FACTORIZATION THEOREM FOR PERFECT MAPS BETWEEN METRIZABLE SPACES

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1. Introduction. We assume that all spaces are normal and all maps are continuous. We write $A \in \text{ANR}$ for a space $A$ if $A$ is an ANR for the class of all compact metrizable spaces.

Given spaces $X$ and $A$ we write $\dim X \leq A$ if for any closed subset $F$ of $X$ any map $f: F \rightarrow A$ can be extended to $X$. For a map $\xi: X \rightarrow X_0$ we write $\dim \xi \leq A$ if $\dim \xi^{-1}(x_0) \leq A$ for any $x_0 \in X_0$. It is known that a space $X$ satisfies the relation $\dim X \leq S^n$ for the $n$-sphere $S^n$ if and only if $X$ satisfies the inequality $\dim X \leq n$ in the sense of the covering dimension.

Our purpose in this paper is to prove the following theorem:

**THEOREM.** Let $A \in \text{ANR}$, let $\xi$ be a closed map of a space $X$ into a paracompact space $X_0$, $\zeta$ be a perfect map of a metrizable space $Z$ into a metrizable space $Z_0$, and let $f: X \rightarrow Z$ and $f_0: X_0 \rightarrow Z_0$ be maps such that $\zeta f = f_0 \xi$ and $\dim \xi \leq A$. Then there are metrizable spaces $Y$ and $Y_0$, a perfect map $\eta: Y \rightarrow Y_0$, and maps $g: X \rightarrow Y$, $g_0: X_0 \rightarrow Y_0$, $h: Y \rightarrow Z$ and $h_0: Y_0 \rightarrow Z_0$ such that $\eta g = g_0 \xi$, $\zeta h = h_0 \eta$, $hg = f$, $h_0 g_0 = f_0$, $\dim \eta \leq A$, $w(Y_0) \leq \max(w(X_0), w(Z_0))$, and $\dim Y_0 \leq \dim X_0$.

![Diagram of maps]

For a map $\zeta: Z \rightarrow Z_0$ we write $w(\zeta) \leq \tau$ if there is an embedding $i: Z \rightarrow Z_0 \times I^\tau$ such that $\zeta = \text{pr} i$, where $I^\tau$ is the Tikhonov cube of weight $\tau$ and $\text{pr}: Z_0 \times I^\tau \rightarrow Z_0$ is the projection.

In [9] Pasynkov proved a similar theorem to the above theorem, in which he added the property that $w(\eta) \leq \tau$, if $w(\xi) \leq \tau$, in the case that $X$, $X_0$, $Z$, $Z_0$ are compact (which are not assumed to be metrizable).

However, in [7] Pasynkov stated that, if $f$ is a perfect map between

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metrizable spaces, the relation $w(f) \leq \omega$ holds. Therefore, in the above theorem we need not to add the property that $w(\eta) \leq \tau$ if $w(\xi) \leq \tau$.

2. Proof of Theorem. The above theorem is an easy consequence of Lemmas 2 and 3 (cf. [9]). We need Lemma 1 to prove Lemma 2. The idea of the proof of Theorem is essentially due to Pasynkov.

**Lemma 1** ([9, (5.2)]). Let $Y \in ANR$. Then for any metric $\rho$ in $Y$ there is an $\varepsilon > 0$ with the following properties; if $f$ is a map of a compact space $X$ into $Y$ and $g$ is a map of a closed set $F$ in $X$ into $Y$ such that $d(g, f|_F) \leq \max\{\rho(g(x), f(x)) ; x \in F\} < \varepsilon$, then $g$ can be extended to $X$.

**Lemma 2.** Under the condition of Theorem there are metrizable spaces $Y$ and $Y_0$, a perfect map $\eta : Y \to Y_0$, and maps $g : X \to Y$, $g_0 : X_0 \to Y_0$, $h : Y \to Z$ and $h_0 : Y_0 \to Z_0$ such that $\eta g = g_0 \xi$, $\xi h = h_0 \eta$, $h_0 g_0 = f_0$, $w(Y) \leq \max(w(X_0), w(Z_0))$, $\dim Y \leq \dim X_0$ and for any $y_0 \in Y_0$, any compact $F \subset \eta^{-1}(y_0)$ and any map $X : h(F) \to A$, the map $\chi h|_F$ can be extended to $\eta^{-1}(y_0)$.

**Proof.** Since $w(\xi) \leq \omega$, there is an embedding $i : Z \to Z_0 \times I^\omega$ such that $\xi \preceq \text{pr}i$, where $I^\omega$ is the Hilbert cube and $\text{pr} : Z_0 \times I^\omega \to Z_0$ is the projection. We denote by $\rho$ the projection of $Z_0 \times I^\omega$ onto $I^\omega$. We choose a countable base $\{O_n : n = 1, 2, \ldots\}$ for $I^\omega$ that is closed under finite unions. We fix a metric $\rho$ on $A$ and choose $\varepsilon > 0$ in accordance with Lemma 1. For any $n$ we fix a countable dense set $C_n$ in $C(\overline{O}_n, A)$, which is the space of maps from $\overline{O}_n$ to $A$ with the metric of uniform convergence.

We fix $n$ and $\varphi \in C_n$. For each $x_0 \in X_0$ we consider the set $\Phi(x_0) = \xi^{-1}(x_0) \cap f^{-1}i^{-1}p^{-1}(\overline{O}_n)$. Since $\dim \xi \leq A$ and $A \in ANR$, the map $\varphi pi f : \Phi(x_0) \to A$ can be extended to $\xi^{-1}(x_0)$ and then to a neighbourhood $V(\xi^{-1}(x_0))$ as a map $\mathcal{V}_x_0 : V(\xi^{-1}(x_0)) \to A$.

Every point $x$ of $\xi^{-1}(x_0)$ has a neighbourhood $O_x \subset V(\xi^{-1}(x_0))$ such that
\[
\text{diam} \varphi pi f(O_x) \cap \overline{O}_n < \varepsilon/4 \quad \text{and} \quad \text{diam} \mathcal{V}_x_0(O_x) < \varepsilon/4.
\]
Since $\xi$ is closed, there is a neighbourhood $V(x_0)$ of $x_0$ such that $\xi^{-1}(V(x_0)) \subset \bigcup \{O_x : x \in \xi^{-1}(x_0)\}$ and $\Phi(x_0) \subset \bigcup \{O_x : x \in \Phi(x_0)\}$ for any $x_0 \in V(x_0)$. Hence, for any $x_0 \in V(x_0)$ and every $x' \in \Phi(x_0)$ we can find a point $x \in \Phi(x_0)$ such that $x' \in O_x$, and hence,
\[
\rho(\mathcal{V}_x_0(x'), \varphi pi f(x')) \leq \rho(\mathcal{V}_x_0(x'), \mathcal{V}_x_0(x)) + \rho(\varphi pi f(x), \varphi pi f(x')) < \varepsilon/4 + \varepsilon/4 = \varepsilon/2.
\]
By paracompactness of \( X_0 \) there is a \( \sigma \)-discrete cozero cover \( \omega(n, \varphi) = \bigcup_{j=1}^{\infty} \{ U_{j,\lambda} : j(\lambda) \in \Gamma_j \} \) of \( X_0 \) such that \( \omega(n, \varphi) \) refines \{ \( V(x_0) : x_0 \in X_0 \) \}. For any \( j \) and each \( j(\lambda) \in \Gamma_j \), we take \( x_{j,\lambda} \in X_0 \) such that \( U_{j,\lambda} \subset V(x_{j,\lambda}) \). For each \( j \) we denote by \( H_j(n, \varphi) \) the Hedgehog space (see [3]) constructed by \{ [0, 1]_{\alpha_{j,\beta}} = [0, 1] : j(\lambda) \in \Gamma_j \}. There is a function \( g_{\alpha,\beta}(n, \varphi) : X_0 \to H_j(n, \varphi) \) such that \( U_{j,\lambda} = g_{\alpha,\beta}(n, \varphi)^{-1}(0, 1]_{\alpha_{j,\beta}} \) for any \( j(\lambda) \in \Gamma_j \). We denote by \( P_j(n, \varphi) \) the partial product (see [6]) with base \( H_j(n, \varphi) \) and fiber \( A \). We perform these constructions for all \( n \) and all \( \varphi \in C_n \). We now set

\[
Y' = Z \times \prod \{ P_j(n, \varphi) : j=1, 2, \ldots, \varphi \in C_n, n=1, 2, \ldots \},
\]

\[
Y_0 = Z_0 \times \prod \{ H_j(n, \varphi) : j=1, 2, \ldots, \varphi \in C_n, n=1, 2, \ldots \}.
\]

Clearly \( Y' \) and \( Y_0 \) are metrizable. We denote by \( h \) (resp. \( h_0 \)) the projection of \( Y' \) onto \( Z \) (resp. \( Y_0 \) onto \( Z_0 \)) and for any \( n, \varphi \in C_n \) and each \( j \) we denote by \( g^{\alpha,\beta}_j(n, \varphi) \) (resp. \( g^{\alpha,\beta}_0(n, \varphi) \) the projection of \( Y' \) onto \( P_j(n, \varphi) \) (resp. \( Y_0 \) onto \( H_j(n, \varphi) \)). We set

\[
\eta = \Pi \{ \xi, \eta(n, \varphi) : j=1, 2, \ldots, \varphi \in C_n, n=1, 2, \ldots \},
\]

\[
g = \Delta(f, g_j(n, \varphi)) = \Pi \{ f, g(n, \varphi) : j=1, 2, \ldots, \varphi \in C_n, n=1, 2, \ldots \} \text{ and}
\]

\[
g_0 = \Delta(f_0, g^\alpha_0(n, \varphi)) = \Pi \{ f, g^\alpha_0(n, \varphi) : j=1, 2, \ldots, \varphi \in C_n, n=1, 2, \ldots \}.
\]

Clearly \( \eta \) is perfect and for any \( n, \varphi \in C_n \) and each \( j \)

\[
\eta(n, \varphi) g^{\alpha,\beta}_j(n, \varphi) = g^{\alpha,\beta}_0(n, \varphi) \eta,
\]

\[
g^{\alpha,\beta}_j(n, \varphi) g = g_j(n, \varphi), \quad g^\alpha_0(n, \varphi) g_0 = g_0(n, \varphi);
\]

\[
h \eta = \eta, \quad h g = f, \quad h g_0 = f_0, \quad \xi h = h \xi, \quad \zeta h = h \zeta.
\]

We set \( Y_0 = g_0(X_0) \) and \( Y = \overline{g(X)} \cap \eta^{-1}(Y_0) \). If we now regard \( \eta, h, g^{\alpha,\beta}_j(n, \varphi) \) and \( h_0, g^{\alpha}_0(n, \varphi) \) as the restrictions of these maps to \( Y \) and \( Y_0 \), respectively, then (2), (3) remain valid, and \( \eta \) is perfect.

We fix a point \( y_0 \in Y_0 \), a compact set \( F \subset \eta^{-1}(y_0) \) and a map \( X : h(F) \to A \). We shall prove that \( Xh \) can be extended to \( \eta^{-1}(y_0) \). Since \( h(F) \subset \zeta^{-1}(h_0(y_0)) \), there is a map \( \varphi' : h \circ h(F) \to A \) such that \( X = \varphi' \pi h \), and hence \( Xh = \varphi' \pi h h \).

Since \( A \subset \ANR \), we may assume that \( \varphi' \) is defined on some \( \tilde{O}_n \) with \( O_n \supset \pi h(F) \). Since \( C_n \) is dense in \( C(\tilde{O}_n, A) \), by [9, Lemma 5.1] there is a map \( \varphi \in C_n \) homotopic to \( \varphi \pi h : F \to A \). Since \( \omega(n, \varphi) \) is a cover of \( X_0 \), there is \( f \) and
$j(\lambda) \in \Gamma_j$ such that $t_0 = g_{t_0}^n(n, \varphi)(y_0) \in (0, 1]_{/\Gamma_1}$. For any $y \in F$ $\pi h(y) \in O_n$, $g(n, \varphi)(y) = \{t_0\} \times A \subset (0, 1]_{/\Gamma_1} \times A$ and $g(X)$ is dense in $Y$, hence there is $y' \in g(X)$ such that $\pi h(y') \in O_n$, $g(n, \varphi)(y') \in (0, 1]_{/\Gamma_1} \times A,$

$$\rho(\pi_{/\Gamma_1}^j g(n, \varphi)(y), \pi_{/\Gamma_1}^j g(n, \varphi)(y')) < \varepsilon/4 \quad \text{and} \quad \rho(\pi h(y), \pi h(y')) < \varepsilon/4.$$  

We take a point $x' \in X$ such that $g(x') = y'$, then $p_{ji}(x') = \pi h(y') \in O_n$, and since $g^j(n, \varphi)\xi(x') = \eta^j(n, \varphi)g(n, \varphi)(y') \in (0, 1]_{/\Gamma_1}$, we have $x' \in \xi^{-1}g(n, \varphi)^{-1}(0, 1]_{/\Gamma_1} = \xi^{-1}U_{/\Gamma_1}$. We set $x'_0 = \xi(x')$ then $x'_0 \in U_{/\Gamma_1} \subset V(x_{/\Gamma_1})$ and $X' \in \varphi(x'_0)$. From (1), we have

$$\rho(\pi_{/\Gamma_1}^j g(n, \varphi)(y'), \varphi p_{ji}(y'))$$

$$= \rho(\pi_{/\Gamma_1}^j g(n, \varphi)(x'), \varphi p_{ji}(x'))$$

$$= \rho(\Theta X_{/\Gamma_1}(x'), \varphi p_{ji}(x')) < \varepsilon/2.$$  

Hence, we see that

$$\rho(\pi_{/\Gamma_1}^j g(n, \varphi)(y), \varphi p_{ji}(y))$$

$$\leq \rho(\pi_{/\Gamma_1}^j g(n, \varphi)(y), \pi_{/\Gamma_1}^j g(n, \varphi)(y'))$$

$$+ \rho(\pi_{/\Gamma_1}^j g(n, \varphi)(y'), \varphi p_{ji}(y'))$$

$$+ \rho(\varphi p_{ji}(y'), \varphi p_{ji}(y))$$

$$< \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon.$$  

The map $\pi_{/\Gamma_1}^j g(n, \varphi)$ is defined on $\eta^{-1}(y_0)$. By Lemma 1, $\varphi p_{ji}(y)$ can be extended to $\eta^{-1}(y_0)$, and by Homotopy extension theorem (see e.g. [4]) $Xh$ can be also extended to $\eta^{-1}(y_0)$.

The fact that $w(Y_0) \leq \max(w(X_0), w(Z_0))$ is evident.

We claim that we may assume that $\dim Y_0 \leq \dim X_0$. By [8, Theorem 2.] there is a metrizable space $Y_0$ and maps $g_0: X_0 \to Y_0$ and $h_0^0: Y_0 \to Y_0$ such that $w(Y_0) \leq w(Y_0)$. $\dim Y_0 \leq \dim X_0$ and $g_0 = h_0^0g_0$. We denote by $Y'$ the fan product of $Y_0$ and $Y$ with respect to $h_0^0$ and $\eta$ (see [1. Supplement to Ch. 1, §2]); by $\eta'$ and $h^*$ we denote that projections of $Y'$ into $Y_0$ and $Y$, respectively, and by $g'$ a map of $X$ into $Y'$ such that $\eta'g' = g_0^0\xi$ and $h^*g' = g$. If we replace $Y, Y_0, g, g_0, h, h_0$ and $\eta$ with $Y', Y_0', g', g_0', h_0^*$, $h_0^0$ and $\eta'$, respectively, then these spaces and maps are what is required (cf. [9]).

Lemma 2 has been proved.

Lemma 3 ([9, Lemma 5.3]). Suppose that $A \in \operatorname{ANR}$ and $\{T_n, h_{n+1.n}\} (n = 0, 1, \cdots)$.
is an inverse sequence of compact spaces such that for any \( n \), any compact \( F \subseteq T_{n+1} \), and any map \( \chi: h_{n+1,n}(F) \to A \), the map \( \chi h_{n+1,n} \mid_F \) has an extension to \( T_{n+1} \). Then \( \dim T \leq A \) for the limit \( T \) of the sequence in question.

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References


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