space \( R^{p+1}_{p+1} \) of index \((p+1)\) can be first regarded as a product manifold of

\[
R^{p+1}_{p+1} \times \cdots \times R^{p+1}_{p+1},
\]

where \( \sum_{r=1}^{p} n_r = n \). With respect to the standard orthonormal basis of \( R^{p+1}_{p+1} \) a class of space-like submanifolds

\[
H^{n_1(c_1)} \times \cdots \times H^{n_{p+1}(c_{p+1})}
\]

of \( R^{p+1}_{p+1} \) is defined as the Pythagorean product

\[
H^{n_1(c_1)} \times \cdots \times H^{n_{p+1}(c_{p+1})} = \left\{ \left(x_1, \ldots, x_{p+1} \right) \in R^{p+1}_{p+1} = R^{p+1}_{p+1} \times \cdots \times R^{p+1}_{p+1} : \ |x_r|^2 = -\frac{1}{c_r} > 0 \right\},
\]

where \( r=1, \ldots, p+1 \). The mean curvature vector \( h \) of \( M \) is given by

\[
(2.3) \quad h = -\frac{1}{n} \sum_{r=1}^{p+1} \left(n_r c_r x_r \right) + cx
\]

at \( x=(x_1, \ldots, x_{p+1}) \in M \), which is parallel in the normal bundle of \( M \). The norm \( H \) of the mean curvature vector \( h \) and the squared norm \( S \) of the second fundamental form are given by

\[
(2.4) \quad h^2 = n^2 H^2 = n^2 c - \sum_{r=1}^{p+1} n_r^2 c_r, \quad S = \sum_{r=1}^{p+1} n_r (c - c_r) = nc - \sum_{r=1}^{p+1} n_r c_r.
\]

When \( M \) is maximal, it satisfies \( n_r c_r = nc \) for any index \( r \) by (2.3), which yields \( S = -nc \). Then we get \( S_r(p) + K(p) = 0 \), because of \( S_r(p) = -nc \) and \( K(p) = 0 \).
Suppose that $H \neq 0$. By a theorem of Ki, Kim and Nakagawa [9], if $p=1$, then we have $S_+(1)-S=0$. On the other hand, we have $S_+(1)>h^2-nc$, because of $c<0$. So it is seen that if $p \geq 2$, then we obtain

$$S_+(p)+K(p)-S=h^2-nc-S \geq 0$$

by (2.4). In order to prove the last inequality, the following lemma is prepared. The proof of this lemma is the only calculus and hence it is omitted.

**Lemma 2.1.** Let $a_1, \ldots, a_{p+1}$ be numbers not less than 1 satisfying $\sum a_r = n$ and $b_1, \ldots, b_{p+1}$ be negative numbers satisfying $\sum (1/b_r) = (1/b)$. Then we have

$$\sum (a_r - p(a_r)^2) b_r \geq n(p+1-pn)b.$$ 

3. **Preliminaries.**

Throughout this paper all manifolds are assumed to be smooth, connected without boundary. We discuss in smooth category. Let $M^{n+p}_p(c)$ be an $(n+p)$-dimensional indefinite Riemannian manifold of constant curvature $c$ whose index is $p$, which is called an *indefinite space form of constant curvature c and with index p*. Let $M$ be an $n$-dimensional submanifold of an $(n+p)$-dimensional indefinite space form $M^{n+p}_p(c)$ of index $p$. The submanifold $M$ is said to be *space-like* if the induced metric on $M$ from that of the ambient space is positive definite. We choose a local field of orthonormal frames $e_1, \ldots, e_{n+p}$ adapted to the indefinite Riemannian metric of $M^{n+p}_p(c)$ and the dual coframes $\omega_1, \ldots, \omega_{n+p}$ in such a way that, restricted to the submanifold $M$, $e_1, \ldots, e_n$ are tangent to $M$. Then connection forms $\{\omega_{AB}\}$ of $M^{n+p}_p(c)$ are characterized by the structure equations

$$\begin{align*}
d\omega_A + \sum \varepsilon_B \omega_{AB} \wedge \omega_B &= 0, \quad \omega_{AB} + \omega_{BA} = 0, \\
d\omega_{AB} + \sum \varepsilon_C \omega_{AC} \wedge \omega_{CB} &= \Omega_{AB}, \\
\Omega_{AB} &= -\frac{1}{2} \sum_{c,d} \varepsilon_c \varepsilon_d R'_{ABCD} \omega_C \wedge \omega_D,
\end{align*}$$

(3.1)

\begin{align*}
R'_{ABCD} &= c \varepsilon_A \varepsilon_B (\delta_{AD} \delta_{BD} - \delta_{AC} \delta_{BD}),
\end{align*}

(3.2)

where $\varepsilon_A = 1$ for an index $A \leq n$, $\varepsilon_A = -1$ for an index $A \geq n+1$, and $\Omega_{AB}$ (resp. $R'_{ABCD}$) denotes the indefinite Riemannian curvature form (resp. the components of the indefinite Riemannian curvature tensor $R'$) of $M^{n+p}_p(c)$. Therefore the components of the Ricci curvature tensor $Ric'$ and the scalar curvature $r'$ of $M^{n+p}_p(c)$ are given as

$$R'_{AB} = c(n+p-1)\varepsilon_A \delta_{AB}, \quad r' = (n+p)(n+p-1)c.$$
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In the sequel, the following convention on the range of indices is used, unless otherwise stated:

\[ 1 \leq A, B, \ldots \leq n + p; \quad 1 \leq i, j, \ldots \leq n; \quad n + 1 \leq \alpha, \beta, \ldots \leq n + p. \]

We agree that the repeated indices under a summation sign without indication are summed over the respective range. The canonical forms \( \{ \omega_A \} \) and the connection forms \( \{ \omega_{AB} \} \) restricted to \( M \) are also denoted by the same symbols. We then have

\[ (3.3) \quad \omega_n = 0 \quad \text{for} \quad \alpha = n + 1, \ldots, n + p. \]

We see that \( e_1, \ldots, e_n \) is a local field of orthonormal frames adapted to the induced Riemannian metric on \( M \) and \( \omega_1, \ldots, \omega_n \) is a local field of its dual coframes on \( M \). It follows from (3.1), (3.3) and Cartan's lemma that we have

\[ (3.4) \quad \omega_{ni} = \sum h_{ij}^{\alpha} \omega_j, \quad h_{ij}^{\alpha} = h_{ij}^{\alpha}. \]

The second fundamental form \( \alpha \) and the mean curvature vector \( h \) of \( M \) are defined by

\[ \alpha = -\sum h_{ij}^{\alpha} \omega_i \omega_j e_n, \quad h = -\frac{1}{n} \sum \left( \sum h_{ij}^{\alpha} \right) e_n. \]

The mean curvature \( H \) is defined by

\[ (3.5) \quad H = |h| = \frac{1}{n} \sqrt{\sum (h_{ij}^{\alpha})^2}. \]

Let \( S = \sum (h_{ij}^{\alpha})^2 \) denote the squared norm of the second fundamental form \( \alpha \) of \( M \). The connection forms \( \{ \omega_{ij} \} \) of \( M \) are characterized by the structure equations

\[ (3.6) \quad \begin{cases} \quad d\omega_i + \sum \omega_{ij} \wedge \omega_j = 0, & \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} + \sum \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \quad \Omega_{ij} = -\frac{1}{2} \sum R_{ijkl} \omega_k \wedge \omega_l, \end{cases} \]

where \( \Omega_{ij} \) (resp. \( R_{ijkl} \)) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor \( R \)) of \( M \). Therefore, from (3.1) and (3.6), the Gauss equation is given by

\[ (3.7) \quad R_{ijkl} = c(\delta_i \delta_{jk} - \delta_{ik} \delta_{jl}) - \sum (h_{ij}^{\alpha} h_{kj}^{\alpha} - h_{ik}^{\alpha} h_{lj}^{\alpha}). \]

The components of the Ricci curvature \( Ric \) and the scalar curvature \( r \) are given by

\[ (3.8) \quad R_{jk} = (n - 1) c \delta_{jk} - \sum h_{ij}^{\alpha} h_{kj}^{\alpha} + \sum h_{ij}^{\alpha} h_{jk}^{\alpha}, \]

\[ (3.9) \quad r = n(n - 1) c - n^2 H^2 + \sum (h_{ij}^{\alpha})^2. \]
We also have
\[(3.10)\]
\[d\omega_{\alpha\beta} - \sum \omega_{\alpha i} \wedge \omega_{\beta j} = -\frac{1}{2} \sum R_{\alpha j k} \omega_i \wedge \omega_j,\]
where
\[R_{\alpha j k} = -\sum (h_{\alpha j}^p h_{\alpha k}^q - h_{\alpha k}^p h_{\alpha j}^q)\]
The Codazzi equation and the Ricci formula for the second fundamental form are given by
\[(3.11)\]
\[h_{\alpha j}^m - h_{\alpha j}^m = 0,\]
\[(3.12)\]
\[h_{\alpha j}^m - h_{\alpha j}^m = -\sum h_{\alpha m}^p h_{\alpha j}^q - \sum h_{\alpha j}^p h_{\alpha k}^q + \sum h_{\alpha j}^p h_{\alpha k}^q R_{\alpha j k},\]
where \(h_{\alpha j}^m\) and \(h_{\alpha j}^m\) denote the components of the covariant differentials \(\nabla \alpha\) and \(\nabla \beta\) of the second fundamental form, respectively. The Laplacian \(\Delta h_{\alpha j}^m\) of the components \(h_{\alpha j}^m\) of the second fundamental form \(\alpha\) is given by
\[\Delta h_{\alpha j}^m = \sum h_{\alpha j}^m h_{\alpha k}^m.\]
From \(3.12\) we get
\[(3.13)\]
\[\Delta h_{\alpha j}^m = \sum h_{\alpha m}^p h_{\alpha j}^q - \sum h_{\alpha j}^p h_{\alpha k}^q + \sum h_{\alpha j}^p h_{\alpha k}^q R_{\alpha j k}.\]
The following generalized maximum principle due to Omori [11] and Yau [15] will play an important role in this paper.

**Theorem 3.1.** Let \(M\) be an \(n\)-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let \(F\) be a \(C^2\)-function bounded from above on \(M\), then for any \(\varepsilon > 0\), there exists a point \(p\) in \(M\) such that
\[F(p) + \varepsilon > \sup F, \quad |\text{grad } F(p)| < \varepsilon, \quad \Delta F(p) < \varepsilon.\]
The following lemma is already known.

**Lemma 3.2.** Let \(a_1, \ldots, a_n\) be real numbers satisfying \(\sum a_i = 0\) and \(\sum a_i^2 = k^2\) for \(k > 0\). Then we have
\[|\sum a_i^2| \leq (n-2) \frac{1}{n(n-1)} k^2,\]
where the equality holds if and only if \(n-1\) of them are equal with each other.


Let \(M\) be an \(n\)-dimensional space-like submanifold with parallel mean curvature vector \(h\) of an indefinite space form \(M_\mathbb{R}^{n+p}(c)\). Because the mean curvature vector is parallel, the mean curvature is constant. Suppose that
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$H \neq 0$. We choose $e_{n+1}$ in such a way that its direction coincides with that of the mean curvature vector. Then it is easily seen that we have

\begin{align}
(4.1) & \quad \omega_{n,n+1} = 0, \quad H = \text{constant}, \\
(4.2) & \quad H^n H^{n+1} = H^{n+1} H^n, \\
(4.3) & \quad \text{tr} H^{n+1} = nH, \quad \text{tr} H^n = 0
\end{align}

for any $\alpha \neq n+1$, where $H^n$ denotes an $n \times n$ symmetric matrix ($h_{ij}$).

A submanifold $M$ is said to be pseudo-umbilic, if it is umbilic with respect to the direction of the mean curvature vector $h$, that is,

\[(4.4) \quad h_{ij}^{n+1} = H \delta_{ij}.\]

We denote by $\mu$ an $n \times n$ symmetric matrix with $\mu_{ij} = h_{ij}^{n+1} - H \delta_{ij}$. Then we have

\[(4.5) \quad \text{tr} \mu = 0, \quad |\mu|^2 = \text{tr}(\mu)^2 = \sum (\mu_{ij})^2 = \text{tr}(H^{n+1})^2 - nH^2.\]

So the pseudo-umbilic submanifolds are characterized by the property $\mu = 0$. A non-negative function $\tau$ is defined by $\tau^2 = \sum_{\alpha = n+1} (h_{ij}^\alpha)^2$. We then have

\[(4.6) \quad S = |\mu|^2 + \tau^2 + nH^2.\]

Hence it is seen that $|\mu|^2$ as well as $\tau^2$ are independent of the choice of the frame fields and they are functions defined globally on $M$.

**Proposition 4.1.** Let $M$ be $n$-dimensional complete space-like submanifold with parallel mean curvature vector of an indefinite space form $S^p_{n+1}(c)$. If it satisfies

\[n^2 c \geq n^2 H^2 \geq 4(n-1)c, \quad S \leq S_1(1),\]

then $M$ is pseudo-umbilic, where $H$ denotes the mean curvature, i.e., the norm of the mean curvature vector.

**Proof.** In order to prove this property it suffices to show $\mu = 0$. From (3.13), the Gauss equation (3.7) and (3.10), we have

\[(4.7) \quad \Delta h_{ij}^{n+1} = nch_{ij}^{n+1} - nH \delta_{ij} + \sum h_{km}^{n+1} h_{m}^{k} h_{ij}^{k} - 2\sum h_{kk}^{n+1} h_{kj}^{k} h_{ij}^{k} \]
\[\quad + \sum h_{km}^{k} h_{m}^{k} h_{ij}^{k} - nH \sum h_{km}^{n+1} h_{m}^{n+1} + \sum h_{kk}^{k} h_{km}^{k} h_{m}^{n+1}.\]

Accordingly we obtain from (4.2)

\[\frac{1}{2} \Delta |\mu|^2 = \sum (h_{ij}^{n+1})^2 + nc \sum (h_{ij}^{n+1})^2 - n^2 c H^2 \]
\[\quad + \sum h_{km}^{n+1} h_{m}^{k} h_{ij}^{k} h_{ij}^{k} - 2\sum h_{kk}^{n+1} h_{km}^{n+1} h_{m}^{k} h_{ij}^{k} + \sum h_{km}^{n+1} h_{m}^{n+1} h_{k}^{k} h_{ij}^{k} h_{ij}^{k}.\]
and hence we see

\[ \frac{1}{2} \Delta |\mu|^2 = \sum (h^{+1}_i h^{+1}_j) + n c \sum (h^{+1}_i)^2 \]

(4.8)

\[ -n^2 c H^2 - n H \text{tr}(H^{n+1})^2 - \sum_{\beta \neq n+1} \text{tr}(H^{n+1}H^\beta - H^\beta H^{n+1})^2 + \{\text{tr}(H^{n+1})^2 + \sum_{\beta \neq n+1} \{\text{tr}(H^{n+1}H^\beta)\}^2. \]

On the other hand, because of

\[ \text{tr}(H^{n+1}) = 3H(\text{tr}(H^{n+1})^2 - nH^2) + nH^3, \]

we get

\[ \frac{1}{2} \Delta |\mu|^2 \geq (|\mu|^2 + nH^3) - nH \{\text{tr} \mu^3 + 3H|\mu|^2 + nH^3\} + nc|\mu|^2 \]

(4.9)

\[ = |\mu|^2 + n H^2 - nH \text{tr} \mu^3. \]

Because of \( \text{tr} \mu = 0 \), we can apply Lemma 3.2 to the eigenvalues of \( \mu \) and obtain

\[ |\text{tr} \mu|^2 \leq \frac{n-2}{n(n-1)} |\mu|^2. \]

(4.10)

Hence we obtain

\[ \frac{1}{2} \Delta |\mu|^2 \geq |\mu|^2 \left( |\mu|^2 - nH^2 \frac{n-2}{n(n-1)} |\mu| + nc - nH^2 \right), \]

where we have used (4.9) and (4.10). From (3.8) we know that the Ricci curvature of \( M \) is bounded from below. Putting \( F = -1/\sqrt{|\mu|^2 + a} \) for any positive number \( a \). Since \( M \) is complete and space-like, we can apply the Generalized Maximum Principle (Theorem 3.1) to the function \( F \). For any given positive number \( \varepsilon > 0 \), there exists a point \( p \) at which \( F \) satisfies

\[ \sup F < F(p) + \varepsilon, \quad |\text{grad} F|(p) < \varepsilon, \quad \Delta F(p) < \varepsilon. \]

(4.12)

Consequently the following relationship

\[ \frac{1}{2} F(p)^2 \Delta |\mu|^2(p) < 3\varepsilon^2 - F(p)\varepsilon \]

(4.13)

can be derived by the simple and direct calculations. For a convergent sequence \( \{\varepsilon_m\} \) such that \( \varepsilon_m \to 0 \) \((m \to \infty)\) and \( \varepsilon_m > 0 \), there exists a point sequence \( \{p_m\} \) such that \( \{F(p_m)\} \) converges to \( F_0 = \sup F \) by (4.12). On the other hand, it follows from (4.13) that we have

\[ \frac{1}{2} F(p_m)^2 \Delta |\mu|^2(p_m) < 3\varepsilon_m^2 - F(p_m)\varepsilon_m \]

(4.14)

The right hand side of (4.14) converges to 0 because \( F \) is bounded. Accordingly,
for any positive number \( \varepsilon > 0 \) (\( \varepsilon < 2 \)) there exists a sufficiently large integer \( m \) for which we have

\[ F(p_m) \Delta |\mu|^2(p_m) < \varepsilon. \]

Hence we get

\[
(2-\varepsilon)|\mu|^4(p_m) - 2nH \frac{n-2}{\sqrt{n(n-1)}} |\mu|^4(p_m) + 2(nc-nH^2-\varepsilon a) |\mu|^2(p_m) - \varepsilon a^2 < 0.
\]

Thus the sequence \( \{ |\mu|^2(p_m) \} \) is bounded and the definition of \( F \) gives rise to

\[ \lim_{m \to \infty} |\mu|^2(p_m) = \sup |\mu|^2. \]

Therefore the supremum of \( F \) satisfies \( F_\varepsilon = \sup F < 0 \). According to (4.14) we have

\[ \lim_{m \to \infty} \sup \Delta |\mu|^2(p_m) \leq 0. \]

Thus (4.11) and (4.16) yield

\[ 0 \leq \sup |\mu|^2 \left( \sup |\mu|^2 - nH \frac{n-2}{\sqrt{n(n-1)}} \sup |\mu| + nc - nH^2 \right). \]

Taking account of (4.5) we have

\[ \sup \sum (h_{ij}^{p-1})^2 = nH^2 \text{ or } S_+ (1) \leq \sup \sum (h_{ij}^{p-1})^2 \leq S_+ (1), \]

from which combining with the assumption of Proposition 4.1 it follows that we have

\[ \sup \sum (h_{ij}^{p-1})^2 = nH^2. \]

This means that \( \mu = 0 \) because of (4.5) and therefore \( M \) is pseudo-umbilic.

The inequality (4.17) holds on the space-like submanifold \( M \) of \( M_\varepsilon^{p+p}(c) \). Accordingly, in this case we have

\[ \sup \sum (h_{ij}^{p-1})^2 = nH^2 \text{ or } \sup \sum (h_{ij}^{p-1})^2 \leq S_+ (1). \]

**Remark.** When \( p = 1 \), the hypersurface \( M \) becomes totally umbilic under the assumption of Proposition 4.1, which means that this property is a generalization of the theorem due to the first author and Nakagawa [6].

5. **Proof of Theorem 1.**

In this section the squared norm \( S \) of the second fundamental form of \( M \) is estimated from above. Let \( M \) be an \( n \)-dimensional space-like submanifold with parallel mean curvature vector \( h \) of an indefinite space form \( M_\varepsilon^{p+p}(c) \).
Proof of Theorem 1. Because the mean curvature vector is parallel, the mean curvature is constant. If \( H = 0 \), then from Theorem 1.1 due to Ishihara [8], we know that \( M \) is totally geodesic if \( c \geq 0 \) and \( S \leq -npc \) if \( c < 0 \). Hence Theorem 1 is true. Next we may suppose \( H \neq 0 \). We choose \( e_{n+1} \) in such a way that its direction coincides with that of the mean curvature vector. Then we get (4.1), (4.2) and (4.3). From (3.13), the Gauss equation (3.7) and (3.10) we get

\[
\frac{1}{2} \Delta \tau^2 = \sum_{\alpha \neq \gamma + 1} (h_{ij}^\alpha)^2 - \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha \Delta h_{ij}^\alpha
\]

\[
= \sum_{\alpha \neq \gamma + 1} (h_{ij}^\alpha)^2 + \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha \Delta h_{ij}^\alpha
\]

\[
- 2 \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha h_{ij}^\beta h_{ij}^\gamma + \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha h_{ij}^\beta h_{ij}^\gamma
\]

\[
- nH \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha h_{ij}^\gamma + \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha h_{ij}^\gamma,
\]

and hence we get

\[
\frac{1}{2} \Delta \tau^2 = \sum_{\alpha \neq \gamma + 1} (h_{ij}^\alpha)^2 + \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha \Delta h_{ij}^\alpha
\]

\[
- 2 \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha h_{ij}^\beta h_{ij}^\gamma + \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha h_{ij}^\beta h_{ij}^\gamma
\]

\[
+ \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha h_{ij}^\beta h_{ij}^\gamma + \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha h_{ij}^\beta h_{ij}^\gamma
\]

\[
- 2 \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha h_{ij}^\beta h_{ij}^\gamma + \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha h_{ij}^\beta h_{ij}^\gamma
\]

\[
- nH \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha h_{ij}^\gamma + \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha h_{ij}^\gamma.
\]

(5.1)

We put \( S_{\alpha \beta} = \sum h_{ij}^\alpha h_{ij}^\beta \) for any \( \alpha, \beta \neq n + 1 \). Then \( (S_{\alpha})_\beta \) is a \((p - 1) \times (p - 1)\) symmetric matrix. It can assumed to be diagonal for a suitable choice of \( e_{n+1}, \ldots, e_{n+p} \). Set \( S_{\alpha} = S_{\alpha \alpha} \). We then have \( \tau^2 = \sum S_{\alpha} \). In general, for a matrix \( A = (a_{ij}) \), we define \( N(A) = \text{tr}(A^2) \). Hence we get

\[
\sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha h_{ij}^\beta h_{ij}^\gamma + \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha h_{ij}^\beta h_{ij}^\gamma
\]

\[
+ \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha h_{ij}^\beta h_{ij}^\gamma + \sum_{\alpha \neq \gamma + 1} h_{ij}^\alpha h_{ij}^\beta h_{ij}^\gamma
\]

\[
= \sum_{\alpha \neq \gamma + 1} (S_{\alpha})_\beta + \sum_{\alpha \neq \gamma + 1} N(H_{\alpha} H_{\beta} - H_{\beta} H_{\alpha}).
\]

(5.2)

Obviously, we see

\[
\sum_{\alpha \neq \gamma + 1} N(H_{\alpha} H_{\beta} - H_{\beta} H_{\alpha}) \geq 0.
\]

Suppose \( p \geq 2 \). Let
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\[(p - 1)\sigma_i = \tau^2 = \sum S_a,\]
\[(p - 1)(p - 2)\sigma_i = 2 \sum_{a \prec \beta, \alpha, \beta \prec n + 1} S_a S_\beta.\]

Then we get
\[
\sum(S_a)^2 = (p - 1)(\sigma_i)^2 + (p - 1)(p - 2)(\sigma_i)^2 - \sigma^2 = \sum_{a \prec \beta, \alpha, \beta \prec n + 1} (S_a - S_\beta)^2.
\]

Hence we obtain
\[
\sum_{a, \beta \prec n + 1} h^a_{k_m} h^{k_m}_{k_j} h^{k_j}_{l_j} \geq \sum_{a, \beta \prec n + 1} h^a_{k_m} h^{k_m}_{k_j} h^{k_j}_{l_j} - 2 \sum_{a, \beta \prec n + 1} h^{n+1}_{k_m} h^{n+1}_{k_j} h^{n+1}_{l_j} - 2 \sum_{a, \beta \prec n + 1} h^{n+1}_{k_m} h^{n+1}_{k_j} h^{n+1}_{l_j}
\]
\[
= (p - 1)(\sigma_i)^2 - \frac{1}{p - 1} \tau^4.
\]

Then the equations (5.1), (5.2) and (5.3) imply
\[
\frac{1}{2} \Delta \tau^2 \geq n c \tau^2 + \frac{1}{p - 1} \tau^4 + \sum_{a, \beta \prec n + 1} h^a_{k_m} h^{n+1}_{k_j} h^{n+1}_{l_j} - 2 \sum_{a, \beta \prec n + 1} h^{n+1}_{k_m} h^{n+1}_{k_j} h^{n+1}_{l_j}
\]
\[
\geq (p - 1)(\sigma_i)^2 - \frac{1}{p - 1} \tau^4.
\]

For a fixed index \(\alpha\), since \(H^a H^{n+1} = H^{n+1} H^a\), we can choose \(\{e_1, \ldots, e_n\}\) such that
\[
h^{e_i}_{e_i} = \lambda^2 \delta_{i,i}, \quad h^{e_i+1}_{e_i+1} = \lambda \delta_{i,i}.
\]

Then we get
\[
\sum h^a_{k_m} h^{n+1}_{k_j} h^{n+1}_{l_j} - 2 \sum h^{n+1}_{k_m} h^{n+1}_{k_j} h^{n+1}_{l_j} + \sum h^{n+1}_{k_m} h^{n+1}_{k_j} h^{n+1}_{l_j}
\]
\[
= (\sum \lambda_i \lambda_i^2)^2 - n H \sum \xi_i (\lambda_i^2)^2.
\]

We notice here that eigenvalues \(\lambda_i\) are bounded by (4.19). In order to estimate the last term on the above equation, the following property is prepared.

**Lemma 5.1.** Let \(a_1, \ldots, a_n\) be real numbers satisfying \(\Sigma a_i = 0\) and let \(b_1, \ldots, b_n\) be also real numbers. Then we have
\[
\sum a_i (b_i)^2 \leq \sqrt{n - 1 \over n} \sqrt{\sum (a_i)^2} \sum (b_i)^2,
\]
where the equality holds if and only if the \(n - 1\) of \(a_i\)'s are equal with each other and the corresponding \(n - 1\) of \(b_i\)'s are equal to 0.
Proof. We consider the function \( f = \sum a_i (b_i)^2 \) with constraint \( \sum a_i = 0, \sum (a_i)^2 = a \) and \( \sum (b_i)^2 = b \). Then there exists a critical point of \( f \) on \( \mathbb{R}^2 \) at which we have

\[
(b_i)^2 + \mu_i + 2\mu_i a_i = 0, \quad 2a_i b_i + 2\mu_i b_i = 0.
\]

From (5.4) we get

\[
\mu_i = -\frac{1}{n} b,
\]

and the critical value of \( f \) is equal to \(-\mu_i b = -2\mu_i a\), and therefore we have

\[
(5.5) \quad a_i = -\mu_i, \quad \text{or} \quad b_i = 0.
\]

If \( a_i = -\mu_i \) for any index \( i \), then we get \( f = 0 \), because of \( \sum a_i = 0 \). If \( a_i = -\mu_i, 1 \leq i \leq m \) and \( b_j = 0, m+1 \leq j \leq n \), then we have from (5.4)

\[
2\mu_i a_j = \frac{1}{n} b, \quad j = m + 1, \ldots, n.
\]

If \( \mu_i = 0 \), then \( f = 0 \). Without loss of generality, we may suppose \( \mu_i \neq 0 \). Thus we see

\[
a_j = \frac{b}{2n\mu_i}, \quad j = m + 1, \ldots, n,
\]

which yields

\[
(5.6) \quad m \mu_i = (n - m) \frac{b}{2n\mu_i}.
\]

From (5.4) and (5.5) it follows that we obtain

\[
\mu_i = \pm \frac{1}{2} \sqrt{\frac{n-m}{nm}} \frac{b}{\sqrt{a}},
\]

which means that we obtain

\[
|f| \leq \frac{1}{n} \sqrt{n-1} \sqrt{\sum (a_i)^2} \sum (b_i)^2.
\]

If the equality holds, then we have \( m = 1 \) and \( a_j = \pm \sqrt{a/(n-1)} \), \( b_j = 0 \), \( 2 \leq j \leq n \).

The converse is obvious. \( \square \)

According to Lemma 5.1 we have

\[
(\sum \lambda_i \lambda_j^2) - nH \sum \lambda_i (\lambda_j^2) \geq -nH \sum \lambda_i - H(\lambda_j^2) - nH^2 \sum (\lambda_j^2)
\]

\[
= -nH (\sum \mu_i (\lambda_j^2)^2 + H \sum (\lambda_j^2)^2)
\]

\[
\geq -nH \left( \sqrt{\frac{n-1}{n}} \sqrt{\sum (h_{ij}^{n+1})^2} - nH^2 + H \right) \text{tr}(H)\).
\]
The right hand side of the inequality above does not depend on the choice of frame fields. Therefore we have

\[
\sum_{\alpha \neq n+1} h_{\alpha m}^* h_{m k}^* h_{ij}^* - 2 \sum_{\alpha \neq n+1} h_{\alpha m}^* h_{m j}^* h_{ij}^* + \sum_{\alpha \neq n+1} h_{\alpha m}^* h_{m k}^* h_{ij}^*
\]

\[= -nH \sum_{\alpha \neq n+1} h_{\alpha m}^* h_{m j}^* h_{ij}^* + \sum_{\alpha \neq n+1} h_{\alpha m}^* h_{m k}^* h_{ij}^* \]

\[\geq -nH \left( \sqrt{\frac{n-1}{n}} \sqrt{\sum (h_{ij}^*)^2} - nH^2 + H \right) \tau^2 .\]

Thus we have

\[\frac{1}{2} \Delta \tau^2 \geq \left\{ n c - nH \left( \sqrt{\frac{n-1}{n}} \sqrt{\sum (h_{ij}^*)^2} - nH^2 + H \right) \right\} \tau^2 + \frac{1}{p-1} \tau^4 .\]

Making use of the same proof as in the proof of \( |\mu|^2 \) above, we have

\[0 \geq \left\{ n c - nH \left( \sqrt{\frac{n-1}{n}} \sqrt{\sum (h_{ij}^*)^2} - nH^2 + H \right) \right\} \tau^2 + \frac{1}{p-1} \tau^4 .\]

Thus from (4.19) we get

\[\text{(5.7)} \quad \text{sup} \tau^2 \leq (p-1) \left\{ nH \left( \sqrt{\frac{n-1}{n}} \sqrt{S_\alpha(1)} - nH^2 + H \right) - nc \right\} .\]

The equality (4.6), the inequalities (4.19) and (5.7) yield

\[S \leq S_\alpha(1) + (p-1)H \{ nH + \sqrt{n(n-1)} \{ S_\alpha(1) - nH^2 \} \} .\]

Hence we complete the proof of Theorem 1. \( \square \)

Remark. When \( M \) is maximal (i.e., \( H=0 \)), Theorem 1 implies \( S \leq -n \rho \). Ishihara [8] obtained this relation for complete maximal space-like submanifolds. When \( p=1 \), Theorem 1 becomes \( S \leq S_\alpha(1) \). This result is obtained by the first author and Nakagawa [6]. Hence Theorem 1 generalizes the results above.

6. Proof of Theorems 2 and 3.

Let \( M \) be an \( n \)-dimensional complete space-like submanifold with parallel mean curvature vector of \( M^p \), \( c \leq 0 \). We assume \( S=S_\alpha(1) + K(p) \). Then the equalities of all inequalities in the previous sections have to hold. Consequently, from (4.8) and (5.7) it is seen that

\[\text{(6.1)} \quad h_{\alpha i}=0\]

for any \( i, j, k \) and \( \alpha \). Also from (4.2) and (5.7) it follows that

\[H^\alpha H^\beta = H^\beta H^\alpha\]

for any \( \alpha \) and \( \beta \). The equations imply that all of \( H^\alpha \) are simultaneously...
diagonalizable and the normal connection in the normal bundle of $M$ is flat. Hence we can choose a suitable basis $\{e_i\}$ such that

$$h_{ij}^\alpha = \lambda_i^\alpha \delta_{ij}$$

for any $i$, $j$ and $\alpha$. The submanifold $M$ is said to be isoparametric [13] if the normal connection is flat and the characteristic polynomial of the shape operator $A_\xi$ has constant coefficients over the domain of any local parallel normal field $\xi$.

**Lemma 6.1.** $M$ is isoparametric.

**Proof.** Since the normal connection is flat, it is seen that there exist locally $p$ mutually orthogonal unit normal vector fields which are parallel in the normal bundle. So we can choose a suitable parallel basis $\{e_\alpha\}$ and then we have $\omega_{\alpha \beta} = 0$. Hence, since we have

$$\sum h_{ij}^\alpha \omega_k = d h_{ij}^\alpha - \sum h_{ij}^\beta \omega_k = \sum h_{ij}^\beta \omega_k + \sum h_{ij}^\beta \omega_{\beta \alpha},$$

setting $i = j$ in the above equation and using (6.1) we get $d h_{ij}^\alpha = 0$. Hence $h_{ij}^\alpha$ is constant and $M$ is isoparametric.

**Lemma 6.2.** $M$ is of non-positive curvature.

**Proof.** Suppose that there exist indices $i$, $j$ and $\alpha$ such that $h_{ij}^\alpha \neq h_{ij}^\beta$. From the equation (6.3) we get

$$\sum h_{ij}^\alpha \omega_k = \sum h_{ij}^\beta \omega_k = (h_{ij}^\alpha - h_{ij}^\beta) \omega_{ij} = 0,$$

from which it follows that $\omega_{ij} = 0$. Accordingly, we have

$$\sum \omega_{ij} \wedge \omega_{kj} = 0.$$ 

In fact, for any fixed indices $i$ and $\alpha$ we denote by $[i]$ the set consisting of indices $k$ such that $h_{ij}^\alpha = h_{kj}^\alpha$. Then we have $[i] \neq [j]$ by the supposition and hence we get

$$\sum_{k \in [i]} \omega_{ij} \wedge \omega_{kj} = \sum_{k \in [i]} \omega_{ij} \wedge \omega_{kj} + \sum_{k \in [j]} \omega_{ij} \wedge \omega_{kj} = \sum_{k \in [i] \wedge [j]} \omega_{ij} \wedge \omega_{kj},$$

each term of which vanishes identically. By the structure equation

$$d \omega_{ij} + \sum \omega_{ij} \wedge \omega_{kj} = -\frac{1}{2} \sum R_{ij \beta} \omega_k \wedge \omega_i,$$

we obtain

$$R_{ij \beta} = c - \sum \lambda_i \lambda_j = 0.$$
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Next, suppose that $h^i_i = h^j_j$ for any distinct indices $i$ and $j$ and for any index $\alpha$. Then the Gauss equation implies

$$R_{ijkl} = c - \sum_\alpha (h^i_i)^\alpha - \sum_\alpha (\lambda^\alpha)^\alpha \leq 0,$$

because of $c \leq 0$.

Thus $M$ is of non-positive curvature. □

**Proof of Theorem 2.** By a theorem due to Koike [10] and Lemmas 6.1 and 6.2 it is seen that $M$ is locally congruent to the product submanifold $H_{n(1)} \times \cdots \times H_{n(2)} \times R^n$ of $R^{n+p}$, where $\sum p_i n_p + m = n$ and $1 \leq q \leq p$. Then $M$ can be naturally regarded as the space-like submanifold of $R^{n+p}$ whose mean curvature vector is given by (2.1). It is also parallel in the normal bundle of $M$ in $R^{n+p}$. The constant $S(1)$ and the squared norm $S$ of the second fundamental form are given by (2.2). Therefore it is seen that we have

$$S_+(p) + K(p) = -p \sum r \alpha c_r = S,$$

which implies $p = q = 1$ and $n_1 = 1$. Accordingly the hyperbolic cylinder $H_{1(1)} \times R^{n-1}$ of $R_{1}^{n+1}$ is the complete space-like hypersurface with constant mean curvature whose squared norm $S$ attaining the maximal value. □

**Proof of Theorem 3.** When $p = 1$ it is seen by a theorem due to Ki, Kim and Nakagawa [9] that the hyperbolic cylinder $H_{1(1)} \times H^{n-1}(\bar{c})$ is the complete space-like hypersurface with constant mean curvature of $H_{1}^{n+1}(\bar{c})$ satisfying the given condition.

Suppose next that $p \geq 2$. By means of Koike’s theorem and Lemmas 6.1 and 6.2 again, $M$ is locally congruent to the product submanifold $H_{n(1)} \times \cdots \times H_{n(2)} \times R^{n+1} \times H_{q}^{n+q}(\bar{c'})$, where $\sum p_i n_p = n$, $\sum q_i (1/c_r) = (1/c') \geq (1/c')$ and $H_{q}^{n+q}(\bar{c'})$ is a totally umbilic submanifold of $H_{p}^{n+p}(c)$. The mean curvature vector of $M$ in $H_{q}^{n+q}(\bar{c'})$ is denoted by $h'$, which is parallel in the normal bundle of $M$ in $H_{q}^{n+q}(\bar{c'})$. Then the mean curvature vector $h$ of $M$ of $H_{p}^{n+p}(\bar{c})$ is given by $h = h' + h''$, where $h''$ is the mean curvature vector of $H_{q}^{n+q}(\bar{c'})$ in $H_{p}^{n+p}(c)$. Consequently the mean curvature vector $h$ is parallel in the normal bundle $N(M)$ and the mean curvature $H$ and the squared norm $S$ of $M$ in $H_{p}^{n+p}(c)$ are given by

$$h^2 = n^2 H^2 = n^2 c - \sum r \alpha c_r,$$

$$S = nc - \sum r \alpha c_r.$$
We have $S,1 \geq h^2 - nc$, because of $c < 0$. So it is seen by Lemma 2.1 that we obtain

\[(6.4) \quad S,1(p) + K(p) - S \geq h^2 - pnc + (p - 1)h^2 - S = ph^2 - pnc - S \geq 0,\]

where the equality holds if and only if $H = 0$. Accordingly, if we have $S = S,1(p) + K(p)$, then $H$ must vanish identically. This implies that Theorem 3 is proved by a theorem due to Ishihara [8]. □

References


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