DEGREE OF THE STANDARD ISOMETRIC MINIMAL IMMERSSIONS OF THE SYMMETRIC SPACES OF RANK ONE INTO SPHERES

Dedicated to Professor Isamu Mogi on his 60th birthday

By

Katsuya Mashimo

Introduction

Let \((M=\mathbb{G}/K,g)\) be an irreducible symmetric space of compact type. We denote by \(\mathcal{A}\) the Laplace-Beltrami operator of \((M,g)\) acting on the space of \(C^\infty\) functions on \(M\). We denote by \(\lambda_k\) the \(k\)-th eigen-value of \(\mathcal{A}\) and by \(V^k\) the corresponding eigen-space. For each integer \(k\), \(k \geq 1\), there exists an isometric minimal immersion \(x_k\) of \(M\) into the unit sphere in \(R^{m(k)+1}\) defined by an orthonormal base of \(V^k\), which we call the \(k\)-th standard isometric minimal immersion of \(M\).

do Carmo and Wallach [2] showed that the \(k\)-th standard isometric minimal immersion of a sphere \(S^n\) has degree \(k\). We showed that the \(k\)-th standard minimal immersion of a complex projective space \(CP^n, n \geq 2\), has degree \(2k\) [5]. In these cases the degree of the immersion coincides with the algebraic degree of the homogeneous polynomials which define the immersion. In this note we determine the degree of the standard isometric minimal immersions of the other symmetric spaces of rank one into spheres.

**Theorem A.** Let \(x_k\) be the \(k\)-th standard isometric minimal immersion of a quaternion projective space \(QP^n, n \geq 2\). Then \(x_k\) has degree \(2k\).

Let \(\pi: S^{n+3} \rightarrow QP^n\) be the Hopf fibration, where we consider \(S^{n+3}\) as the unit sphere in \(Q^{n+1} = C^{2n+2}\). Then for each eigen-function \(f\) on \(QP^n\) which belongs to \(V^k\), there exists a homogeneous harmonic polynomial \(P_f\) of type \((k,k)\) on \(C^{2n+2}\) which induces \(f\) through \(\pi\). So the degree of the immersion coincides with the algebraic degree of the homogeneous polynomials which define the immersion.

**Theorem B.** Let \(x_k\) be the \(k\)-th standard isometric minimal immersion of the Cayley projective plane \(CayP^n\). Then \(x_k\) has degree \(2k\).

Received May 11, 1981.
The author is grateful to his advisor Professor T. Takahashi and Professors H. Nakagawa and R. Takagi for their hearty guidances.

1. Preliminaries

Let \((M^n, g)\) be an irreducible symmetric space of compact type. We denote by \(\Delta\) the Laplace-Beltrami operator of \((M, g)\). Let \(\lambda_k\) be the \(k\)-th eigen-value of \(\Delta\), \(0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots\), and \(V^k\) be the corresponding eigen-space. We define an inner product \((\cdot, \cdot)\) in \(V^k\) as follows:

\[
(f, h) = \int_M f \cdot h \, d\mu, \quad f, h \in V^k
\]

where \(d\mu\) is the canonical measure of \((M, g)\) normalized in such a way that

\[
\int_M d\mu = \dim. V^k = m(k) + 1.
\]

Let \(f_0, f_1, \ldots, f_{m(k)}\) be an orthonormal base of \(V^k\). Then

\[
x_k : M \to R^{m(k)+1}; \quad p \mapsto (f_0(p), f_1(p), \ldots, f_{m(k)}(p))
\]

realizes an isometric minimal immersion of \((M, (\lambda_k)_{k=0}^\infty)\) into the unit sphere in \(R^{m(k)+1}\). We call this isometric minimal immersion the \(k\)-th standard minimal immersion of \(M\).

Let \((G, K)\) be a symmetric pair corresponding to \(M\). Then \(G\) acts on \(V^k\) by

\[
(\sigma \cdot f)(p) = f(\sigma^{-1} \cdot p), \quad \sigma \in G, \quad p \in M.
\]

Under this action of \(G\) on \(V^k\), \(v_0 = \sum_i f_i(eK) f_i\) is a \(K\)-fixed unit vector. Then \(x_k\) is equivalent to the following isometric minimal immersion of \(M\) into the unit sphere \(S^{m(k)}\) in \(V^k\) under the identification \(V^k \to R^{m(k)}; f \mapsto (x_0)\) where \(f = \sum_i x_i f_i\);

\[
x_k : M \to S^{m(k)}; \quad \sigma K \mapsto \sigma \cdot v_0, \quad \sigma \in G.
\]

Thus \(x_k\) is an equivariant immersion, i.e., there exists a Lie group homomorphism \(\rho\) of \(G\) into the isometry group \(I(S^{m(k)})\) of the unit sphere \(S^{m(k)}\) in \(R^{m(k)+1}\) such that

\[
x_k(\sigma \cdot p) = \rho(\sigma) \cdot x_k(p), \quad \sigma \in G, \quad p \in M.
\]

Let \(B_{j1p} = (x_k)_p\) and let \(B_{j1p}\) be the second fundamental form of \(x_k\) at \(p\). Inductively we define the higher fundamental forms. Here we denote by \(O_p(M)\), \(j \geq 2\), the linear span of Image \(B_{j1p}\), and by \(N_{j1p}\) the orthogonal projection of the tangent space \(T_p(S^{m(k)})\) to the orthogonal complement of \((T_p M + O_p(M) + \cdots + O_p(M))\):

\[
N_{j1p} : T_p(S^{m(k)}) \to (T_p M + O_p(M) + \cdots + O_p(M))^{\perp}.
\]
Then the \((j+1)\)-th fundamental form \(B_{j+1;p}\) is defined by

\[
B_{j+1;p}(u_0, u_1, \ldots, u_j) = \left( (\nabla_{u_i} B_j(U_1, U_2, \ldots, U_j)) \right)^{x_i}_{p},
\]

where \(\nabla\) is the covariant derivative in \(S^{x}(\kappa)\) and \(U_i, 1 \leq i \leq j\), are local extensions of \(u_i\). The smallest positive integer \(d\) such that \(B_d|^p = 0\) and \(B_{d+1;p} = 0\) is called the degree of \(x_k\). By the following Lemma 1 the definition of the degree is independent of the choice of \(p\).

**Lemma 1.** (1) \(B_j\) is \(G\)-invariant, i.e.,

\[
B_{j;e;p}(\sigma \cdot u_1, \ldots, \sigma \cdot u_j) = \rho(\sigma) \cdot B_{j;p}(u_1, \ldots, u_j),
\]

\[\sigma \in G, ~ u_1, u_2, \ldots, u_j \in T_pM.\]

So we get

\[
\rho(\sigma) \cdot O_k(M) = O_k(M), \quad \sigma \in G.
\]

(2) \(B_{j;p}\) is a symmetric \(j\)-multilinear mapping of \(T_pM\) onto \(O_k(M)\).

Since \(B_{j;p}\) is symmetric we regard it as a linear mapping of the \(j\)-th symmetric power \(S^j(T_pM)\) onto \(O_k(M)\) as follows:

\[
B_{j;p}(u_1, \ldots, u_j) = B_{j;p}(u_1, \ldots, u_j)
\]

where \(u_1, \ldots, u_j\) is the symmetric product of \(u_i, \ldots, u_j \in T_pM\). Extend the isotropy action of \(K\) on \(T_{eK}M\) to \(S^j(T_{eK}M)\) in a natural manner. Hereafter we only consider the fundamental forms at the origin \(eK \in M\). So we omit the subscript \(eK\). We have the following

**Lemma 2.** (1) \(B_j\) is a \(K\)-homomorphism of \(S^j(T_{eK}M)\) to \(O_k(M)\).

(2) \(V^k\) admits the following orthogonal direct sum decomposition as \(K\)-module

\[
V^k = R_{p} + T_{eK}M + O_k^1(M) + \cdots + O_k^d(M),
\]

where \(d\) is the degree of \(x_k\).

For the proof of Lemma 1 and Lemma 2 we refer, for instance, to [5].

Let \(\mathfrak{g}\) and \(\mathfrak{k}\) be the Lie algebras of \(G\) and \(K\) respectively, and \(\mathfrak{p}\) be the orthogonal complement of \(\mathfrak{k}\) in \(\mathfrak{g}\) with respect to the Killing form \(B\) of \(\mathfrak{g}\). We identify \(\mathfrak{p}\) with the tangent space \(T_{eK}M\) in a natural manner. Then the isotropy action of \(K\) on \(\mathfrak{p}\) is the adjoint action of \(K\) on \(\mathfrak{p}\). We identify the \(j\)-th symmetric power \(S^j(T_{eK}M) = S^j(\mathfrak{p})\) with the space of real valued homogeneous polynomials on \(\mathfrak{p}\) of degree \(j\) in a natural manner. Take an orthonormal base \(e_1, \ldots, e_n\) of \(\mathfrak{p}\) and put
Since the adjoint action of $K$ on $\mathfrak{p}$ leaves invariant the inner product $B_{\mathfrak{p} \times \mathfrak{p}}$ induced by the Killing form $B$ of $\mathfrak{g}$ we have the following decomposition of $S^J(\mathfrak{p})$ as $K$-module ([8], p. 255)

$$S^J(\mathfrak{p}) = \begin{cases} H_j, & j=0,1, \\ H_j + r \cdot S^{J-2}(\mathfrak{p}) \text{ (direct sum)}, & j \geq 2. \end{cases}$$

where $H_j$ is the space of homogeneous harmonic polynomials of degree $j$ on $\mathfrak{p}$. We have the following Lemma due to do Carmo and Wallach [2].

**Lemma 3.** Ker $B_{\mathfrak{f}} \supseteq r \cdot S^{J-2}(\mathfrak{p})$, $j \geq 2$.

2. Proof of Theorem B

Let $\mathfrak{g}$ be the compact real form of the complex simple Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{f}$. Let $G$ be the connected Lie group with the Lie algebra $\mathfrak{g}$. Then $G$ contains a Lie subgroup $K$ which is isomorphic to Spin $(9)$ and $(G, K)$ is a symmetric pair corresponding to the Cayley projective plane $\text{Cay} P^3$ [6]. Let $\mathfrak{f}$ be the Lie algebra of $K$ and $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{f}$ in $\mathfrak{g}$ with respect to the Killing form $B$ of $\mathfrak{g}$. Let $\tilde{\mathfrak{f}}$ [resp. $\tilde{\mathfrak{p}}$] be the complexification of $\mathfrak{f}$ [resp. $\mathfrak{p}$] in $\tilde{\mathfrak{g}}$ and $\theta$ be the automorphism of $\tilde{\mathfrak{g}}$ such that $\theta|\tilde{\mathfrak{f}} = \text{Id}$ and $\theta|\tilde{\mathfrak{p}} = -\text{Id}$.

Let $\mathfrak{h}$ be a maximal abelian subalgebra of $\mathfrak{f}$. Then $\mathfrak{h}$ is also a maximal abelian subalgebra of $\mathfrak{g}$ and the complexification $\tilde{\mathfrak{h}}$ of $\mathfrak{h}$ in $\tilde{\mathfrak{g}}$ is a common Cartan subalgebra of $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{f}}$. We regard roots and weights as elements of $(-1)^{1/2} \mathfrak{h}$ via $B$. We denote by $\varpi$ the set of non-zero roots of $\tilde{\mathfrak{g}}$ with respect to $\tilde{\mathfrak{f}}$. We fix a lexicographic order in $(-1)^{1/2} \mathfrak{h}$ and take simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of $\varpi$ such that

$$\begin{pmatrix} 2B(\alpha_i, \alpha_j)/B(\alpha_j, \alpha_j) \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$ 

Take fundamental weight system $\mu_1, \mu_2, \mu_3, \mu_4$ of $\tilde{\mathfrak{g}}$. Then it is easily seen that $\theta = \exp(2\pi(-1)^{1/2} \mu_1)$ and that

$$\beta_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4, \quad \beta_2 = -\alpha_1, \quad \beta_3 = -\alpha_2, \quad \beta_4 = -\alpha_3,$$

is a fundamental root system of $\tilde{\mathfrak{f}}$ ([4], p. 121), i.e., each root of $\tilde{\mathfrak{f}}$ is written uniquely in the form $\beta = \sum m_i \beta_i$ with all non-positive or all non-negative integers $m_i$. Let $\nu_1, \nu_2, \nu_3, \nu_4$ be the fundamental weight system of $\tilde{\mathfrak{f}}$. From (2.1) and [3] (p. 69) we get

$$\mu_i = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 = \beta_1 + \beta_2 + \beta_3 + \beta_4,$$
Since Cay \( P^k \) is a symmetric space of rank one, each eigen-space \( V^k \) of \( \Delta \) acting on real valued \( C^\infty \) functions is a class one orthogonal representation of \((G, K)\), i.e., \( V^k \) is \( G \)-irreducible and contains a non-zero \( K \)-fixed vector \([1]\). It is known that \((V^k)^G\) is also \( G \)-irreducible. So \((V^k)^G\) is a class one unitary representation. We know that the highest weight of irreducible \( G \)-module \((V^k)^G\) is \( k\mu_4 \) \([8]\).

To prove Theorem B we need the following

**Lemma 4 (Smith [7]).** Let \( e_1, \ldots, e_{16} \) be an orthonormal base of \( \mathfrak{p} \) with respect to \( B_{12 \times 3} \). Let \( r = \sum_{i=1}^{16} e_i \in S^4(\mathfrak{p}) \). Then \((S^4(\mathfrak{p}))^G\) is decomposed under adjoint representation as follows:

\[
(S^4(\mathfrak{p}))^G = \bigoplus_{j=1}^{2j} V(\nu_j) + \sum_{2p+q+j} \bigoplus_{(p,q) \in I} V(p\nu_1 + q\nu_4) + \text{(direct sum), } j \geq 2,
\]

where \( V(\nu) \) is the irreducible \( K \)-module with highest weight \( \nu \).

Extend the \( j \)-th fundamental form \( B_j : S^4(\mathfrak{p}) \to O_{16}(\text{Cay } P^n) \) to a complex linear mapping \( B_j : (S^4(\mathfrak{p}))^G \to (O_{16}(\text{Cay } P^n))^G \). Then by Lemma 3 and Lemma 4 we get

\[
(O_{16}(\text{Cay } P^n))^G = B_j(\sum_{2p+q+j} V(p\nu_1 + q\nu_4)).
\]

By Schur's Lemma \( B_j | V(p\nu_1 + q\nu_4) \) is zero or an isomorphism. So we denote by \( I \) the set of indices \((p, q)\) such that \( B_j | V(p\nu_1 + q\nu_4) \) is an isomorphism. Then we get the following decomposition of \((V^k)^G\) from (1.1)

\[
(V^k)^G = V(0) + V(\nu_4) + \sum_{j=2}^{2j} \sum_{2p+q+j \in I} V(p\nu_1 + q\nu_4),
\]

where \( d \) is the degree of the \( k \)-th standard minimal immersion. From (2.5) we get

\[
d = \text{Max}_{(p,q) \in I} (2p + q).
\]

Let \((V^k)^G = \sum_{s \in K} V_s\) be the weight space decomposition of \((V^k)^G\) as a \( G \)-module. Then it is also the weight space decomposition as a \( K \)-module since \( K \) has the same rank as \( G \).

Let \((p, q)\) be a pair contained in \( I \). Then by the above remark \( p\nu_1 + q\nu_4 \) is a weight of \((V^k)^G\) as a \( G \)-module. Each weight of \((V^k)^G\) as a \( G \)-module is written in the following form

\[
k\mu_4 - \sum_{i=1}^{4} m_i \mu_i,
\]

with non-negative integers \( m_i \). Choose \( m_i \) such that
(2.7) \( p\nu_1 + q\nu_4 = k\mu_4 - \sum \iota_{i+1} m\alpha_i \).

By (2.2) and (2.3) we get
\[
\begin{align*}
    p\nu_1 + q\nu_4 &= (p+q/2)\beta_1 + (p+q)\beta_2 + (p+3q/2)\beta_3 + (p+2q)\beta_4 \\
    k\mu_4 - \sum \iota_{i+1} m\alpha_i &= (k-m_i/2)\beta_1 + (k+m_i-m_2)\beta_2 + (k+m_2-3m_i/2)\beta_3 + (k+m_3-2m_i)\beta_4.
\end{align*}
\]

Comparing the coefficient of \( \beta_i \) on both sides of (2.7) we get
\[
    p + q/2 = k - m_i/2, \text{ for any } (p, q) \epsilon I.
\]

So we get \( d \geq 2k \) from (2.6).

Let \( \pi_{p, q} : (V^*)^p \rightarrow V(p\nu_1 + q\nu_4) \), \( (p, q) \epsilon I \), be the orthogonal projection with respect to the decomposition (2.5). Let \( v_0 \) be the weight vector corresponding to \( k\mu_4 \).

Since \( v_0 = \sum_{(p, q) \epsilon I} \pi_{p, q} v_0 \) there exists a pair \( (p, q) \epsilon I \) such that \( \pi_{p, q} v_0 \neq 0 \). Since \( \pi_{p, q} \) is a \( K \)-homomorphism \( k\mu_4 \) is a weight of \( V(p\nu_1 + q\nu_4) \). Choose non-negative integers \( n_i \) such that

(2.8) \( k\mu_4 = p\nu_1 + q\nu_4 - \sum \iota_{i+1} n_i \beta_i \).

By (2.2) and (2.3) we get
\[
\begin{align*}
    k\mu_4 &= k(\beta_1 + \beta_2 + \beta_3 + \beta_4) \\
    p\nu_1 + q\nu_4 - \sum \iota_{i+1} n_i \beta_i &= (p+q/2-n_i)\beta_1 + (p+q-n_2)\beta_2 + (p+3q/2-n_3)\beta_3 + (p+2q-n_4)\beta_4.
\end{align*}
\]

Comparing the coefficient of \( \beta_i \) on both sides of (2.8) we get
\[
    p + q/2 - n_i = k.
\]

So we get \( d \geq 2k \) from (2.6). So the Theorem B is proved.

We can prove Theorem A in the same manner as the proof of Theorem B. Necessary facts are found in [7].

References

[5] Mashimo, K., Degree of the standard isometric minimal immersions of complex pro-
Degree of the standard isometric minimal immersions


Institute of Mathematics
University of Tsukuba
Sakura-mura Niihari-gun
Ibaraki, 305 Japan