PROPERTIES OF AN L-CARDINAL

By
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When we study the set theory ZF(aa), (Ref. [1] or [3]) it may be natural to consider a cardinal \( \kappa \) such that for every formula in the language of usual set theory,

\[
R(\kappa) \models \text{aa}\alpha \phi \iff V \models \text{aa}\alpha \phi.
\]

Let \( \kappa \) be measurable, \( M \) a transitive isomorph of \( V^\kappa/U \) where \( U \) is a normal ultrafilter on \( \kappa \), and \( j \) the canonical elementary embedding of \( V \) into \( M \). If "aa" is interpreted by the closed unbounded filter of \( \kappa \) and \( j(\kappa) \) respectively, in \( M \),

\[
R(\kappa) \models \text{aa}\alpha \phi \iff R(j(\kappa)) \models \text{aa}\alpha \phi.
\]

Therefore measurability is sufficient to show the consistency of the desired situation. But when we want \( \kappa \) to have this property in full \( V \), a new cardinal axiom is needed.

1. Definitions of an L-cardinal and its basic properties.

Definition. Let \( \phi \) be a formula in set theory whose constants are all in \( R(\kappa) \), and \( \lambda \) be an ordinal \( \geq \kappa \).

a) A cardinal \( \kappa \) is a \((\phi, \lambda)\)-cardinal, if there exists an elementary embedding \( j: V \to M \) such that

(i) \( j(\kappa) > \lambda \) and \( \kappa \) is the least ordinal moved by \( j \),
(ii) for every \( x \) in \( R(j(\kappa))^M \), \( M \models \phi(x) \iff V \models \phi(x) \).

b) \( \kappa \) is a \( \phi \)-cardinal if for every \( \lambda > \kappa \), \( \kappa \) is a \((\phi, \lambda)\)-cardinal.

c) \( \kappa \) is a \( \Sigma^\alpha_n \)-cardinal if for every \( \Sigma^\alpha_n \) formula \( \phi \), \( \kappa \) is a \( \phi \)-cardinal.

d) Let \( A \) be a set of formulas, \( \kappa \) is a \((A, \lambda)\)-cardinal if for every formula in \( A \), \( \kappa \) is a \((\phi, \lambda)\)-cardinal.

e) \( \kappa \) is an L-cardinal if for every formula \( \phi \), \( \kappa \) is a \( \phi \)-cardinal.

The axiom of an L-cardinal definitely cannot be formulated in ZFC. However, all the arguments can be carried out in ZFC within some \( R(\kappa) \) where \( \kappa \) is inaccessible.

The first lemma is trivial but basic in the development.

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Lemma 1.1. Let \( \phi \) be a formula such that \( \{ \alpha < \kappa | \phi(\alpha) \} \) is unbounded in \( \kappa \) and all constants are in \( R(\kappa) \). If \( \kappa \) is a \((\phi - \lambda)\)-cardinal, there exists an ordinal \( \beta > \lambda \) such that \( \phi(\beta) \) holds.

Proof. Let \( j \) be an elementary embedding from \( V \) into \( M \) as in the definition of \((\phi - \lambda)\)-cardinal. By assumption,

\[
V \models \forall \alpha < \kappa \exists \beta < \kappa \{ \alpha < \beta \land \phi(\beta) \}
\]

Hence,

\[
M \models \forall \alpha < j(\kappa) \exists \beta < j(\kappa) \{ \alpha < \beta \land \phi(\beta) \}
\]

Since \( \lambda < j(\kappa) \), there is a \( \beta < j(\kappa) \) such that \( M \models \phi(\beta) \) and \( \beta > \lambda \). Also \( V \models \phi(\beta) \) because \( \beta \in R(j(\kappa))^{\mathcal{M}} \).

Corollary 1.2. Let \( \kappa \) be a \( \{ \phi, \neg \phi \} \)-cardinal. If \( \{ \alpha < \kappa | \phi(\alpha) \} \) is closed unbounded in \( \kappa \), then \( \{ \alpha | \phi(\alpha) \} \) is closed unbounded in \( OR \).

Proof. Assume \( \{ \alpha < \kappa | \phi(\alpha) \} = C \) is closed unbounded in \( \kappa \). By Lemma 1.1, \( \{ \alpha | \phi(\alpha) \} = C' \) is an unbounded class. Let \( \alpha \) be a limit point of \( C' \) and \( j : V \to M \) be an elementary embedding that satisfies the definition of \((\phi, \neg \phi) - \alpha\)-cardinal.

\[
V \models \forall \beta < \alpha \exists \gamma < \alpha \{ \beta < \gamma \land \phi(\gamma) \}
\]

implies

\[
M \models \forall \beta < \alpha \exists \gamma < \alpha \{ \beta < \gamma \land \phi(\gamma) \}.
\]

As \( C \) is closed, \( M \models \forall \alpha < j(\kappa) \{ \forall \beta < \alpha \exists \gamma < \alpha \{ \beta < \gamma \land \phi(\gamma) \} \to \phi(\alpha) \} \). Hence \( M \models \phi(\alpha) \).

Also \( V \models \phi(\alpha) \).

For simplicity we consider a fixed formula \( \Phi_\phi(x) \equiv \exists \gamma \{ x = R(\alpha) \} \). If \( \kappa \) is a \((\Phi_\phi - \lambda)\)-cardinal, \( R(j(\kappa))^{\mathcal{M}} = R(j(\kappa)) \). Of course some results follow by the weaker assumption that \( \kappa \) is a \((\phi - \lambda)\)-cardinal which is reduced from the fact that \( \kappa \) is a \((\Phi_\phi - \lambda)\)-cardinal.

Lemma 1.3. If \( \kappa \) is a \( \Phi_\phi \)-cardinal, \( \{ \alpha | \alpha \text{ is strongly inaccessible} \} \) is a proper class.

Proof. Since \( \kappa \) is measurable, \( \{ \alpha < \kappa | \alpha \text{ is strongly inaccessible} \} \) is unbounded in \( \kappa \). Since \( R(j(\kappa))^{\mathcal{M}} = R(j(\kappa)) \), the strongly inaccessible in \( M \) which are less than \( j(\kappa) \) are also strongly inaccessible in \( V \). Now the conclusion is clear by Corollary 1.2.

Lemma 1.4. If \( \kappa \) is a \((\Phi_\phi - \kappa)\)-cardinal, \( \kappa \) is the \( \kappa \)-th measurable.

Proof. Let \( j : V \to M \) be an associated embedding. \( j(\kappa) > \kappa \) and \( \lim(\kappa) \) implies
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\( j(\kappa) > \kappa + \omega \). Hence \( R(\kappa + 2)^n = R(\kappa + 2) \). Let \( U \) be a normal ultrafilter on \( \kappa \). Since \( U \subseteq R(\kappa + 2), M = (U \) is a normal ultrafilter on \( \kappa \). Hence \( M \models (\kappa \text{ is measurable}) \). As usual, define \( U' \)

\[ X \subseteq U' \iff X \subseteq \kappa \land \kappa \in j(X) \]

\( U' \) is a normal ultrafilter on \( \kappa \) and \( \{ \alpha < \kappa | \alpha \text{ is measurable} \} \subseteq U' \).

**Corollary 1.5.** If \( \kappa \) is a \( \Phi_0 \)-cardinal, \( \{ \alpha | \alpha \text{ is measurable} \} \) is a proper class.

**Proof.** By Corollary 1.2 and Lemma 1.4.

1.3 and 1.5 follow from the assumption of an extendible cardinal. 1.4 follows from \( 2^\kappa \)-supercompactness. As our definition of \( (\phi - \lambda) \)-cardinal does not assume \( ^1 M \in M \), it is not known whether \( \kappa \) carries a normal ultrafilter on \( P_\kappa \lambda \). Even it is not clear whether \( \kappa \) is \( \lambda \)-compact (Ref. [2]). The next definition is also due to [2].

**Definition.** \( \Sigma_n \) relativizes down to \( R(\kappa) \) iff for each formula \( \phi \in \Sigma_n \),

\[ \forall a \in R(\kappa) (\phi(a) \rightarrow R(\kappa) \models \phi(a)) \]

**Lemma 1.6.** If \( \kappa \) is a \( \Sigma_n \)-cardinal, \( \Sigma_n \) relativizes down to \( R(\kappa) \).

**Proof.** By induction on \( n \). \( \Pi_1 \) relativizes down to \( R(\kappa) \) since \( \kappa \) is strongly inaccessible (Ref. [4]). Thus \( \Sigma_1 \) relativizes down to \( R(\kappa) \). Assume \( \phi \in \Sigma_n \) and \( a \in R(\kappa), \phi(a) \) holds. \((n \geq 2)\) There is a \( \Sigma_{n-1} \) formula \( \psi \) such that \( \phi(a) \rightarrow \exists x \psi(x, a) \). Since \( \phi(a) \), for some \( b \), \( V \models \phi(b, a) \). Choose \( \lambda \) large enough to \( b \in R(\lambda) \). Let \( j : V \rightarrow M \) be an associated embedding of \( (\neg \phi - \lambda) \)-ness of \( \kappa \). (Note that \( \neg \phi \in \Sigma_n \).) Since \( b \in R(j(\kappa))^M \) and \( V \models \phi(b, a), M \models \phi(b, a) \). (As \( \phi_0 \) is a \( \Sigma_1 \) formula, we can assume \( b \in M \).) But in \( V, \Sigma_{n-1} \) relativizes down to \( R(\kappa) \). Hence in \( M, \Sigma_{n-1} \) relativizes down to \( R(j(\kappa)) \). Therefore \( M \models (R(j(\kappa)) = \phi(b, a)) \). Hence \( M \models R(j(\kappa)) = \phi(a) \). By elementarity of \( j \) and \( j(a) = a, R(\kappa) = \phi(a) \) in \( V \).

Now the followings are all clear.

**Theorem 1.7.** If \( \kappa \) is an \( L \)-cardinal, \( \Sigma_n \) relativizes down to \( R(\kappa) \).

**Corollary 1.8.** If \( \kappa \) is an \( L \)-cardinal, \( R(\kappa) \) is an elementary substructure of \( V \).

**Corollary 1.9.** If \( \kappa \) is an \( L \)-cardinal and \( \alpha \) is a definable cardinal, then \( \alpha < \kappa \).
Note that: If \( \kappa \) is supercompact, \( \Sigma_k \) relativizes down to \( R(\kappa) \). If \( \kappa \) is extendible, \( \Sigma_k \) relativizes down to \( R(\kappa) \). (Ref. [2])

Also recall the notion of "ghost cardinal" of M. Takahashi. It is the least cardinal not definable by \( \mathcal{J}_1 \)-formula. Of course an \( L \)-cardinal is not definable in set theory, \( ZFC \).

**Corollary 1.10.** Let \( \kappa \) be an \( L \)-cardinal and \( \phi \) be a formula whose constants are all in \( R(\kappa) \). If there exists an ordinal \( \gamma \geq \kappa \), such that \( \phi(\gamma) \) holds, then \( \{\alpha \mid \phi(\alpha)\} \) is a proper class and \( \{\alpha < \kappa \mid \phi(\alpha)\} \) is unbounded in \( \kappa \).

**Proof.** By Corollary 1.2, it suffices to show \( \{\alpha < \kappa \mid \phi(\alpha)\} \) is unbounded in \( \kappa \). If not, there is an \( \alpha < \kappa \) such that \( \forall \beta < \kappa (\alpha < \beta \rightarrow \neg \phi(\beta)) \). By the above Theorem, \( \forall \beta < \kappa (\alpha < \beta \rightarrow R(\kappa) \models \neg \phi(\beta)) \). Then \( R(\kappa) \models \forall \beta (\alpha < \beta \rightarrow \neg \phi(\beta)) \). Using Corollary 1.9, we have \( \forall \beta (\alpha < \beta \rightarrow \neg \phi(\beta)) \). Contradicting \( \phi(\gamma) \).

1.7-1.10 are too strong and give us suspicion about consistency.

2. Cohen extension and an \( L \)-cardinal.

Once \( L \)-cardinal is defined, many problems are raised. If \( V \) is a model of \( ZFC + \exists \kappa : L \)-cardinal, is there a Cohen extension of \( V \) where \( ZFC + \exists \kappa : L \)-cardinal + G.C.H. hold? Is not \( j(\kappa) \) necessarily measurable? (When \( \kappa \) is extendible, \( j(\kappa) \) is always measurable.) Can an \( L \)-cardinal be strongly compact?

All these questions are unclear now. We need some technics to preserve an \( L \)-cardinal. We only get a quite easy fact that is not useful to solve the above problems.

**Lemma 2.1.** If \( |P| < \kappa \) and \( \kappa \) is a \((\|\phi\| - \lambda)\)-cardinal, then \( \kappa \) is a \((\phi - \lambda)\)-cardinal in \( V[G] \).

**Proof.** We can assume \( P \subseteq R(\kappa) \). Let \( j : V \rightarrow M \) be an elementary embedding that witnesses \( \kappa \) is a \((\|\phi\| - \lambda)\)-cardinal. We extend \( j \) to \( j' \) as usual.

\[
j'(K_\alpha(x)) = K_\alpha(j(x))
\]

(We use the notations of [7].) \( j' \) is an elementary embedding of \( V[G] \) into \( M[G] \). (Ref. [5]) If \( x \in R(j(\kappa))^M \), there is a name \( x \in R(j(\kappa))^M \) such that \( K_\alpha(x) = x \). \( M[G] \models \phi(x) \) if \( M \models (p \models \phi(x)) \) for some \( p \in G \). The latter implies \( \exists p \in G (p \models \phi(x)) \) in \( V \). Therefore \( V[G] \models \phi(x) \).

**Theorem 2.2.** If \( \kappa \) is an \( L \)-cardinal in \( V \) and \( |P| < \kappa \), \( \kappa \) is an \( L \)-cardinal in \( V[G] \).
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Using Theorem 2.2, we consider the relation between an $L$-cardinal and strongly compact cardinals.

**Theorem 2.3.** Let $\kappa$ be an $L$-cardinal.

(i) If $\kappa$ is strongly compact, $\kappa$ is the $\kappa$-th strongly compact.

(ii) If $\kappa$ is not strongly compact, there is no strongly compact cardinal greater than $\kappa$, and it is consistent that there is an $L$-cardinal and there is no strongly compact cardinal.

**Proof.** (i) By Corollary 1.10.

(ii) At first we assert that $\kappa$ is not a limit of strongly compacts. For a measurable cardinal that is a limit of strongly compacts is strongly compact and $\kappa$ is clearly measurable. (Ref. [6])

Thus there is no strongly compact above $\kappa$ by Corollary 1.10. And there exists a regular cardinal $\alpha < \kappa$ such that there is no strongly compact cardinal between $\alpha$ and $\kappa$. We use the forcing condition that collapses $\alpha$ to $\omega_1$. In the extended universe there is no strongly compact cardinal and $\kappa$ remains an $L$-cardinal by Theorem 2.2.

**References**


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