ON A SUFFICIENT CONDITIONS FOR
MULTIVALENTLY STARLIKENESS

By

Mamoru Nunokawa

Let \( q \in \mathbb{N} = \{1, 2, 3, \ldots \} \) and \( A(q) \) denote the class of function
\[
f(z) = z^q + \sum_{n=q+1}^{\infty} a_n z^n
\]
which are analytic in the open disk \( E = \{z : |z| < 1\} \).

A function \( f(z) \in A(q) \) is called \( q \)-valently starlike with respect to the origin if and only if
\[
Re \left( \frac{zf'(z)}{f(z)} \right) > 0 \quad \text{in} \quad E.
\]

There are many papers in which various sufficient conditions for multivalently starlikeness were obtained, but almost these results were got by using real part of some analytic functions.

Recently, Mocanu [3] obtained the following result by using the imaginary part of \( zf''(z)/f'(z) \).

**Theorem A.** If \( f(z) \in A(1) \) and
\[
\left| \text{Im} \left( \frac{zf''(z)}{f'(z)} \right) \right| < \sqrt{3} \quad \text{in} \quad E,
\]
then \( f(z) \) is univalently starlike in \( E \).

We need the following lemma due to [1, 2].

**Lemma 1.** Let \( w(z) \) be analytic in \( E \) and suppose that \( w(0) = 0 \). If \( |w(z)| \) attains its maximum value on the circle \( |z| = r < 1 \) at a point \( z_0 \), then we can write
\[
z_0 w'(z_0) = kw(z_0)
\]
where \( k \) is a real number and \( k \geq 1 \).

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Applying the same method as the proof of [4, Theorem 1], we can prove the following lemma:

**Lemma 2.** Let \( p(z) \) be analytic in \( E \), \( p(0)=q \) and suppose that there exists a point \( z_0 \in E \) such that

\[
(1) \quad \Re p(z) > 0 \quad \text{for} \quad |z| < |z_0|
\]

\( \Re p(z_0)=0 \) and \( p(z_0)=ia \) where \( a \) is a real number and not zero.

Then we have

\[
\frac{z_0p'(z_0)}{p(z_0)} = ik
\]

where

\[
k = \frac{1}{2} \left( \frac{q^2 + a^2}{a} \right) \geq q \quad \text{if} \quad a > 0,
\]

and

\[
k = -\frac{1}{2} \left( \frac{q^2 + a^2}{a} \right) \leq -q \quad \text{if} \quad a < 0.
\]

**Proof.** Let us put

\[
\phi(z) = \frac{q - p(z)}{q + p(z)}.
\]

Then we have that \( \phi(0) = 0 \), \( |\phi(z)| < 1 \) for \( |z| < |z_0| \) and \( |\phi(z_0)| = 1 \). From (1), (2) and Lemma 1, we have

\[
\frac{z_0\phi'(z_0)}{\phi(z_0)} = \frac{2z_0p'(z_0)}{q^2 - p(z_0)^2} = \frac{-2z_0p'(z_0)}{q^2 + |p(z_0)|^2} \geq 1.
\]

This shows that

\[
-z_0p'(z_0) \geq \frac{1}{2}(q^2 + |p(z_0)|^2)
\]

and \( z_0p'(z_0) \) is a negative real number.

Applying the same method as the proof of [4, Theorem 1], for \( a > 0 \), we have

\[
\Im \frac{z_0p'(z_0)}{p(z_0)} \geq \frac{1}{2} \left( \frac{q^2 + a^2}{a} \right) \geq q
\]

and for \( a < 0 \), we have

\[
\Im \frac{z_0p'(z_0)}{p(z_0)} \leq -\frac{1}{2} \left( \frac{q^2 + a^2}{|a|} \right) \leq -q.
\]

This completes our proof.
On a Sufficient Condition for Starlikeness

Applying Lemma 2, we will obtain a generalized result of Theorem A.

**Main Theorem.** Let \( f(z) \in A(q) \) and suppose that

\[
1 + \frac{zf''(z)}{f'(z)} \neq ik \quad \text{in } E,
\]

where \( k \) is a real number and \( |k| \geq \sqrt{3}q \).

Then \( f(z) \) is \( q \)-valently starlike in \( E \).

**Proof.** Let us put

\[
p(z) = \frac{zf'(z)}{f(z)}
\]

where \( p(0) = q \). From the assumption (3), we easily have

\[
p(z) \neq 0 \quad \text{in } E.
\]

In fact, if \( p(z) \) has a zero of order \( n \) at \( z = \alpha \in E \), then we can put

\[
p(z) = (z - \alpha)^n p_1(z), \quad (n \in \mathbb{N})
\]

where \( p_1(z) \) is analytic in \( E \) and \( p_1(\alpha) \neq 0 \).

Then we have

\[
1 + \frac{zf''(z)}{f'(z)} = \frac{zp'(z)}{p(z)} + p(z)
\]

\[
= \frac{nz}{z - \alpha} + \frac{zp_1'(z)}{p_1(z)} + (z - \alpha)^n p_1(z).
\]

But, the imaginary part of (4) can take any infinite values when \( z \) approaches \( \alpha \).

This contradicts (3). Hence we have

\[
p(z) \neq 0 \quad \text{in } E.
\]

Therefore, if there exists a point \( z_0 \in E \) such that \( \Re p(z) > 0 \) for \( |z| < |z_0| \),

\[
\Re p(z_0) = 0 \quad \text{and} \quad p(z_0) = ia,
\]

then we have

\[
p(z_0) \neq 0 \quad \text{and} \quad a \neq 0.
\]

From Lemma 2 and (4), for \( a > 0 \), we have

\[
1 + \frac{z_0f''(z_0)}{f'(z_0)} = \frac{z_0p'(z_0)}{p(z_0)} + p(z_0)
\]

\[
= i \left( \text{Im} \frac{z_0p'(z_0)}{p(z_0)} + \text{Im} p(z_0) \right)
\]
and
\[ \text{Im}\left( \frac{z_0 \beta'(z_0)}{\rho(z_0)} + p(z_0) \right) \geq \frac{1}{2} \left( \frac{q^2 + 3a^2}{a} \right) \geq \sqrt{3} q. \]

For \( a < 0 \), we have
\[ 1 + \frac{z_0 \beta'(z_0)}{\rho(z_0)} = i \left( \text{Im} \frac{z_0 \beta'(z_0)}{\rho(z_0)} + p(z_0) \right) \]
and so
\[ \text{Im} \left( \frac{z_0 \beta'(z_0)}{\rho(z_0)} + p(z_0) \right) \leq -\frac{1}{2} \left( \frac{q^2 + 3a^2}{|a|} \right) \leq -\sqrt{3} q. \]
These contradict (3). Hence we have
\[ \text{Re} \ p(z) > 0 \quad \text{in} \ E. \]
This shows that \( f(z) \) is \( q \)-valently starlike in \( E \).
This completes our proof.
From Main theorem, we easily have the following result.

**Corollary.** Let \( f(z) \in A(q) \) and suppose that there exists a real number \( R \) for which
\[ \left| \frac{zf''(z)}{f'(z)} - R \right| < \sqrt{(R+1)^2 + 3q^2} \quad \text{in} \ E. \]
Then \( f(z) \) is \( q \)-valently starlike in \( E \).

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**References**


Department of Mathematics
University of Gunma
Aramaki, Maebashi 371
Japan
e-mail, nunokawa@la.gunma-u.ac.jp