THE MODULI SPACE OF ANTI-SELF-DUAL CONNECTIONS OVER HERMITIAN SURFACES

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1 Introduction

Let \((M, g)\) be a compact Hermitian surface with an orientation induced by the complex structure of \(M\), and \(P\) a principal bundle over \(M\) with structure group \(SU(n)\). Then a canonical representation \(\rho\) of \(SU(n)\) induces a smooth complex vector bundle \(E = P \times_\rho \mathbb{C}^n\). A necessary and sufficient condition for a \(SU(n)\)-connection \(D\) on \(E\) to be an anti-self-dual connection is that the curvature of \(D\) is a differential 2-form of type \((1,1)\), and is orthogonal to the fundamental form \(\Phi\) of \((M, g)\). Hence, a holomorphic structure is induced on \(E\) and hence on \(\text{End}^0E\) (the subbundle of \(\text{End}E\) consisting of endomorphisms with trace 0) by an anti-self-dual connection \(D\). Itoh ([4]) showed that the moduli space of anti-self-dual connections over Kähler surfaces is a complex manifold. We will extend this result over Kähler surfaces to over Hermitian surfaces, which are not necessarily Kählerian.

Let \(K_M\) be a canonical line bundle over \(M\). We define \(\tilde{H}_D = H^0_0(M; /\langle \text{End}^0E \otimes K_M \rangle)\) as the space of holomorphic sections, where \(\text{End}^0E\) is endowed with the holomorphic structure induced from the irreducible anti-self-dual connection \(D\). We denote by \(\mathcal{M}\) the moduli space of irreducible anti-self-dual connections (the quotient space of irreducible anti-self-dual connections by the gauge transformation group \(SU(E)\)), and set \(\mathcal{M}_0\) as follows: \(\mathcal{M}_0 = \{[D] \in \mathcal{M} : \tilde{H}_D = (0)\}\). Then we obtain the following

**Theorem 1.** Let \(M\) be a compact Hermitian surface. If \(\mathcal{M}_0\) is not empty, then \(\mathcal{M}_0\) is a complex manifold.

We can make \(H_D\) vanish under a certain condition. On a Hermitian manifold

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$$(M, g)$$, $\text{Scal}(g)$ denotes the scalar curvature of the Hermitian connection with respect to $g$. Then we have the following vanishing theorem.

**Proposition 1.** Let $(M, g)$ be a compact Hermitian surface with fundamental form $\Phi$ which satisfies $\partial \overline{\partial} \Phi = 0$. If $\int_M \text{Scal}(g) d\nu \geq 0$, then $\tilde{H} = 0$.

With this proposition, Theorem 1 implies the following.

**Theorem 2.** Let $(M, g)$ be a compact Hermitian surface which satisfies the same condition as proposition 1. If $\mathcal{M}$ is not empty, then $\mathcal{M}$ is a complex manifold.

### 2. Two moduli spaces

In this section we will recall the moduli spaces of anti-self-dual connections and holomorphic semi-connections following [1], [4], and [5].

Let $(M, g)$ be a compact oriented Hermitian surface with fundamental form $\Phi = \sqrt{-1} \sum g_{ab} dz^a \wedge d\bar{z}^b$. We will denote by $A^p$ (resp. $A^{p,q}$) the space of real valued smooth $p$-forms (resp. $(p,q)$-forms) on $M$. Then we have the decomposition of the space of 2-forms,

$$A^2 \otimes \mathbb{C} = A^{2,0} \oplus A^{1,1} \oplus A^{0,2}. \quad (2.1)$$

The fundamental form $\Phi$ decomposes $A^{1,1}$ further:

$$A^{1,1} = A^{1,1} \oplus (A^{1,1})^\perp, \quad (2.2)$$

where

$$A^{1,1} = \{ f \Phi : f \in C^\infty(M; \mathbb{C}) \}, \quad (2.3)$$

and

$$(A^{1,1})^\perp = \{ \psi = \sum g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta : \sum g^{a\bar{\beta}} \psi_{a\bar{\beta}} = 0 \}. \quad (2.4)$$

$(A^{1,1})^\perp$ is the space of all primitive $(1,1)$-forms in $(M, g)$. We put

$$A_1^2 = (A^{2,0} + A^{1,1} + A^{0,2}) \cap A^2, \quad (2.5)$$

and

$$A_2^2 = (A^{1,1})^\perp \cap A^2. \quad (2.6)$$

Then $A_2^2$ (resp. $A_2^1$) is the self-dual part (resp. the anti-self-dual part) of $A^2$ ([1]). Then projection from $A^2$ onto $A_2^2$ is denoted by $p_\perp$. 

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Let \( P \) be a principal bundle over \( M \) with structure group \( SU(n) \). Then the canonical representation \( \rho \) of \( SU(n) \) induces a smooth complex vector bundle \( E = P \times_{\rho} \mathbb{C}^n \). We denote by \( h \) and \( \omega \), the Hermitian structure and the \( n \)-form on \( E \) defined by the \( SU(n) \)-structure of \( P \), respectively. Let \( GL(E) \) denote the group of \( C^\infty \)-bundle automorphisms of \( E \) (inducing the identity transformations on the base manifold \( M \)). Let \( SL(E) \) (resp. \( SU(E) \)) denote the subgroup of \( GL(E) \) consisting of bundle automorphisms of \( E \) (resp. unitary automorphisms of \( (E, h) \)) with determinant 1. They are called the gauge transformation groups of \( E \). Let \( \text{End}^{0}(\rho, \omega) \) (resp. \( \text{End}^{0}(E, h) \)) the subbundle of the endomorphism bundle \( E \) consisting of endomorphisms (resp. skew-Hermitian endomorphisms) with trace 0. \( \text{End}^{0}(E, h) \) is the real subbundle of \( \text{End}^{0}E \) and we have

\[
\text{End}^{0}E = \text{End}^{0}(E, h) \oplus \sqrt{-1}\text{End}^{0}(E, h).
\] (2.7)

For \( \psi = \psi_0 + \sqrt{-1}\psi_1, \psi_0 - \sqrt{-1}\psi_1 \), we denote the complex conjugate by \( \overline{\psi} \), which is defined by \( \overline{\psi} = \psi_0 - \sqrt{-1}\psi_1 \).

An \( SU(n) \)-connection \( D \) in \( (E, h) \) is a connection in \( E \) preserving \( h \) and \( \omega \), i.e., a homomorphism \( D: \text{End}^{0}(E, h) \rightarrow \text{End}^{1}(E, h) \) over \( \mathbb{C} \) such that

\[
D(f\sigma) = \sigma df + f.D\sigma \quad \text{for} \quad f \in \mathcal{A}^{0}_{\mathbb{C}}, \sigma \in \mathcal{A}^{1}_{\mathbb{C}}(E),
\]

\[
Dh = 0,
\]

\[
D\omega = 0.
\] (2.8)

The set of \( SU(n) \)-connections has an affine structure. Namely, it is given by \( \{D + v: v \in \mathcal{A}^{1}(\text{End}^{0}(E, h))\} \) for a fixed \( SU(n) \)-connection \( D \). We can extend an \( SU(n) \)-connection \( D \) to a connection in \( \text{End}^{0}(E, h) \). We call \( D \) irreducible when the kernel of \( D: \text{End}^{0}(E, h) \rightarrow \mathcal{A}^{1}(\text{End}^{0}(E, h)) \) is trivial. An \( SU(n) \)-connection \( D \) is called anti-self-dual, if the curvature form \( R(D) \) belongs to \( \mathcal{A}^{2}_{\mathbb{C}}(\text{End}^{0}(E, h)) \), namely \( p, R(D) = 0 \). Let \( \text{Asd} \) be the set of all anti-self-dual \( SU(n) \)-connections in \( (E, h) \). The gauge transformation group \( SU(E) \) acts on the space of \( SU(n) \)-connections and leaves \( \text{Asd} \) invariant. Thus we obtain the moduli space \( \text{Asd}/SU(E) \) of anti-self-dual \( SU(n) \)-connection in \( (E, h) \).

A semi-connection \( D'' \) in \( E \) is a linear map \( D'': A^{0}(E) \rightarrow A^{0,1}(E) \) satisfying \( D''(f\sigma) = D''f\sigma + f D''\sigma \) for \( \sigma \in A^{0}(E), f \in C^\infty(M; \mathbb{C}) \). Moreover we assume that \( D'' \) preserves the \( n \)-form \( \omega \), i.e.; \( D''\omega = 0 \). The set of semi-connections has an (complex) affine space. Namely, it is given by \( \{D'' + v: v \in A^{0,1}(\text{End}^{0}E)\} \) for a fixed semi-connection \( D'' \). We can extend \( D'' \) to a semi-connection in \( \text{End}^{0}E \). We call \( D'' \) simple when the kernel of \( D'': A^{0}(\text{End}^{0}E) \rightarrow A^{0,1}(\text{End}^{0}E) \) is trivial. A semi-connection \( D'' \) which satisfies \( D'' \circ D'' = 0 \) defines a unique holomorphic structure on \( E \). We call such a semi-connection holomorphic. Let \( \text{Hol} \) be the set
of all holomorphic semi-connections in $E$. The gauge transformation group $SL(E)$ acts on the space of semi-connections and leaves $\text{Hol}$ invariant. Thus we obtain the moduli space $\text{Hol}/SL(E)$ of holomorphic semi-connections in $E$.

Let $D$ be an $SU(n)$-connection in $(E, h)$. Set $D = D' + D''$ where $D': A^0(E) \to A^{0,1}(E)$. Then $D''$ is a semi-connection in $E$. This natural map $D \mapsto D''$ is a bijective map of the set of $SU(n)$-connections onto the set of semi-connections. If $D$ is anti-self-dual, $D''$ is holomorphic. In fact the $(0,2)$-component of $R(D) = D'' \circ D'$. Thus we obtain a natural map $f: \text{Asd}/SU(E) \to \text{Hol}/SL(E)$. It is known that $f$ is an injective map (cf. [5, p.243]). Moreover we have

**Lemma 1.** If an anti-self-dual connection $D$ is irreducible, then $D''$ is simple.

**Proof** Suppose $\phi \in A^0(\text{End}^0 E)$ be a holomorphic section of $\text{End}^0 E$. Then $D''\phi = 0$. By the vanishing theorem of the holomorphic sections ([5]), we obtain $D\phi = 0$. By the assumption that $D$ is an irreducible connection, we conclude $\phi = 0$.

In order to consider infinitesimal deformations, we introduce two complexes (2.9), (2.10), and their cohomology groups. For $D \in \text{Asd}$ set

$$0 \to A^0(\text{End}^0 E, h) \xrightarrow{D} A^1(\text{End}^0 E, h) \xrightarrow{D^2} A^2(\text{End}^0 E, h) \to 0$$

(2.9)

where $D_\pm = p_\pm \circ D$. Their cohomology groups are denoted by $H^p_D (p = 0, 1, 2)$. For $D'' \in \text{Hol}$, we consider the Dolbeault complex

$$0 \to A^{0,0}(\text{End}^0 E) \xrightarrow{D^*} A^{0,1}(\text{End}^0 E) \xrightarrow{D^2} A^{0,2}(\text{End}^0 E) \to 0$$

(2.10)

and their cohomology groups are denoted by $H^{p,0}_D (p = 0, 1, 2)$. We set

$$\mathcal{M}_0 = \{ [D] \in \text{Asd}/SU(E) : D \text{ is irreducible and } H^1_{D'} \text{ vanishes} \}$$

(2.11)

Then it is known that $\mathcal{M}_0$ is a smooth manifold, and its tangent space at $[D]$ is naturally isomorphic to $H^1_{D'}$. We set

$$\mathcal{M}_0' = \{ [D''] \in \text{Hol}/SU(E) : D'' \text{ is simple and } H^{0,2}_{D''} \text{ vanishes} \}$$

(2.12)

Similarly it is known that $\mathcal{M}_0'$ is a complex manifold and its tangent space at $[D'']$ is naturally isomorphic to $H^{0,1}_{D''}$.

Now we consider the following natural homomorphism between two complexes (2.9), (2.10) for an irreducible anti-self-dual $SU(n)$-connection $D$ and its corresponding holomorphic semi-connection $D$:
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\[ \begin{array}{c}
0 \rightarrow A^0(\text{End}^0(E, h)) \xrightarrow{h_0} A^1(\text{End}^0(E, h)) \xrightarrow{h_1} A^2(\text{End}^0(E, h)) \rightarrow 0 \\
\downarrow h_0 \quad \downarrow h_1 \quad \downarrow h_2
\end{array} \]

\[ \begin{array}{c}
0 \rightarrow A^{0,0}(\text{End}^0(E)) \xrightarrow{h_0} A^{0,1}(\text{End}^0(E)) \xrightarrow{h_1} A^{0,2}(\text{End}^0(E)) \rightarrow 0 \\
\downarrow h_0 \quad \downarrow h_1 \quad \downarrow h_2
\end{array} \]  

(2.13)

where

- \( h_0 \) : inclusion
- \( h_1 : \alpha \rightarrow \alpha^{0,1} \)
- \( h_2 : \alpha \rightarrow \alpha^{0,2} \)

and \( \alpha^{0,p} \) represents the \((0, p)\)-component of \( \alpha \). Itoh showed that \( h_p \) induces an isomorphism of \( H^p_D \) onto \( H^{0,p}_D \) \((p = 0, 1, 2)\) when \((M, g)\) is a Kähler surface. We can extend this result to the case of a Hermitian surface. Its proof will be given in section 3. Therefore we have \( f(\mathcal{K}_0) \subset \mathcal{N}_0 \) for the natural map \( f \). Moreover it is known that \( f \) is a differentiable map. Since we can regard the differential \( f_* \) of \( f \) at \([D]\) as \( h_1, f \) is a diffeomorphism of \( \mathcal{N}_0 \) into \( \mathcal{N}_0 \). Thus it has been shown that \( \mathcal{N}_0 \) is a complex manifold. We note that \( H^{0,2}_D \) is isomorphic to \( \tilde{H} = H^0(M, \text{End}^0(E \otimes K_M)) \) by the Serre duality. Hence our Theorem 2 has been proved.

3. Isomorphisms between cohomology groups \( H^p_D \) and \( H^{0,p}_D \)

In this section, we prove that for an irreducible anti-self-dual connection the cohomology groups \( H^p_D \) are isomorphic to \( H^{0,p}_D \) \((p = 0, 1, 2)\) in the diagram (2.13).

We first begin with the preparation for the proof. On a Hermitian surface \((M, g)\), we define differential 1-forms \( \theta = -d^* \Phi \eta = \theta J \), and \((1, 0)\)-form \( \varphi = \eta + \sqrt{-1} \theta \). Here \( J \) is the complex structure of \((M, g)\). Then we obtain following formulas by direct calculation.

**Lemma 2.** For the operators acting on \( A^p(\text{End}^0E) \), the following formulas hold:

\[ D^* = -\sqrt{-1}(D^2 \Lambda - \Lambda D^2) + \frac{1}{2} (p-2)i(\varphi) - \frac{\sqrt{-1}}{2} e(\varphi) \Lambda \]  

(3.1)

\[ D^* = \sqrt{-1}(D^2 \Lambda - \Lambda D^2) + \frac{1}{2} (p-2)i(\varphi) - \frac{\sqrt{-1}}{2} e(\varphi) \Lambda \]  

(3.2)

It is known that there is a unique Hermitian metric up to the homothety such that
$d^* \eta = 0$ in the conformal class of the given Hermitian metric ([3]). Moreover the anti-self-duality is preserved by a conformal change of the metric. Therefore we may assume that $d^* \eta = 0$ on the given Hermitian surface. Define a mapping $\mathcal{J} : A^0(\text{End}^0 E) \to A^0(\text{End}^0 E)$ by $\mathcal{J} = -\sqrt{-1} \Lambda D'D''$. Then we have

**Lemma 3.** On $A^0(\text{End}^0 E)$

$$\mathcal{J} = \frac{1}{2} (\Delta_D + i(\eta) D),$$

(3.3)

where $\Delta_D = D^* D$.

**Proof** In fact

$$\Delta_D = D^* D = (D^* + D'') (D' + D'')$$

(3.4)

Using equations (3.1) and (3.2), we see that

$$D^* D' + D' D'' = \sqrt{-1} \Lambda D'' D' - \frac{1}{2} i(\Lambda \phi) D' - \sqrt{-1} \Lambda D'' D' - \frac{1}{2} i(\phi) D'$$

(3.5)

$$= \sqrt{-1} \Lambda (D'' D' - D' D'') - i(\eta) D.$$

Since $D$ is an anti-self-dual connection, for $\psi \in A^0(\text{End}^0 E)$, we have

$$\Lambda(D'' D' + D' D'')\psi = \Lambda R(D)(\psi)$$

$$= \Lambda(R(D) \circ \psi - \psi \circ R(D))$$

$$= (\Lambda R(D))\psi - \psi(\Lambda R(D))$$

(3.6)

$$= 0.$$

It follows that

$$\Delta_D = -2 \sqrt{-1} \Lambda D'' D' - i(\eta) D.$$

(3.7)

Then we obtain (3.3).

From Lemma 3 we see that $\mathcal{J}(A^0(\text{End}^0 E, h)) \subset A^0(\text{End}^0 E, h))$. Let $\mathcal{J}^*$ be the formal adjoint operator of $\mathcal{J}$. For $\phi, \psi \in A^0(\text{End}^0 (E, h))$,

$$(\mathcal{J}\phi, \psi)_M = \left( \frac{1}{2} \Delta_\phi + \frac{1}{2} i(\eta) D \phi, \psi \right)_M$$

$$= \left( \phi, \frac{1}{2} \Delta_\psi + \frac{1}{2} D^* \varepsilon(\eta) \psi \right)_M, \quad (3.8)$$

Consequently we have
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\[ \mathcal{G}^* = \frac{1}{2} (\Delta_D + D^* \varepsilon(\eta)). \] (3.9)

By the direct calculation on \( \mathcal{A}^0(\text{End}^0(E,h)) \), we have

\[ D^* \varepsilon(\eta) = \varepsilon(d^* \eta) - i(\eta)D = -i(\eta)D. \] (3.10)

Consequently, we obtain

\[ \mathcal{G}^* = \frac{1}{2} (\Delta_D - i(\eta)D). \] (3.11)

Lemma 4. On \( \mathcal{A}^0(\text{End}^0(E,h)) \), we have

\[ \ker \mathcal{G} = \ker \mathcal{G}^* = \ker D \] (3.12)

Proof: It is clear that \( \ker D \subset \ker \mathcal{G} \), and \( \ker D \subset \ker \mathcal{G}^* \) by (3.3), (3.11). Conversely suppose that \( \mathcal{G} \phi = 0 \), for \( \phi \in \mathcal{A}^0(\text{End}^0(E,h)) \). Then

\[ 0 = (\mathcal{G} \phi, \phi)_M = \left( \frac{1}{2} \Delta_D \phi + \frac{1}{2} i(\eta)D \phi, \phi \right)_M \]
\[ = \frac{1}{2} (D\phi, D\phi)_M + \frac{1}{2} (i(\eta)D\phi, \phi)_M \] (3.13)

Using (3.10), we see that

\[ (i(\eta)D\phi, \phi)_M = (\phi, D^* \varepsilon(\eta)\phi)_M \]
\[ = -(\phi, i(\eta)D\phi)_M \]
\[ = -(i(\eta)D\phi, \phi)_M. \] (3.14)

Then

\[ (i(\eta)D\phi, \phi)_M = 0, \] (3.15)

From (3.13) it follows that \( D\phi = 0 \). Noting that \( \mathcal{G}(\mathcal{A}^0(\text{End}^0(E,h))) \subset \mathcal{A}^0(\text{End}^0(E,h)) \), we obtain

\[ \ker \mathcal{G} \subset \ker D \] (3.16)

Owing to (3.11), we obtain \( \ker \mathcal{G}^* \subset \ker D \) similarly.

Theorem 3. Let \( D \) be an irreducible anti-self-dual \( \text{SU}(n) \)-connection. Then the homomorphisms of the cohomology groups \( h_p : H^0_{\mathcal{G}} \to H^{0,p}_{\mathcal{G}^*}(p = 0,1,2) \) induced from the diagram (2.13) are isomorphisms.
Proof)

$h_0$: By Lemma 1, we have $H^0 \rightarrow H^{0,0} = 0$. Therefore it is trivial that $h_0$ is isomorphic.

$h_1$: First we show the injectivity of $h_1$. Suppose $[\alpha] \in H^1$ and $h_1([\alpha]) = 0$. That is, $\alpha \in A^1(\text{End}^0(E, h))$ satisfies $D_\alpha = 0$ and there exists $\phi \in A^0(\text{End}^0 E)$ such that $h_1(\alpha) = \alpha^{0,1} = D^*\phi$. Since $D_\alpha = 0$, $\Lambda(D''D\overline{\phi} + D'D\overline{\phi}) = 0$. We set $\phi = \phi_0 + \sqrt{-1}\phi_1$ and $\overline{\phi} = \phi_0 \sqrt{-1}\phi_1$ for $\phi_0, \phi_1 \in A^0(\text{End}^0(E, h))$. Then

$$0 = \Lambda(D''D\phi_0 - \sqrt{-1}D'\phi_1 + D'D\phi_0 + \sqrt{-1}D'D\phi_1)$$
$$\quad = \Lambda(D'D\phi_0 + D'D\phi_0 - \sqrt{-1}\Lambda(D''D\phi_1 - D'D\phi_1)$$

(3.17)

Since $D$ is an anti-self-dual connection,
$$\Lambda(D''D + D'D)\phi_0 = (\Lambda R(D))\phi_0 = 0,$$

(3.18)

and
$$\sqrt{-1}\Lambda(D'D'' - D''D')\phi_1 = 2\gamma\phi_1.$$

(3.19)

Therefore we have $2\gamma\phi_1 = 0$. Together with Lemma 4, the irreducibility of $D$ implies $\phi_1 \equiv 0$. Consequently
$$\alpha = \alpha^{0,0} + \alpha^{0,1} = D'\phi_0 + D''\phi_0 = D\phi_0$$

(3.20)

and then $[\alpha] = 0$ in $H^1$. It is shown that $h_1$ is injective.

Next, in order to prove the surjectivity of $h_1$, given $\beta \in A^{0,1}(\text{End}^0 E)$ satisfying $D^*\beta = 0$, we will find $[\alpha] \in H^1$ such that $h_1([\alpha]) = [\beta]$ in $H^{0,1}$. To do so, we put $\alpha = \overline{\beta} + D'\overline{\psi} + \beta + D''\psi \in A^1(\text{End}^0(E, h))$. The equation $D_\alpha = 0$ means
$$D''\alpha^{0,1} = D''(\beta + D''\psi) = 0$$

(3.21)

and
$$\Lambda(D''\alpha^{1,0} + D'\alpha^{0,1}) = \Lambda(D''\beta + D''D'\overline{\psi} + D'\beta + D'D''\psi)$$
$$\quad = \Lambda(D''\beta + D'\beta + 2\sqrt{-1}\Lambda D'D''\psi) = 0,$$

(3.22)

where $\psi = \psi_0 + \sqrt{-1}\psi_1$. Therefore we have
$$2\gamma\psi_1 = \Lambda(D''\beta + D'\beta)$$

(3.23)

By Lemma 4 and the irreducibility of $D$, the kernel of $\gamma^k$ is trivial. Then we can find $\psi_1$ which satisfies the equation (3.23). Taking $\psi_0$ suitably, we obtain
\( \alpha \in A^1(\text{End}^0 E, h) \) satisfying \( h_1([\alpha]) = [\beta] \).

\( h_2 \) is surjective. So we show the injectivity. Let \( \psi \) be an element of \( A^2(\text{End}^0 E, h) \). We decompose \( \psi \) as follows: \( \psi = \psi^{0,2} + (1/2)\Phi \wedge \phi + \psi^{0,2} \) for \( \phi \in A^0(\text{End}^0 E, h) \). Suppose \( h_2([\psi]) = 0 \). That is, there exists a \( \beta \in A^{0,1}(\text{End}^0 E, h) \) such that \( h_2(\psi) = \psi^{0,2} = D''\beta \). We will find \( \alpha \in A^1(\text{End}^0 E, h) \) such that \( \psi = D_0\alpha \). To do so, we put \( \alpha = \beta + D'\gamma + D''\phi \) for some \( \gamma \in A^0(\text{End}^0 E) \). Then we have

\[
\psi = D_0\alpha = D'\beta + D'\gamma + \frac{1}{2} \Phi \wedge \Lambda [D''(\beta + D'\gamma) + D'(\beta + D''\gamma)] + D''(\beta + D''\gamma)
\]

(3.24)

We set \( \gamma = \gamma_0 + \sqrt{-1}\gamma_1 \) for \( \gamma_0, \gamma_1 \in A^0(\text{End}^0 E, h) \). Then

\[
\phi = \Lambda [D''\beta + D''D'\gamma + D'\beta + D'D''\gamma] = \Lambda [D''\beta + D'\beta] + 2\Lambda D'D''\gamma_1.
\]

(3.25)

Therefore we have

\[
2 \phi \gamma_1 = \Lambda (D''\beta + D'\beta) - \phi.
\]

(3.26)

The solution \( \gamma_1 \) of (3.26) exists since \( D \) is irreducible and \( \ker \phi^0 = \{0\} \). We have found \( \alpha \) satisfying \( \psi = D_0\alpha \).

4. Vanishing of \( \tilde{H}_p \)

In this section, we will prove Proposition 1 in the introduction. First we recall the results obtained by Gauduchon in [2]. Let \((M, g)\) be an \( m \)-dimensional compact Hermitian manifold with \( \partial \bar{\partial} \Phi = 0 \). Let \( L \) be a holomorphic line bundle over \((M, g)\), and \( h \) be its Hermitian structure. We denote by \( k \) the mean curvature of \((L, h)\). We use the notation "mean curvature" following Kobayashi [5, p. 51] and it is called the Ricci-scalar in Gauduchon [2]. Then the following holds ([2]):

1. \( \int_M k d\nu \) is independent of the Hermitian structure \( h \).

2. There exists a unique Hermitian structure \( h_0 \) on \( L \) (up to the homothety) such that its mean curvature \( k_0 \) is constant.

In particular, applying the above results to the canonical line bundle \( K_M \), we obtain the Hermitian structure with constant mean curvature \( k_0 \). We note that
\[ k_0 \text{Vol}(M, g) = -\int_M \text{Scal}(g) \, dv, \]
where \( \text{Scal}(g) \) denotes the scalar curvature of the Hermitian connection with respect to \( g \).

Now we return to the proof of Proposition 1. The \( C^\infty \)-Hermitian vector bundle \((E, h)\) has a holomorphic structure defined by the anti-self-dual \( SU(n) \)-connection \( D \). \( D \) is the Hermitian connection of \((E, h)\) with respect to this holomorphic structure and it has mean curvature 0 and so for \( \text{End}^0 E \). Together with the former, it implies that the tensor product \( F = \text{End}^0 E \otimes K_M \) admits a Hermitian structure with mean curvature \( k_0 I_L \). If \( k_0 < 0 \), by the vanishing theorem of the holomorphic sections ([5, pp. 49–53]), \( \text{End}^0 E \otimes K_M \) admits no nonzero holomorphic sections. Further, if \( k_0 = 0 \), then every holomorphic section is parallel. Let \( f \) be a nonzero holomorphic section section of \( \text{End}^0 E \otimes K_M \). For each point \( x \) on \( M \), consider the eigenspace of the homomorphism \( f_x \). These eigenspaces define a parallel subbundle of \( E \). This contracts that \( D \) is an irreducible connection. Consequently, even if \( k_0 = 0 \), \( \text{End}^0 E \otimes K_M \) has no nonzero holomorphic sections.

**Remark:** Let \((M, g)\) be a compact anti-self-dual Hermitian surface (i.e., its Weyl conformal curvature tensor \( W \) belongs to \( A^2 \)) with \( \partial\bar{\partial}\Phi = 0 \). Then we have
\[ \int_M \text{Scal}(g) \, dv \geq 0 \]
and the equality holds if and only if \((M, g)\) is Kählerian (cf. Boyer [6]).

**References**


