CARDINAL FUNCTIONS OF SPACES WITH ORTHO-BASES

By

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§ 1. Introduction.

Throughout this paper, "space" will mean $T_1$-space. Let $\mathcal{B}$ be a base of a space $X$. $\mathcal{B}$ is said to be an ortho-base if for every $\mathcal{B}' \subseteq \mathcal{B}$, $\cap \mathcal{B}'$ is open or $\mathcal{B}'$ is a neighborhood base of some point. $\mathcal{B}$ is said to have subinfinite rank if for every $\mathcal{B}' \subseteq \mathcal{B}$ such that $\cap \mathcal{B}' \neq \emptyset$ and $\mathcal{B}'$ is infinite, at least two elements of $\mathcal{B}'$ are related by set inclusion. Spaces having an ortho-base, and spaces having a base of subinfinite rank were introduced by Nyikos as natural generalizations of non-archimedean spaces [4] [5].

Concerning cardinal functions of spaces with special bases, Gruenhage showed that for each regular space $X$ having a base of subinfinite rank, $d(X)=hd(X) \geq hl(X) = s(X)$ holds [3]. $d(X)$ is the density of $X$, $hd(X)$ is the hereditary density, $hl(X)$ is the hereditary Lindelöf degree, and $s(X)$ is the spread (i.e., the supremum of the discrete subspaces of $X$). In this paper we investigate cardinal functions of spaces having ortho-bases. We shall show that $hd(X) \geq hl(X) = s(X)$ holds for each space $X$ having an ortho-base.

§ 2. Main result.

We need two lemmas. For convenience, for a cardinal $\tau$, we say a space $X$ to be $\tau$-developable if there exist $\tau$ open covers $\{\mathcal{H}_\alpha\}_{\alpha<\tau}$ such that for each $x \in X$ $\{\text{St}(x, \mathcal{H}_\alpha)\}_{\alpha<\tau}$ is a neighborhood base of $x$.

**Lemma 1.** Let $X$ be a space having an ortho-base $\mathcal{B}$ and $D$ be the set of isolated points of $X$. If $D$ is dense in $X$, then $X$ is $|D|$-developable.

**Proof.** Set $D=\{d_\alpha|\alpha<\tau\}$, where $\tau$ is a cardinal. For each $x \in X-D$ and $\alpha<\tau$, we take $B_\alpha(x) \in \mathcal{B}$ such that $x \notin B_\alpha(x)$ and $d_\alpha \notin B_\alpha(x)$. Put $\mathcal{H}_\alpha=\{d_\alpha|\alpha<\tau\} \cup \{B_\alpha(x)\mid x \in X-D\}$. $\mathcal{H}_\alpha$ is obviously an open cover of $X$. Let $x$ be a point of $X$ and $W$ be a neighborhood of $x$. If $x \in D$, then $\text{St}(x, \mathcal{H}_\alpha)=\{x\} \subseteq W$ for some $\alpha$. So, we assume $x \notin X-D$. Suppose that $\text{St}(x, \mathcal{H}_\alpha) \subseteq W$ for any $\alpha<\tau$. Then for each $\alpha$, we can take $H_\alpha \in \mathcal{H}_\alpha$ such that $x \notin H_\alpha$ and $H_\alpha \subseteq W$. Since $\{H_\alpha\}_{\alpha<\tau}$ can not be a neigh-

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borhood base of $x$, $H=\cap_{x\in X} H_x$ must be open. But $H \cap D = \emptyset$, because $H \not \subseteq D$. Since $D$ is dense in $X$, this is a contradiction.

The following lemma is well known in the countable case and can be easily carried over to the general case. So we omit the proof.

**Lemma 2.** Let $X$ be $\tau$-developable. If the cardinality of each closed discrete subspace is at most $\tau$, then $X$ is $\tau$-Lindelöf (i.e., every open cover has a subcover of the cardinality $\tau$).

**Theorem 3.** Let $X$ be a space having an ortho-base $\mathcal{B}$. Then $hd(X) \geq s(X) = \text{hl}(X)$ holds.

**Proof.** Since $hd(X) \geq s(X)$ and $\text{hl}(X) \geq s(X)$ are obvious, we show $s(X) \geq \text{hl}(X)$. Let $s(X) = \tau$. Since for each subspace $Y$ of $X$, $s(Y) \leq \tau$ and $Y$ has an ortho-base, the proof is complete if we show that $X$ is $\tau$-Lindelöf. Suppose that there exists an open cover $\mathcal{U}$ of $X$ which has not a subcover of the cardinality $\tau$. Firstly we take $x_0 \in X$ and $\mathcal{U}_0 \in \mathcal{U}$ such that $x_0 \in U_0$. Put $V_0 = U_0$. Let $\gamma < \tau$. We assume that for each $\beta < \gamma$ we could take $x_\beta \in X$ and an open set $V_\beta$ such that the following (*) is satisfied.

\[ V_\beta \cap \{x_\alpha | \alpha < \gamma\} = \{x_\beta\} \quad \text{for each } \beta < \gamma. \]

(*) There exists $U_\beta \in \mathcal{U}$ such that $V_\beta \subseteq U_\beta$ for each $\beta < \gamma$.

Then, if we set $A = \{x_\alpha | \alpha < \gamma\}$, since $|A| \leq \tau$, $\text{Cl} A$ is $\tau$-Lindelöf by Lemma 1 and Lemma 2. Thus $\text{Cl} A \cup (\cup_{\beta < \gamma} V_\beta)$ is covered by $\tau$ elements of $\mathcal{U}$. So we can take $x_\tau \in X - \text{Cl} A \cup (\cup_{\beta < \gamma} V_\beta)$. We take $U_\tau \in \mathcal{U}$ and an open set $V_\tau$ such that $x_\tau \in V_\tau \subseteq U_\tau$ and $V_\tau \cap A = \emptyset$. Now by the induction we get the discrete space $\{x_\alpha | \alpha < \tau\}$. This is a contradiction to $s(X) = \tau$.

There exists a space having an ortho-base such that $hd(X) \neq d(X)$. In fact the space in [6, 3.6.1] is such a space.

Concerning SH (Souslin's hypothesis), we note the following theorem.

**Theorem 4.** The following (a), (b) and (c) are equivalent.

(a) SH is false.

(b) There exists a non-metrizable non-archimedean space such that $s(X)$ is countable.

(c) There exists a non-metrizable regular space having an ortho-base such that $s(X)$ is countable.
Proof. The equivalence of (a) and (b) is due to [1]. Also, refer [5, Theorem 1.7]. (b)→(c) is trivial. We show (c)→(b). Let X be a space of (c). Since by Theorem 3 X is regular Lindelöf, it is paracompact. Therefore X is a protometrizable space (i.e., paracompact space having an ortho-base). It follows from Fuller’s result [2, Theorem 6] that X is the perfect irreducible image of a non-archimedean space Y. Since metrizability is an invariant of perfect maps, Y is not metrizable. Since the spread of a non-archimedean space is equal to the cellularity, by the irreducibility of the map, s(Y) must be countable. Thus Y is the desired space.

Corollary 5. The following (a) and (b) are equivalent.
(a) SH.
(b) Each regular space having an ortho-base is metrizable if the spread is countable.

References

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