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CHARACTERIZATIONS OF REAL HYPERSURFACES OF A COMPLEX HYPERBOLIC SPACE IN TERMS OF RICCI TENSOR AND HOLOMORPHIC DISTRIBUTION

By

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§ 0. Introduction.

Let CP^n and CH^n denote the complex projective n-space with constant holomorphic sectional curvature 4, and the complex hyperbolic n-space with constant holomorphic sectional curvature -4, respectively. Let M be a real hypersurface of CP^n or CH^n. M has an almost contact metric structure (ϕ, ξ, η, g) induced from the complex structure J of CP^n or CH^n. Real hypersurfaces in CP^n and CH^n have been studied by many authors (cf. [1], [2], [3], [11], [12], [13], [14], [15] and [17]). For real hypersurfaces in CP^n, Takagi ([16]) showed that all homogeneous real hypersurfaces in CP^n are realized as the tubes of constant radius over compact Hermitian symmetric spaces of rank 1 or 2 (cf. [2] and [5]). He proved that all homogeneous real hypersurfaces in CP^n could be classified into six types which are said to be of type A_1, A_2, B, C, D and E. Kimura ([5]) also proved that a real hypersurfaces M in CP^n is homogeneous if and only if M has constant principal curvatures and ξ is principal. Other interesting results in real hypersurfaces of CP^n are shown by Kimura-Maeda ([8]) and Maeda-Udagawa ([10]):

THEOREM A ([8]). *Let M be a real hypersurface in CP^n. Then the following inequality holds:*

$$\|\nabla S\|^2 \geqq 1/(n-1)\{2n(h-\eta(A\xi)\phi+(\phi A\xi)h+trace((\nabla_\xi A)A\phi)\}^2,$$

where S is the Ricci tensor of M and $k=trace\,A$. Moreover, the equality holds if and only if M is locally congruent to a geodesic hypersphere of CP^n.

Let TCP^n be the tangent bundle of CP^n. For a real hypersurface M of CP^n, let TM be the tangent bundle of M. Then, $T^\circ M=\{X\in TM \mid X\perp\xi\}$ is a subbundle of TM. Thus each of TM and $T^\circ M$ has a connection induced from

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TCP^n. The orthogonal complement of $T^\circ M$ in TCP^n with respect to the metric on TCP^n is denoted by $N^\circ M$, which is also a subbundle of TCP^n with the induced metric connection. Denote by ∇° and $\nabla^\perp$ the connections of $T^\circ M$ and $N^\circ M$, respectively. Let A be the second fundamental form of $T^\circ M$ in TCP^n. Then, A is a smooth section of $\mathrm{Hom}(TM, \mathrm{Hom}(T^\circ M, N^\circ M))$. Set $A^\circ = A|_{T^\circ M}$. We say that A° is η-parallel if $\nabla^\circ_X A^\circ \equiv 0$ for any $X \in T^\circ M$.

THEOREM B ([10]). *Let M be a real hypersurface of CP^n. Assume that A° is η-parallel. Then M is locally congruent to one of the following:*

(i) *a geodesic hypersphere,*

(ii) *a tube over a totally geodesic $CP^k (1 \leq k \leq n-2)$,*

(iii) *a tube over a complex quadric Q_{n-1},*

(iv) *a real hypersurface in which $T^\circ M$ is integrable and its integral manifold is a totally geodesic CP^{n-1} (that is, M is a ruled real hypersurface),*

(v) *a real hypersurface in which $T^\circ M$ is integrable and its integral manifold is a complex quadric Q_{n-1}.*

Note that the cases (i), (ii) and (iii) in Theorem B are homogeneous but (iv) and (v) are not homogeneous. Although as in ([16]), homogeneous real hypersurfaces of CP^n has been given a complete classification, it is still open for the question of the classification of that of CH^n.

Montiel ([12]) constructed five examples of homogeneous real hypersurfaces in CH^n using the techniques similar to Cecil and Ryan ([2]). Berndt ([1]) gives a characterization of real hypersurface in CH^n which corresponds to the result in ([5]):

THEOREM C([1]). *Let M be a real hypersurface in CH^n. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following:*

(A_0) *a horosphere in CH^n,*

(A_1) *a geodesic hypersphere (that is, a tube over a point),*

(A_1') *a tube over a complex hyperplane CH^{n-1},*

(A_2) *a tube over a totally geodesic CH^k $(1 \leq k \leq n-2)$,*

(B) *a tube over a totally real hyperbolic space RH^n.*

The purpose of this paper is to investigate the real hypersurfaces of CH^n corresponding to the results in Theorem A and Theorem B. Namely, we first show the following:

THEOREM 1. *Let M be a real hypersurface in CH^n. Then the following inequality hold.*

$$\|\nabla S\|^2 \geqq 1/(n-1)\{2n(h-\eta(A\xi))+(\phi A\xi)\cdot h-trace\,((\nabla_\xi A)A\phi)\}^2, \tag{2.30}$$

where S is the Ricci tensor of M and $h=trace\,A$. Moreover, equality of (2.30) *holds if and only if M is locally congruent to one of type* (A_0), (A_1) *or* (A_1').

Similarly as in CP^n, we may define the A° and notion of η-parallelism of A° for a real hypersurface in CH^n. Corresponding to Theorem B, we obtained the following result for CH^n.

THEOREM 2. *Let M be a real hypersurface of CH^n. Assume that A° is η-parallel. Then M is locally congruent to one of type* (A_0), (A_1), (A_1'), (A_2), (B) *or a ruled real hypersurface.*

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§ 1. Preliminaries

We begin with recalling fundamental formulas on real hypersurfaces of a complex hyperbolic space CH^n, endowed with the Bergman metric g of constant holomorphic sectional curvature -4, and J the complex structure of CH^n. Now, let M be a real hypersurface of CH^n and let N be a unit normal vector on M. For any X tangent to M, we put

$$JX=\phi X+\eta(X)N$$

where ϕX and $\eta(X)N$ are, respectively, the tangent and normal components of JX. Then ϕ is a (1, 1)-tensor and η is a 1-form. Moreover, $\eta(X)=g(X, \xi)$ with $\xi=-JN$ and (ϕ, η, ξ, g) determines an almost contact metric structure on M.

Then we have

$$\phi^2X=-X+\eta(X)\xi, \qquad g(\xi, \xi)=1, \qquad \phi\xi=0, \tag{1.1}$$

$$(\nabla_X\phi)Y=\eta(Y)AX-g(AX, Y)\xi, \tag{1.2}$$

$$\nabla_X\xi=\phi AX. \tag{1.3}$$

(1.2) and (1.3) follow from $\bar{\nabla}_XY=\nabla_XY+g(AX, Y)N$ and $\bar{\nabla}_XN=-AX$, where $\bar{\nabla}$ and ∇ are, respectively, the Levi-Civita connections of CH^n and M, and A is the shape operator of M. Let R be the curvature tensor of M. Then the

Gauss and Codazzi equations are the following:

$$R(X, Y)Z=-g(Y, Z)X+g(X, Z)Y-g(\phi Y, Z)\phi X+g(\phi X, Z)\phi Y +2g(\phi X, Y)\phi Z+g(AY, Z)AX-g(AX, Z)AY, \tag{1.4}$$

$$(\nabla_X A)Y-(\nabla_Y A)X=-\eta(X)\phi Y+\eta(Y)\phi X+2g(\phi X, Y)\xi. \tag{1.5}$$

From (1.1), (1.3), (1.4) and (1.5), we get

$$SX=-(2n+1)X+3\eta(X)\xi+hAX-A^2X, \tag{1.6}$$

$$(\nabla_X S)Y=3\{g(\phi AX, Y)\xi+\eta(Y)\phi AX\}+(X\cdot h)AY +(hI-A)(\nabla_X A)Y-(\nabla_X A)AY, \tag{1.7}$$

where h=trace A, S is the Ricci tensor of type (1.1) on M and I is the identity map, respectively.

We here recall the notion of an η-parallel Ricci tensor S of M, which is defined by $g((\nabla_X S)Y, Z)=0$ for any X, Y and Z orthogonal to ξ. Also, we consider similarly the η-parallel shape operator A of M in CH^n, which is defined by $g((\nabla_Y A)Y, Z)=0$ for any X, Y and Z orthogonal to ξ. Now we state the following theorems without proof for later use.

THEOREM D([15]). *Let M be a real hypersurface of CH^n. Then the Ricci tensor of M is η-parallel and ξ is principal if and only if M is locally congruent to one of homogeneous real hypersurfaces of type* (A_0), (A_1), (A_1'), (A) *and* (B).

THEOREM E([15]). *Let M be a real hypersurface of CH^n. Then the shape operator A of M in CH^n is η-parallel and ξ is principal if and only if M is locally congruent to one of homogeneous real hypersurfaces of type* (A_0), (A_1), (A_1'), (A_2) *and* (B).

It is easily seen that if the shape operator is η-parallel, then so is the Ricci tensor, under the condition such that ξ is principal.

THEOREM F([3]). *Let M be a real hypersurface of CH^n. Then the following are equivalent*: (i) *M is locally congruent to one of homogeneous real hypersurfaces of type* (A_0), (A_1), (A_1') *and* (A_2).

(ii) $(\nabla_X A)Y=\eta(Y)\phi X+g(\phi X, Y)\xi$ *for any* $X, Y\in TM$.

PROPOSITION A([17]). *Assume that ξ is a principal curvature vector and the corresponding principal curvature is α. If $AX=rX$ for $X\perp\xi$, then we have* $A\phi X=(\alpha r-2)/(2r-\alpha)\phi X$.

§ 2. Characterizations of real hypersurfaces of CH^n in terms of Ricci tensor.

We have the following

PROPOSITION 1. *Let M be a real hypersurface of CH^n ($n \geqq 3$). If the Ricci tensor S of M satisfies for some λ*

$$(2.1) \qquad (\nabla_X S)Y = \lambda\{g(\phi X, Y)\xi + \eta(Y)\phi X\} \quad \text{for any } X, Y \in TM,$$

then λ is constant and ξ is a principal vector.

PROOF. Suppose that the condition (2.1) holds. First of all we shall show that grad $\lambda = 3\lambda\phi A\xi$. Erom (2.1), (1.2) and (1.3), we have

$$(2.2) \qquad (\nabla_W(\nabla_X S))Y - (\nabla_{\nabla_W X} S)Y$$
$$= (W\cdot\lambda)\{g(\phi X, Y)\xi + \eta(Y)\phi X\} + \lambda\{\eta(X)g(AW, Y)\xi - 2\eta(Y)g(AW, X)\xi$$
$$+ g(\phi X, Y)\phi AW + g(\phi AW, Y)\phi X + \eta(X)\eta(Y)AW\},$$

from which we get

$$(2.3) \qquad (\nabla_X(\nabla_W S))Y - (\nabla_{\nabla_X W} S)Y$$
$$= (X\cdot\lambda)\{g(\phi W, Y)\xi + \eta(Y)\phi W\} + \lambda\{\eta(W)g(AX, Y)\xi - 2\eta(Y)g(AX'' W)\xi$$
$$+ g(\phi W, Y)\phi AX + g(\phi AX, Y)\phi W + \eta(W)\eta(Y)AX\}.$$

It follows from (2.2) and (2.3) that

$$(2.4) \qquad (R(W, X)S)Y$$
$$= (W\cdot\lambda)\{g(\phi X, Y)\xi + \eta(Y)\phi X\} - (X\cdot\lambda)\{g(\phi W, Y)\xi + \eta(Y)\phi W\}$$
$$+ \lambda\{\eta(X)g(AW, Y)\xi - \eta(W)g(AX, Y)\xi + g(\phi X, Y)\phi AW - g(\phi W, Y)\phi AX$$
$$+ g(\phi AW, Y)\phi X - g(\phi AX, Y)\phi W + \eta(Y)(\eta(X)AW - \eta(W)AX)\},$$

where R is the curvature tensor of M.

Let $e_1, e_2, \cdots, e_{2n-1}$ be local fields of orthonormal vectors on M. From (2.4) and (1.1), we find

$$(2.5) \qquad \textstyle\sum_{i=1}^{2n-1} g((R(e_i, X)S)Y, e_i)$$
$$= (e_i\cdot\lambda)\{g(\phi X, Y)g(\xi, e_i) + \eta(Y)g(\phi X, e_i)\} + \lambda\{\eta(X)g(AY, \xi) - g(AX, Y)$$
$$+ g(\phi Y, \phi AX) - g(A\phi Y, \phi X) - \eta(Y)g(AX, \xi) + (\text{trace } A)\eta(X)\eta(Y)\}.$$

Exchanging X and Y in (2.5), we see

(2.6) $\sum_{i=1}^{2n-1} g((R(e_i, Y)S)X, e_i)$

$$=(e_i\cdot\lambda)\{g(\phi Y, X)g(\xi, e_i)+\eta(X)g(\phi Y, e_i)\}+\lambda\{\eta(Y)g(AX, \xi)-g(AY, X)$$
$$+g(\phi X, \phi AY)-g(A\phi X, \phi Y)-\eta(X)g(AY, \xi)+(\text{trace}\,A)\eta(X)\eta(Y)\}.$$

Here we see that

$$(\text{the left hand side of (2.5)})=\sum g(R(e_i, X)(SY), e_i)-\sum g(R(e_i, X)Y, Se_i)$$
$$=g(SX, SY)-\sum g(R(e_i, X)Y, Se_i)$$

and

$$-\sum g(R(e_i, X)Y, Se_i)=\sum g(R(X, Y)e_i, Se_i)+\sum g(R(Y, e_i)X, Se_i)$$
$$=\text{trace}\,(S\cdot R(X, Y))-\sum g(R(e_i, Y)X, Se_i)$$
$$=-\sum g(R(e_i, Y)X, Se_i)$$

that is, the left hand side of (2.5) is symmetric with respect to X, Y. And hence equations (2.5) and (2.6) yield

(2.7) $0=2(\xi\cdot\lambda)g(\phi X, Y)+(\phi X\cdot\lambda)\eta(Y)-(\phi Y\cdot\lambda)\eta(X)+3\lambda\{\eta(X)\eta(AY)-\eta(Y)\eta(AX)\}.$

Putting $Y=\phi X$ in (2.7), we get

$$0=2(\xi\cdot\lambda)g(\phi X, \phi X)-\{-X\cdot\lambda+\eta(X)\xi\cdot\lambda\}\eta(X)+3\lambda\eta(X)\eta(A\phi X).$$

Contracting with respect to X in the above equations, we have

$$4(n-1)(\xi\cdot\lambda)=0$$

thus

$$\xi\cdot\lambda=0$$

Putting $Y=\xi$ in (2.7), we have

$$\phi X\cdot\lambda+3\lambda\{\eta(X)\eta(A\xi)-\eta(AX)\}=0.$$

Putting $X=\phi X$ in above equation, we have

$$X\cdot\lambda=3\lambda g(\phi A\xi, X),$$

that is,

(2.8) $$\text{grad}\,\lambda=3\lambda\phi A\xi.$$

Using (2.8), we can write (2.4) in the following.

(2.9) $$(R(W, X)S)Y=3\lambda\{g(\phi A\xi, W)(g(\phi X, Y)\xi+\eta(Y)\phi X)-g(\phi A\xi, X)(g(\phi W, Y)\xi$$
$$+\eta(Y)\phi W)\}+\lambda\{\eta(X)g(AW, Y)\xi-\eta(W)g(AX, Y)\xi$$
$$+g(\phi X, Y)\phi AW-g(\phi W, Y)\phi AX+g(\phi AW, Y)\phi X$$
$$-g(\phi AX, Y)\phi W+\eta(X)\eta(Y)AW-\eta(W)\eta(Y)AX\}.$$

From (2.9),

(2.10) $$\sum g((R(e_i, X)S)\xi, \phi e_i)=3(2n-3)\lambda g(\phi A\xi, X),$$

(2.11) $$\sum g((R(e_i, \phi e_i)S)\xi, X)=-6\lambda g(\phi A\xi, X).$$

On the other hand by Gauss equation (1.4), the left hand side of (2.10) is

(2.12) $$\sum g((R(e_i, X)S)\xi, \phi e_i)=2ng(\phi S\xi, X)-g(A\phi AS\xi, X)+g(AS\phi A\xi, X).$$

Similarly using Gauss equation (1.4), we see that the left hand side of (2.11) is

(2.13) $$\sum g((R(e_i, \phi e_i)S)\xi, X)=4ng(\phi S\xi, X)-2g(A\phi AS\xi, X)+2g(SA\phi A\xi, X)$$

From (2.10) and (2.12), we have

(2.14) $$-3(2n-3)\lambda\phi A\xi=2n\phi S\xi-A\phi AS\xi+AS\phi A\xi$$

From (2.11) and (2.13), we have

(2.15) $$-3\lambda\phi A\xi=2n\phi S\xi-A\phi AS\xi+SA\phi A\xi$$

From (2.14) and (2.15), we have

(2.16) $$6\lambda(2-n)\phi A\xi=AS\phi A\xi-SA\phi A\xi.$$

On the other hand, from (1.6), we have $SX=-(2n+1)X+3\eta(X)\xi+hAX-A^2X$ and $ASX-SAX=3\eta(X)A\xi-3\eta(AX)\xi$. Hence $AS(\phi A\xi)-SA(\phi A\xi)=0$, which, together with (2.16), implies that $(2-n)\lambda\phi A\xi=0$. Therefore if $n\geqq 3$ we conclude that $\lambda\phi A\xi=0$. This, together with (2.8), implies λ is constant. If λ is not non-zero, we have $\phi A\xi=0$, which is equivalent to that ξ is a principal vector. If $\lambda=0$, then $\nabla S=0$, which is impossible by [4]. Q. E. D.

Using Proposition 1, we have the following

PROPOSITION 2. *Let M be a real hypersurface of CH^n. Then the following are equivalent:*

(1) *The Ricci tensor S of M satisfies*
(2.1) $(\nabla_X S)Y=\lambda\{g(\phi X, Y)\xi+\eta(Y)\phi X\}$
for any $X, Y\in TM$, where λ is a function.

(2) *M is locally congruent to one of type the following:*
(A_0) *a horosphere,*
(A_1) *a geodesic hypersphere in CH^n,*
(A_1') *a tube over a complex hyperplane CH^{n-1}.*

PROOF. From proposition 1, we know that the ξ is a principal vector satisfying (1). Moreover, equation (2.1) shows that the Ricci tensor of our real hypersurfaces M is η-parallel. Therefore Theorem D asserts that M is one of the homogeneous real hypersurfaces of type (A_0), (A_1), (A_1'), (A_2) and (B).

Next we shall check (2.1) for real hypersurfaces above one by one.

Let M be of type (A_0):

Principal curvatures and their multiplicities of type (A_0) are given by the following table.

principal curvatures	1	2
multiplicities	$2n-2$	1.

The shape operator A is as

$$AX=X+\eta(X)\xi \quad \text{for } X\in TM. \tag{2.17}$$

Substituting the condition (ii) in Theorem F and (2.17) into (1.7), we can see that our real hypersurface M satisfies (2.1), that is,

$$(\nabla_X S)Y=2n\{g(\phi X, Y)+\eta(Y)\phi X\}. \tag{2.18}$$

Let M be of type (A_1):

Setting $t=\coth(\theta)$. Then principal curvatures and their multiplicities of type (A_1) are given by the following table.

principal curvatures	t	$t+(1/t)$
multiplicities	$2n-2$	1.

The shape operator A is as

$$AX=tX+(1/t)\eta(X)\xi \quad \text{for } X\in TM. \tag{2.19}$$

Substituting the condition (ii) in Theorem F and (2.19) into (1.7), we can see that our real hypersurface M satisfies (2.1), that is,

$$(\nabla_X S)Y=2nt\{g(\phi X, Y)\xi+\eta(Y)\phi X\}. \tag{2.20}$$

Let M be of type (A_1'):

Setting $t=\tanh(\theta)$. Then principal curvatures and their multiplicities of type (A_1') are given by the following table.

principal curvatures	t	$t+(1/t)$
multiplicities	$2n-3$	1.

By a similar computation we can see that our real hypersurface M satisfies (2.1), that is,

$$(\nabla_X S)Y=2nt\{g(\phi X, Y)\xi+\eta(Y)\phi X\}. \tag{2.21}$$

Let M be of type (A_2):

Setting $t=\tanh(\theta)$. Then principal curvatures and their multiplicities of type (A_2) are given by the following table.

principal curvatures	t	$(1/t)$	$t+(1/t)$
multiplicities	$2k$	$2(n-k-1)$	1.

Now, we put $k=p$, $n-k-1=q$ so, $p+q=n-1$.

Let X be a principal curvature vector orthogonal to ξ with principal curvature t. Note that $A\phi X=t\phi X$ (cf, proposition A). Substituting the condition (ii) in Theorem F into (1.7), we find

$$(\nabla_X S)\phi X=\{(2p+2)t+2q(1/t)\}\xi. \tag{2.22}$$

On the other hand, let X be a principal curvature vector orthogonal to ξ with principal curvature $(1/t)$. By similar computations we see

$$(\nabla_X S)\phi X=\{2pt+(2q+2)(1/t)\}\xi. \tag{2.23}$$

From (2.22) and (2.20), we conclude that our manifold does not satisfy (2.1).

Let M be of type (B):

Setting $t=\cos^2(2\theta)$. Then principal curvatures and their multiplicities of type (B) are given by the following table.

principal curvature	$(\sqrt{t}-1)/(\sqrt{t-1})$	$(\sqrt{t}+1)/(\sqrt{t-1})$	$2\sqrt{t-1}/\sqrt{t}$
multipricities	$n-1$	$n-1$	1.

We put $(\sqrt{t}-1)/(\sqrt{t-1})=r_1$, $(\sqrt{t}+1)/(\sqrt{t-1})=r_2$, $2\sqrt{t-1}/\sqrt{t}=\alpha$.

From proposition A if X be a principal curvature vector orthogonal to ξ with principal curvature r_1, then $A\phi X=r_2\phi X$. With respect to such X, the next formula (cf. [6])

$$(\nabla_X A)\phi X=(\alpha-r_2)r_1\xi \tag{2.24}$$

being satisfied, we see

$$(\nabla_X A)A\phi X=(\alpha-r_2)r_1r_2\xi. \tag{2.25}$$

With respect to this X, substituting (2.24) and (2.25) into (1.7), we find

$$(\nabla_X S)\phi X=(3+h\cdot\alpha-h\cdot r_2-\alpha^2+r_2^2)r_1\xi. \tag{2.26}$$

On the other hand, if X be a corresponding principal curvature vector to principal curvature r_2, then from proposition A $A\phi X=r_1\phi X$. With respect to this X, the next formula (cf. [6])

$$(\nabla_X A)\phi X=(\alpha-r_1)r_2\xi \tag{2.27}$$

being satisfied, we see

$$(\nabla_X A)A\phi X=(\alpha-r_1)r_1r_2\xi\,. \tag{2.28}$$

With respect to this X, substituting (2.27) and (2.28) into (1.7), we find

$$(\nabla_X S)\phi X=(3+h\cdot\alpha-h\cdot r_1-\alpha^2+r_1^2)r_2\xi \tag{2.29}$$

From (2.26) and (2.29) we conclude that our manifold does not satisfy (2.1).

Q. E. D.

Motivated by Proposition 2, we prove the following.

THEOREM 1. *Let M be a real hypersurface in CH^n. Then the following inequality hold.*

$$\|\nabla S\|^2\geqq 1/(n-1)\{2n(h-\eta(A\xi))-(\phi A\xi)\cdot h-\mathrm{trace}\,((\nabla_\xi A)A\phi)\}^2 \tag{2.30}$$

*where S is the Ricci tensor of M and $h=$*trace A. *Moreover, the equality of* (2.30) *holds if and only if M is locally congruent to one of type* (A_0), (A_1') *or* (A_1).

PROOF. We define a tensor T on M by the following:

$$T(X, Y)=(\nabla_X S)Y-\lambda\{g(\phi X, Y)\xi+\eta(Y)\phi X\}\,.$$

Let $e_1, e_2, \cdots, e_{2n-1}$ be local fields of orthonormal vector on M. Now we calculate the length of T. From (1.1) we have

$$\|T\|^2=\|\nabla S\|^2-4\lambda\sum g((\nabla_{e_i}S)\xi, \phi e_i)+4(n-1)\lambda^2\geqq 0\,. \tag{2.31}$$

Regarding (2.31) as quadratic inequality with respect to λ, we calculate the discriminant of the quadric equation and we have

$$1/(n-1)(\sum g((\nabla_{e_i}S)\xi, \phi e_i))^2\leqq\|\nabla S\|^2\,. \tag{2.32}$$

It follows from (1.1), (1.5) and (1.7) that

$$\begin{aligned}
&\sum g((\nabla_{e_i}S)\xi, \phi e_i)\\
&\quad=3g(\phi Ae_i, \phi e_i)-g(\mathrm{grad}\,h, \phi A\xi)+h\cdot g((\nabla_{e_i}A)\xi, \phi e_i)\\
&\qquad-g(A(\nabla_{e_i}A)\xi, \phi e_i)-g((\nabla_{e_i}A)A\xi, \phi e_i)\\
&\quad=3g(A\phi e_i, \phi e_i)-g(\mathrm{grad}\,h, \phi A\xi)+(2n-2)\cdot h-\mathrm{trace}\,((\nabla_\xi A)A\phi)\\
&\qquad-g(A\phi e_i, \phi e_i)-2\eta(A\xi)+2g(A\xi, \xi)-(2n-2)\eta(A\xi)\\
&\quad=2n(h-\eta(A\xi))-(\phi A\xi)\cdot h-\mathrm{trace}\,((\nabla_\xi A)A\phi)\,,
\end{aligned}$$

that is,

$$\sum g((\nabla_{e_i}S)\xi, \phi e_i)=2n(h-\eta(A\xi))-(\phi A\xi)\cdot h-\mathrm{trace}\,((\nabla_\xi A)A\phi)\,. \tag{2.33}$$

Therefore we substitute (2.33) into (2.32) and get inequality (2.30). And, Proposition 2 shows that the equality of (2.30) holds if and only if M is locally congruent to one of type (A_0), (A_1) or (A_1'). Q. E. D.

COROLLARY 1 ([4]). *There are no real hypersurfaces with parallel Ricci tensor of complex hyperbolic space* CH^n.

PROOF. From Proposition 2, if M is not type (A_0), (A_1) or (A_1'), then $\|\nabla S\|^2 > 0$. Thus it follows $\nabla S \neq 0$. If M is type (A_0), (A_1) or (A_1') then, from $\phi A = A\phi$, $\phi\xi = 0$ and $\nabla_\xi A = 0$,

$$\|\nabla S\|^2 = 1/(n-1)\{2n(h-\eta(A\xi))\}^2 .$$

If M be of type (A_0), then

$$\|\nabla S\|^2 = 16n^2(n-1) > 0 .$$

If M be of type (A_1), then

$$\|\nabla S\|^2 = 16n^2(n-1)\coth^2(\theta) > 0 .$$

If M be of type (A_1'), then

$$\|\nabla S\|^2 = 16n^2(n-1)\tanh^2(\theta) > 0 .$$

Thus, it follows $\nabla S \neq 0$. Q. E. D.

§ 3. Characterizations of real hypersurfaces in CH^n in terms of holomorphic distribution.

Now let M be a real hypersurface of CH^n. Let TCH^n and TM be the tangent bundles of CH^n and M, respectively. Let $T^\circ M$ be a subbundle of TM defined by $T^\circ M = \{X \in TM \mid X \perp \xi\}$. Thus each of TM and $T^\circ M$ has a connection induced from TCH^n. The orthogonal complement of $T^\circ M$ in TCH^n with respect to the metric on TCH^n is denoted by $N^\circ M$, which is also a subbundle of TCH^n with the induced metric connection. Denote by ∇° and $\nabla^\perp$ the connections of $T^\circ M$ and $N^\circ M$, respectively. We have

$$\tilde{\nabla}_X Y = \nabla^\circ_X Y + A^\circ(X, Y) \qquad \text{for any } X, Y \in T^\circ M .$$

Let A be the second fundamental form of $T^\circ M$ in TCH^n. A is a smooth section of Hom (TM, Hom ($T^\circ M$, $N^\circ M$)). Set $A = A|_{T^\circ M}$. The covariant derivative of A is defined by

$$(\nabla_X A)(Y, Z) := \nabla^\perp_X(A^\circ(Y, Z)) - A^\circ(\nabla_X Y, Z) - A^\circ(Y, \nabla^\circ_X Z)$$
$$\text{for any } X \in TM,\ Y, Z \in T^\circ M .$$

Now we prepare without proof the following fundamental relations.

PROPOSITION B ([10]).
(i) $A^{\circ}(X, Y)=g(AX, Y)N-g(\phi AX, Y)\xi$,
(ii) $\nabla_X^{\circ}\phi=0$,
(iii) $\nabla_X^{\perp}\xi=g(AX, \xi)N$,
(iv) $\nabla_X^{\perp}N=-g(AX, \xi)\xi$,
where $X, Y\in T^{\circ}M$.

PROPOSITION C ([10]). *For any* $X, Y, Z\in T^{\circ}M$,

$$(\nabla_X^{\circ}A^{\circ})(YZ)=\Psi(X, Y, Z)N+\Psi(X, Y, \phi Z)\xi,$$

where Ψ *is the trilinear tensor defined by*

$$\Psi(X, Y, Z)=g((\nabla_X A)Y, Z)-\eta(AX)g(\phi AY, Z)$$
$$-\eta(AY)g(\phi AX, Z)-\eta(AZ)g(\phi AX, Y).$$

We show the following fundamental result.

PROPOSITION 3. *Let* M *be a real hypersurface of* CH^n. *Then the following are equivalent*:
(i) *The holomolphic distribution* $T^{\circ}M=\{X\in TM|X\perp\xi\}$ *is integrable,*
(ii) $g((\phi A+A\phi)X, Y)=0$ *for any* $X, Y\in T^{\circ}M$.

PROOF. The distribution $T^{\circ}M$ is integrable

$$\longleftrightarrow [X, Y]\in T^{\circ}M \quad \text{for any } X, Y\in T^{\circ}M$$
$$\longleftrightarrow g([X, Y], \xi)=0$$
$$\longleftrightarrow g(\nabla_X Y-\nabla_Y X, \xi)=0$$
$$\longleftrightarrow g(Y, \phi AX)-g(X, \phi AY)=0$$
$$\longleftrightarrow g((\phi A+A\phi)X, Y)=0 \quad \text{for any } X, Y\in T^{\circ}M.$$

Q. E. D.

Recall the definition of η-parallel of A. We say that A° is η-parallel if $\nabla_X^{\circ}A^{\circ}\equiv 0$ for any $X\in T^{\circ}M$. Using the notions defined above, we obtained the following result.

THEOREM 2. *Let* M *be a real hypersurface of* CH^n. *Assume that* A° *is* η*-parallel. Then* M *is locally congruent to ond of type* (A_0), (A_1), (A_1'), (A_2), (B) *or a ruled real hypersurface (that is, a real hypersurface in which $T^{\circ}M$ is integrable and its integral manifold is totally geodesic CH^{n-1}.)*

PROOF. By proposition C, A° is η-parallel if and only if $\Psi(X, Y, Z)=0$ for any $X, Y, Z\in T^{\circ}M$, that is,

$$(3.1)\qquad g((\nabla_X A)Y, Z)=\eta(AX)g(\phi AY, Z)+\eta(AY)g(\phi AX, Z)+\eta(AZ)g(\phi AX, Y)$$

for any $X, Y, Z\in T^{\circ}M$. Since the Codazzi equation (1.5) tells us that $g((\nabla_X A)Y, Z)$ is symmetric for any $X, Y, Z\in T^{\circ}M$, exchanging X and Y in (3.1), we obtain

$$(3.2)\qquad \eta(AZ)g((A\phi+\phi A)X, Y)=0 \qquad \text{for any } X, Y, Z\in T^{\circ}M.$$

Now we assume that $\eta(AZ)=0$ for any $Z\in T^{\circ}M$, that is, ξ is a principal curvature vector. Then the equation (3.1) shows that $g((\nabla_X A)Y, Z)=0$ for any $X, Y, Z\in T^{\circ}M$, that is, the shape operator A of M is η-parallel. And hence our real hypersurface M is locally congruent to one of type (A_0), (A_1), (A_1'), (A_2) or (B) by Theorem E.

Next, if there exists $Z\in T^{\circ}M$ such that $\eta(AZ)\neq 0$, that is, ξ is not a principal curvature vector. Then the equation (3.2) tells us that the holomorphic distribution $T^{\circ}M$ is integrable (cf., Proposition 3) and the integral manifold M° of $T^{\circ}M$ is a complex hypersurface in CH^n. Moreover, the second fundamental form A° of M° is parallel. Therefore we conclude that M° is locally congruent to CH^{n-1} (cf. [9].) Q. E. D.

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