YANG-MILLS CONNECTIONS IN ORTHONORMAL FRAME BUNDLE OVER SU(2)

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Abstract. The main result in this note is that the connection form with respect to frames in the orthonormal frame bundle for a left invariant Riemannian metric g on SU(2) becomes a Yang-Mills connection if and only if g is bi-invariant.

1. Introduction and statement of results

To find Yang-Mills connections in a principal fibre bundle is important. In this paper, we get necessary and sufficient conditions for the connection form in the orthonormal frame bundle on SU(2) with respect to an arbitrary given left invariant Riemannian metric to be a Yang-Mills connection. We get the following main Theorem and Corollary.

THEOREM. Let g be left invariant Riemannian metric on SU(2). Then the connection form in the orthonormal frame bundle defined by the Levi-Civita connection of g becomes a Yang-Mills connection if and only if g is bi-invariant.

COROLLARY. The connection in the above Theorem is a Yang-Mills connection if and only if the Lie group SU(2) with a left invariant Riemannian metric g is a space of constant curvature.

2. The proof of the main theorem

Let G denote the Lie group SU(2). Let the Lie algebra of all left invariant vector fields on SU(2) denote g. The Killing form B of g satisfies $B(X, Y) = 4 \text{Trace}(XY)$, $(X, Y \in g)$. We define an inner product $\langle, \rangle_0$ on g by

\[
\langle X, Y \rangle_0 := -B(X, Y), \quad (X, Y \in g).
\]

The following Lemma is known (cf. [4, p. 154]).
Lemma 1. Let $g$ be an arbitrary given left invariant Riemannian metric on $G$. Let $\langle \cdot, \cdot \rangle$ be an inner product on $\mathfrak{g}$ defined by $\langle X, Y \rangle := g_e(X, Y)$, where $X, Y \in \mathfrak{g}$ and $e$ is the identity matrix of $G$. Then there exists an orthonormal basis $(X_1, X_2, X_3)$ of $\mathfrak{g}$ with respect to $\langle \cdot, \cdot \rangle$ such that

$$
[X_1, X_2] = (1/\sqrt{2})X_3, \quad [X_2, X_3] = (1/\sqrt{2})X_1, \\
[X_3, X_1] = (1/\sqrt{2})X_2,
$$

where $a_i, (i = 1, 2, 3)$, are positive constant real numbers determined by the given left invariant Riemannian metric $g$ of $G$.

The connection function $\alpha$ on $\mathfrak{g} \times \mathfrak{g}$ corresponding to the left invariant Riemannian connection of $(G, g)$ is given as follows (cf. [3, p. 52]):

$$
\left\{ \begin{array}{ll}
\alpha(X, Y) = 1/2[X, Y] + U(X, Y), & (X, Y) \in \mathfrak{g}, \\
\langle X, Y \rangle := g_e(X, Y), &
\end{array} \right.
$$

where $U(X, Y)$ is determined by

$$
2 \langle U(X, Y), Z \rangle = \langle [Z, X], Y \rangle + \langle X, [Z, Y] \rangle, \quad (X, Y, Z) \in \mathfrak{g}).
$$

Now, putting $Y_1 := 2\sqrt{2}X_1$, $Y_2 := 2\sqrt{2}X_2$, and $Y_3 := 2\sqrt{2}X_3$ for the orthonormal basis $(X_1, X_2, X_3)$ with respect to $\langle \cdot, \cdot \rangle$ in Lemma 1, we have the following:

$$
\left\{ \begin{array}{ll}
[Y_1, Y_2] = 2Y_3, & [Y_2, Y_3] = 2Y_1, & [Y_3, Y_1] = 2Y_2, \\
(V_1, V_2, V_3): \text{orthonormal frame fields on } (G, g),
\end{array} \right.
$$

where $V_i := X_i/a_i, (i = 1, 2, 3)$.

Using (4) and (5), we get

Lemma 2.

$$
\left\{ \begin{array}{ll}
U(Y_1, Y_1) = U(Y_2, Y_2) = U(Y_3, Y_3) = 0, \\
U(Y_1, Y_2) = (a_2^2 - a_3^2)a_1^{-2}Y_3, \\
U(Y_2, Y_3) = (a_3^2 - a_2^2)a_1^{-2}Y_1, \\
U(Y_3, Y_1) = (a_3^2 - a_1^2)a_2^{-2}Y_2.
\end{array} \right.
$$

The connection function $\alpha$ on $\mathfrak{g} \times \mathfrak{g}$ which is corresponding to a given left invariant Riemannian connection of $(G, g)$ is uniquely expressed as

$$
\alpha(V_i, V_j) = \sum_k \Gamma_{ij}^k V_k, \quad (i, j = 1, 2, 3).
$$

Let $(\theta^1, \theta^2, \theta^3)$ be the dual 1-forms to the orthonormal frame basis $(V_1, V_2, V_3)$. Then, the connection form $\omega$ and the curvature form $\Omega$ with respect to frames
in the orthonormal frame bundle of \((G, g)\) are defined as follows:

(8) \[ \omega^1_j = \sum_{i=1}^3 \Gamma^i_{jk} \theta^k, \]

(9) \[ \Omega^1_j = \sum_{i=1}^3 \theta^i (\alpha(V_k, \alpha(V_i, V_j)) - \alpha(V_k, \alpha(V_i, V_j)) - \alpha([V_k, V_i], V_j)) \theta^k \wedge \theta^i. \]

From (3), (4), (7) and Lemma 2, the non-zero terms of \(\Gamma^i_{jk}\) are

\[ \Gamma^i_{12} = -\Gamma^i_{21} = (a_i^2 + a_i^3 - a_i^2)/2 \sqrt{2} a_1 a_2 a_3, \]

\[ \Gamma^i_{13} = -\Gamma^i_{31} = (a_i^1 + a_i^3 - a_i^3)/2 \sqrt{2} a_1 a_2 a_3, \]

\[ \Gamma^i_{23} = -\Gamma^i_{32} = (a_i^3 + a_i^2 - a_i^1)/2 \sqrt{2} a_1 a_2 a_3. \]

Using (7)-(10), we get

\[ \omega^1_i = \omega^2_i = \omega^3_i = 0, \]

\[ \omega^2_i = -\omega^1_i = \frac{(a_i^2 + a_i^3 - a_i^2)}{2 \sqrt{2} a_1 a_2 a_3} \theta^2, \]

\[ \omega^3_i = -\omega^1_i = \frac{(a_i^1 + a_i^3 - a_i^3)}{2 \sqrt{2} a_1 a_2 a_3} \theta^3. \]

We denote \(\nabla \Omega(V_j, V_i), \omega(V_j)\) and \(\Omega(V_j, V_i)\) by \(\nabla_j \omega_{ji}, \omega_j\) and \(\Omega_{ji}\), respectively. The connection in (8) is a Yang-Mills connection (cf. [1, p. 107]) if and only if

(13) \[ \langle \delta_\Omega(V_i) \rangle = -\sum_j \langle \nabla_j \omega_{ji} + [\omega_j, \Omega_{ji}] \rangle = 0, \quad (i = 1, 2, 3). \]

We put

\[ \langle \delta_\Omega(V_1) \rangle : = (b_{(1)}), \quad \langle \delta_\Omega(V_2) \rangle : = (b_{(2)}), \quad \langle \delta_\Omega(V_3) \rangle : = (b_{(3)}). \quad (i, j = 1, 2, 3). \]

From (7) and (10)-(14), the non-zero terms of \((b_{(1)}), (b_{(2)})\) and \((b_{(3)})\) are

(15) \[ b_{(1)}^3 = -b_{(1)}^3 = c^{-1} (2a_1^4 - a_1^4 a_2^2 - a_1^4 a_3^2 + a_2^4 a_3^2 + a_1^4 a_3^2 - a_2^4 - a_3^4), \]

(16) \[ b_{(2)}^3 = -b_{(2)}^3 = c^{-1} (2a_2^4 - a_1^4 a_2^2 - a_1^4 a_3^2 - a_2^4 a_3^2 + a_1^4 a_3^2 + a_1^4 a_2^2 - a_3^4), \]

(17) \[ b_{(3)}^3 = -b_{(3)}^3 = c^{-1} (2a_3^4 - a_1^4 a_2^2 - a_1^4 a_3^2 + a_2^4 a_3^2 - a_1^4 a_3^2 + a_3^4 a_2^2 - a_3^4), \]

where \( c = 4 \sqrt{2} (a_1 a_2 a_3)^2 \).

Hence, positive numbers \( a_1, a_2 \) and \( a_3 \) satisfy (15)=(16)=(17)=0 if and only
if $\omega$ in (8) is a Yang-Mills connection. Moreover, positive numbers $a_1$, $a_2$ and $a_3$ satisfy (15)=(16)=(17)=0 if and only if $a_1=a_2=a_3$. Thus, the proof of the main theorem is completed.

On the other hand, using (12) and the main Theorem, we get the Corollary.

Bibliography


