GRADED COALGEBRAS
AND MORITA-TAKEUCHI CONTEXTS

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0. Introduction

Viewing a $G$-graded $k$-coalgebra over the field $k$ as a right $kG$-comodule coalgebra it is possible to use a Hopf algebraic approach to the study of coalgebras graded by an arbitrary group that was started in [NT].

Let $C = \bigoplus_{g \in G} C_g$ be a $G$-graded coalgebra. The graded $C$-comodules may be viewed as comodules over the smash product $C \rtimes kG$, the general definition of which was given in [M]. Coalgebras graded by an arbitrary group have been considered in [FM] in order to introduce the notion of $G$-graded Hopf algebras. On the other hand, M. Takeuchi introduced in [T] the sets of pre-equivalence data connecting categories of comodules over two coalgebras (we call such a set a Morita-Takeuchi context). The main result of this note is a coalgebra version of a result established by M. Cohen, S. Montgomery in [CM] for group-graded rings: for a graded coalgebra $C$ the coalgebras $C$ and $C \rtimes kG$ are connected by a Morita-Takeuchi context in which one of the structure maps is injective. Most of the results in this note are consequences of the foregoing. As a first application we find that a coalgebra $C$ is strongly graded if and only if the other structure map of the context is also injective. The final section provides analogues of the Cohen-Montgomery duality theorems: if $C$ is a coalgebra graded by the finite group $G$ of order $n$, then $G$ acts on the smash coproduct as a group of automorphisms of coalgebras and $(C \rtimes kG) \rtimes kG^*$ is coalgebra isomorphic to the comatrix coalgebra $M^c(n, C)$. If $G$ is a finite group of order $n$, acting on the coalgebra $D$ as a group of coalgebra automorphisms, then the smash coproduct $D \rtimes kG^*$ is strongly graded by $G$ and moreover: $(D \rtimes kG^*) \rtimes kG \cong M^c(n, D)$. The second duality theorem is again a direct consequence of the Morita-Takeuchi context mentioned above.

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1. Graded Coalgebras and the Smash Coproduct

Throughout this paper $k$ is a field. We use Sweedler’s “sigma” notation [S] and further notation and conventions in [T], [D]. Let $G$ be a group with identity element 1. Recall that a $k$-coalgebra $(C, \Delta, \varepsilon)$ is graded by $G$ if $C$ is a direct sum of $k$-subspaces, $C = \bigoplus_{g \in G} C_g$, such that $\Delta(C_g) \subseteq \sum_{x+y=g} C_x \otimes C_y$, for all $g \in G$, and $\varepsilon(C_g) = 0$ for $g \neq 1$. A right $C$-comodule $M$ with structure map $\rho : M \rightarrow M \otimes C$ is a graded $C$-comodule if $M = \bigoplus_{g \in G} M_g$ as $k$-subspaces, such that $\rho(M_g) \subseteq \sum_{x+y=g} M_x \otimes C_y$ for all $g \in G$. For graded right $C$-comodules $M$ and $N$ a graded comodule morphism is a $C$-comodule morphism $f : M \rightarrow N$ such that $f(M_g) \subseteq N_g$ for all $g \in G$. The category of graded right $C$-comodules, denoted by $gr\mathcal{C}$, is a Grothendieck category, cf. [NT]. The main purpose of this section is to develop a Hopf algebraic approach to the graded theory. First we recall, see [S] or [A], some definitions.

1.1. Definition. Let $H$ be a bialgebra over the field $k$, $A$ a $k$-algebra and $(C, \Delta_c, \varepsilon_c)$ a $k$-coalgebra. Then:

i. $A$ is said to be a (right) $H$-module algebra if $A$ is a right $H$-module such that $(ab) \cdot h = \sum (a \cdot h_1)(b \cdot h_2)$ and $1_A \cdot h = \varepsilon(h)1_A$ for any $h \in H$, and $a, b \in A$.

ii. $C$ is a right $H$-comodule coalgebra if $C$ is an $H$-comodule by $c \rightarrow \sum c_{(0)} \otimes c_{(1)}$ such that we have:

$$\sum c_{1(0)} \otimes c_{2(0)} \otimes c_{1(1)} c_{2(1)} = \sum c_{(0)1} \otimes c_{(0)2} \otimes c_{1(1)},$$

$$\sum \varepsilon_c(c_{(0)}) c_{(1)} = \varepsilon_c(c) 1_H \text{ for all } c \in C,$$

iii. $C$ is a (left) $H$-module coalgebra if $C$ is a left $H$-module such that:

$$\Delta_c(h \cdot c) = \sum h_1 c \otimes h_2 c, \quad \varepsilon_c(h \cdot c) = \varepsilon_H(h) \varepsilon_c(c) \text{ for } c \in C, \ h \in H.$$

In the sequel we shall not refer to “right” of “left” as in the above definitions, the choice of “sides” shall remain fixed throughout.

For any group $G$ the group algebra $kG$ has a bialgebra structure defined by $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ for all $g \in G$. The next result establishes the connection between $G$-graded coalgebras and $kG$-comodule coalgebras.

1.2. Proposition. A coalgebra $C$ graded by $G$ many in a natural way be viewed as a $kG$-comodule coalgebra; conversely every $kG$-comodule coalgebra is a $G$-graded coalgebra.

Proof. For a $G$-graded $C$ the map $\rho : C \rightarrow C \otimes kG$, $c \rightarrow c \otimes \sigma$ for all $\sigma \in G$,
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c∈C, defines a kG-comodule coalgebra structure on C. Conversely, if C is a
kG-comodule coalgebra then any c∈C has a unique presentation ρ(c) =
Σg∈G c_g ⊗ g. Put C_g = {c ∈ C, g ∈ G}; C_g is a k-subspace of C. From
(I⊗δ)p(c) = c⊗1 we derive that c = Σg∈G c_g and C = Σg∈G C_g. For c∈C, g∈G
we have that c∈C_g if and only if ρ(c) = c⊗g. If Σc_g = 0 for some c_g∈C_g
then by applying ρ we obtain Σc_g⊗g = 0 or c_g = 0 for all g∈G. Therefore
C = Σg∈G C_g. Consider c∈C and A(c) = ∑c_1⊗c_2 with homogeneous
c_i's and c_2's. From 1.1 we retain that ∑c_1⊗c_2 equals ∑c_1⊗c_2⊗deg c_1·deg c_2, or in
other words Δ(c) is the sum of all terms with deg c_1·deg c_2, establishing that
C is a G-graded coalgebra.

We say that the group G acts on the coalgebra D whenever there is a
group morphism ϕ: G→ Aut(D), the latter denoting the set of all coalgebra
automorphisms of D with group structure defined as follows: if f, g∈Aut(D),
f·g = f◦g.

1.3. Proposition. If G acts on the coalgebra D then D has the structure
of a kG-module coalgebra; conversely any kG-module coalgebra has a natural
G-action.

Proof. Suppose that ϕ: G→ Aut(D) determines that G acts on D then
the map kG⊗D→D, g⊗d→ϕ(g)(d) defines a kG-module structure on D as
desired. Conversely, if D is a kG-module coalgebra then we may define a G-
action on D by ϕ: G→ Aut(D), ϕ(g)(d) = g·d for g∈G, d∈D.

1.4. Remark. Let, for a finite group G, kG* be the dual bialgebra for the
finite dimensional bialgebra kG. If the finite group G acts on the coalgebra D
then D is also a kG*-comodule coalgebra. If {p_g, g∈G} is the dual basis of
{g, g∈G} then {p_g, g∈G} is a system of orthogonal idempotents of kG*.
The coalgebra structure of kG* is given in the usual way by: Δ(p_g) = Σ_g·g p_g⊗p_g,
ε(p_g) = δ_g·1.

The right comodule structure of D is given by ρ: D→D⊗kG*, ρ(d) =
Σg∈G (g·d)⊗p_g.

In the sequel, the smash coproduct plays a central part. For a bialgebra
H and an H-module coalgebra C the smash-coproduct C⊗H is defined as the
k-space C⊗H with Δ: C⊗H→(C⊗H)⊗(C⊗H) given by Δ(c⊗h) = Σ(c_1⊗c_2)·h_3
⊗(c_2·h_4), and ε: C⊗H→k given by ε(c⊗h) = ε(c)ε_H(h).
1.5. Proposition. \( C \times H \) with \( \Delta \) and \( \varepsilon \) as above is a coalgebra.

Proof. This is just the right hand version of Theorem 2.11 of [M], a proof is given in Proposition 2.3 of [FM]. \( \square \)

The smash coproduct is useful in general but has particular interest in some special cases frequently considered:

i. Graded smash coproduct

If the coalgebra \( C \) is graded by \( G \) then the coalgebra structure of \( C \times kG \) is given by: \( \Delta(c \times g) = \sum (c_i \times \deg c_i \cdot g) \otimes (c_i \otimes g) \), for any homogeneous \( c \in C \) and \( g \in G \) (where we assumed, as we will always do in the sequel, that we have used the homogeneous decomposition \( \sum c_i \otimes c_i \)), whereas for all \( c \in C, g \in G \) we have that \( \varepsilon(c \times g) = \varepsilon_c(c) \).

ii. If the finite group \( G \) acts on the coalgebra \( D \), i.e. \( D \) is a \( kG^\ast \)-comodule coalgebra, then the coalgebra structure of \( D \times kG^\ast \) is given by:

\[
\Delta(d \times p_\delta) = \sum_{u \in G} (d_1 \times p_\delta) \otimes (d_2 \times p_u),
\]

and

\[
\varepsilon(d \times p_\delta) = \varepsilon_p(d) \delta_{k,1}, \quad \text{for all } d \in D, g \in G.
\]

Note that the graded smash coproduct appears in a natural way when one studies graded comodules. Recall that a \( k \)-Abelian category is \( k \)-equivalent to a category of comodules \( \mathcal{M} \) over some coalgebra \( C \) if and only it is of finite type (Theorem 5.1 of [T]). The coalgebra giving the category as a category of comodules may, in general, be a somewhat mystical object. However for a \( G \)-graded coalgebra \( C \) the \( k \)-Abelian category of graded comodules, say \( \text{gr}^C \), is of finite type and it is therefore, equivalent to a category of comodules over the coalgebra given in the following.

Theorem 1.6. If \( C \) is a coalgebra graded by \( G \) then the categories \( \text{gr}^C \) and \( \mathcal{M}^{C \times \mathbb{G}} \) are isomorphic.

Proof. Take \( M \in \text{gr}^C \) with \( \rho : M \to M \otimes C \), \( \rho(m) = \sum m_0 \otimes m_1 \). We make \( M \) into a right \( C \times kG \)-comodule by defining \( \rho' : M \to M \otimes (C \times kG) \), \( m \mapsto \sum m_0 \otimes \sum m_1 \otimes (\deg m)^{-1} \) for homogeneous \( m \in M \). A morphism \( f : M \to N \) of \( G \)-graded \( C \)-comodules is also a morphism of \( C \times kG \)-comodules and we have defined a functor \( T : \text{gr}^C \to \mathcal{M}^{C \times \mathbb{G}} \).

Conversely, starting from an \( M \in \mathcal{M}^{C \times \mathbb{G}} \) we obtain on \( M \) a right \( C \)-comodule
structure and a right $kG$-comodule structure because the linear maps $\alpha : C \times kG \to C$, $c \times g \mapsto c$, and $\beta : C \times kG \to kG$, $c \times g \mapsto c(c)g^{-1}$ for $c \in C$, $g \in G$, are coalgebra morphisms. As in the proof of Proposition 1.2 it follows that $M = \bigoplus_{g \in G} M_g$ and a straightforward verification learns that $M$ becomes a graded $C$-comodule. Now, for $M, N \in \mathcal{M}_{G}^{C \ltimes kG}$ and a morphism of $C \times kG$-comodules $f : M \to N$ it follows that $f$ is also a morphism of $G$-graded $C$-comodules when $M$ and $N$ are viewed as such. This defines the functors $S : \mathcal{M}_{G}^{C \ltimes kG} \to \mathcal{G}$ and it is easily seen that $T$ and $S$ are isomorphisms of categories and inverse to each other.

1.7. Remarks. 1. If the coalgebra $C$ is graded by a finite group $G$, then the dual algebra $C^*$ is graded by $G$ with $C^*_x = \{ f \in C^*, f(C) = 0 \text{ for all } x \neq g \}$. Hence $C^*$ is a $kG^*$-module algebra and we may construct the smash product $C^* \# kG^*$ with multiplication given by $(c^* \# h^*)(d^* \# g^*) = \sum (c^*(d^* \cdot h^*)) \# g^* h^*_g$, for all $c^*, d^* \in C^*$ and $h^*, g^* \in kG^*$. It is easy to see that the algebra $C^* \# kG^*$ is algebra-isomorphic to the dual algebra of $C \times kG$.

2. If $G$ acts on the coalgebra $D$ via $\varphi : G \to \text{Aut}(D)$, then the group morphism $\tilde{\varphi} : G \to \text{Aut}(D^*)$ given by $\tilde{\varphi}(g)(d^*) = d^* \varphi(g)$ for $g \in G$, $d^* \in D^*$, defines an action of $G$ on the algebra $D^*$. Note that $\text{Aut}(D^*)$ is a group with respect to $\sigma \cdot \tau = \tau \cdot \sigma$ for $\sigma, \tau \in \text{Aut}(D^*)$. Thus $D$ is a $kG$-module coalgebra and $D^*$ is a $kG$-module algebra. If $G$ is finite then $D$ is a $kG^*$-comodule coalgebra and the dual algebra of the smash product $D \times kG^*$ is isomorphic to the skew group ring $D^* \# kG$.

2. The Morita-Takeuchi Context Associated to a Graded Coalgebra

The Morita-theorems for categories of comodules have been proved by M. Takeuchi in [T]; we call a set of pre-equivalence data as in [T] a Morita-Takeuchi context.

2.1. Definition. A Morita-Takeuchi context $(C, D, cP_D, \theta Q_C, f, g)$ consists of coalgebras $C$ and $D$, bicomodules $cP_D, \theta Q_C$ and bicolinear maps $f : C \to P \square_D Q$, $g : D \to Q \square_C P$ making the following diagrams commute:

$$
\begin{array}{ccc}
\begin{array}{c}
P \\
\downarrow \cong
\end{array} & \cong & \begin{array}{c}
P \square_D D \\
\downarrow \cong \quad I \square g
\end{array} & \cong & \begin{array}{c}
Q \\
\downarrow \cong \quad I \square f
\end{array} \\
\begin{array}{c}
C \square_C P \\
\downarrow f \square I
\end{array} & \cong & \begin{array}{c}
P \square_D Q \square_C P \\
\downarrow g \square I
\end{array} & \cong & D \square_D Q \square_C P
\end{array}
$$

The context is called strict if $f$ and $g$ are injective, hence isomorphisms. In
this case the categories $\mathcal{M}^C$ and $\mathcal{M}^D$ of comodules over $C$, resp. $D$, are equivalent categories.

The following remark extends a corresponding one for Morita contexts given in [CRW].

2.2. Proposition. Let $(C, D, cP_d, \rho Q_c, f, g)$ be a Morita-Takeuchi context such that $f$ is injective. Then $\mathcal{M}^C$ is equivalent to a quotient category of $\mathcal{M}^D$.

Proof. Theorem 2.5 of [T] yields that $f$ is an isomorphism and the exact functor $S = -\square_dQ : \mathcal{M}^D \rightarrow \mathcal{M}^C$, has a right adjoint $T = -\square_cP : \mathcal{M}^C \rightarrow \mathcal{M}^D$ such that the natural transformation $f^{-1} : ST \rightarrow Id$ is an isomorphism. By a result of P. Gabriel (cf. [G] or Proposition 15.18 of [F]) we have: $\ker S = \{X \in \mathcal{M}^D, X \square_dQ = 0\}$ is a localizing subcategory of $\mathcal{M}^D$ and $S$ induces an equivalence from the quotient category $\mathcal{M}^C / \ker S$ to $\mathcal{M}^C$.

2.3. Corollary. Let $(C, D, cP_d, \rho Q_c, f, g)$ be a Morita-Takeuchi context such that $f$ is injective then $g$ is injective (i.e. the context is strict) if and only if $\rho Q$ is faithfully coflat.

Proof. By Proposition 2.2 the injectivity of $g$ is equivalent to $S$ being an equivalence, again equivalent to $\ker S = \{0\}$ or $\rho Q$ being faithfully coflat.

Before establishing the main result of this section let us point out that there is a natural way to associate a graded coalgebra to a given Morita-Takeuchi context. Indeed, if we have a Morita-Takeuchi context $(C, D, cP_d, \rho Q_c, f, g)$ let $x \mapsto \Sigma x_{-1} \otimes x_0$, resp. $x \mapsto \Sigma x_{(0)} \otimes x_{(1)}$, be the left, resp. right, comodule structure of $P$, resp. $Q$. The image of $u \in C$ (resp. $D$) under $f$ (resp. $g$) in $P \square_dQ$ (resp. $Q \square_cP$) will be denoted by $\Sigma f(u)_i \otimes f(u)_2$, (resp. $\Sigma g(u)_1 \otimes g(u)_2$).

Put $\Gamma = \begin{pmatrix} C & P \\ Q & D \end{pmatrix} = \{(c, p, d, q) \in C \times D \times P \times Q \}.$

We make $\Gamma$ into a coalgebra by defining $\Delta : \Gamma \rightarrow \Gamma \otimes \Gamma$ as follows:

$$
\Delta^{(c, 0, d, 0)} = \Sigma^{(c_1, 0, d_1, 0)} \otimes \Sigma^{(c_2, 0, d_2, 0)} + \Sigma^{(f(c), 0, d, 0)} \otimes \Sigma^{(0, 0, 0, 0)} \\
\Delta^{(0, 0, d, 0)} = \Sigma^{(0, f(d), 0, 0)} \otimes \Sigma^{(0, g(d), 0, 0)} + \Sigma^{(0, 0, 0, 0)} \otimes \Sigma^{(0, 0, 0, 0)} \\
\Delta^{(0, p, 0, 0)} = \Sigma^{(p_{-1}, 0, 0, 0)} \otimes \Sigma^{(p_0, 0, 0, 0)} + \Sigma^{(0, 0, 0, 0)} \otimes \Sigma^{(0, 0, 0, 0)} \\
\Delta^{(0, 0, 0, q)} = \Sigma^{(0, q_{-1}, 0, 0)} \otimes \Sigma^{(0, q_0, 0, 0)} + \Sigma^{(0, 0, 0, 0)} \otimes \Sigma^{(0, 0, 0, 0)}
$$
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for \( c \in C, d \in D, p \in P, q \in Q, \) and extended linearly, \( \varepsilon: \Gamma \to k \) given by \( \varepsilon^{(c, p)}_{q, d} = \varepsilon_c(c) + s_p(d) \). Moreover \( \Gamma \) is \( Z \)-graded by putting \( \Gamma_0 = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}, \Gamma_0 = \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix} \) and \( \Gamma_1 = \begin{pmatrix} 0 & P \\ 0 & 0 \end{pmatrix}, \Gamma_k = 0 \) for \( k = -1, 0, 1 \).

Let \( C = \oplus_{c \in G} C_c \) be a coalgebra, graded by \( G \). Recall from [NT] that \( C_1 \) is a coalgebra with comultiplication \( \Delta_1: C_1 \to C_1 \otimes C_1 \) given by \( \Delta_1(c) = \sum \pi(c_1) \otimes \pi(c_2) = \sum \pi(c_1) \otimes c_2 = \sum c_1 \otimes \pi(c_2) \) for all \( c \in C_1 \), where \( \pi: C \to C_1 \) is the natural projection. The co-unit of \( C_1 \) is just \( \varepsilon_c \) restricted to \( C_1 \). Since \( \pi \) is a coalgebra map, \( C \) becomes a right \( C_1 \)-comodule via the structure map \( \rho_1^1: C \to C \otimes C_1, c \mapsto \sum c_1 \otimes \pi(c_2) + (c \text{ homogeneous}) \) and it becomes a right \( C_1 \)-comodule via \( \rho_1^2: C_1 \to C \otimes C_1, c \mapsto \sum c_1 \otimes \pi(c_2) + (c \text{ homogeneous}) \). Now \( C \) is a graded right \( C \)-comodule, so by Theorem 1.6 \( C \) is a right \( C \times kG \)-comodule via the map

\[
\rho_1^1: C \to C \otimes (C \times kG), \quad c \mapsto \sum c_1 \otimes (c \times \deg c)^{-1}
\]

for \( c \) homogeneous. For any homogeneous \( c \in C \), we have \( (I \otimes \rho_1^1) \rho_1^1(c) = (\rho_1^1 \otimes I) \rho_1^1(c) = \sum \pi(c_1) \otimes c_2 \otimes (c \times \deg c)^{-1} \); thus \( C \) becomes a left \( C_1 \), right \( C \times kG \)-bicomodule. In a similar way \( C \) becomes a left \( C \times kG \), right \( C_1 \)-bicomodule where the left \( C \times kG \)-comodule-structure of \( C \) is given by \( \rho_1^1(c) = \sum (c_1 \times \deg c_2) \otimes c_3 \), for any homogeneous \( c \in C \).

Define \( f: C_1 \to C \square_{C \times kG} C, c \mapsto \sum c_1 \otimes c_2 = \Delta_C(c) \). Observe that for any \( c \in C_1 \) we obtain:

\[
\sum \rho_1^1(c_1) \otimes c_3 = \sum c_1 \otimes c_3 (\deg c_3)^{-1} (\deg c_1)^{-1} \otimes c_3
\]

so the definition of \( f \) above is satisfactory. Moreover, \( f \) is a morphism of left and right \( C_1 \)-comodules as is easily verified. Note also that \( f \) is injective because it is the restriction of the comultiplication of \( C \) to \( C_1 \).

Next define \( g: C \times kG \to C \square_{C_1} C, c \times x \mapsto \sum c_1 \otimes \pi_{x^{-1}}(c_2) \) for \( x \in G \) and homogeneous \( c \in C \), where \( \pi_x \) denotes the projection from \( C \) to \( C_x \). In order to have that \( g \) is well-defined it is necessary that: \( \sum \pi(c_1) \otimes \pi_{x^{-1}}(c_2) = \sum \pi_{x^{-1}}(c_1) \otimes \pi_{x^{-1}}(c_2) \). However the left hand side is obtained from \( \sum c_1 \otimes c_2 \otimes c_3 \) by collecting the terms with \( \deg c_3 = 1 \) and \( \deg c_3 = x^{-1} \); on the other hand the right hand sum is an expression of the same thing. Moreover \( g \) is a morphism of right (and left) \( C \times kG \)-comodules; this follows from: \( \Sigma_{\deg c_3 = x^{-1}} c_1 \otimes c_2 \otimes c_3 \)
(c_2)_1 \otimes (c_2)_2 \otimes x = \sum_{\text{deg} c_2 = x^{-1} \text{deg} c_2} (c_1) \otimes (c_1)_3 \otimes (c_2) \otimes x) \text{ because both members are actually equal to: } \sum_{\text{deg} c_2 = x} (c_1 \otimes c_2) \otimes (c_2) \otimes x). \text{ The other assertion (left) follows in a similar way.}

2.4. Theorem. With notation as above: \((C_1, C \times kG, c_1, C \otimes kG, C \otimes C_1, f, g)\) is a Morita-Takeuchi context. The map \(f\) is injective hence an isomorphism.

Proof. The only thing left to be proved is that \(f\) and \(g\) do satisfy the compatibility conditions, i.e. the following diagrams are commutative:

\[
\begin{array}{ccc}
C & \xrightarrow{\theta} & C \square C \times kG \\
\downarrow \phi & & \downarrow \phi' \\
C_1 \square C_1 & \xrightarrow{g} & C \square C \times kG
\end{array}
\]

Now for \(c \in C_1\) we have: \((I \square g)\theta(c) = (I \square g)(\sum c_1 \otimes (c_2) \otimes x^{-1}))) = \sum c_1 \otimes c_2 \otimes \pi_x(c_3)) = \sum_{\text{deg} c_2 = x} c_1 \otimes c_2 \otimes c_3\), and also \((f \square I)\phi(c) = (f \square I)(\sum \pi(c_1) \otimes c_3)) = (f \square I)(\sum_{\text{deg} c_1 = 1} c_1 \otimes c_2) = (f \square I)(\sum_{\text{deg} c_2 = x} c_1 \otimes c_2)\).

That proves commutativity of the first diagram. For the second diagram we just compute: \((I \square f)\theta'(c) = (I \square f)(\sum c_1 \otimes \pi(c_2)) = (I \square f)(\sum_{\text{deg} c_1 = 1} c_1 \otimes c_2) = (I \square f)(\sum_{\text{deg} c_2 = x} c_1 \otimes c_2)\).

2.5. Corollary. If \(C = \bigoplus_{u \in G} C_u\) is a graded coalgebra then \(\mathcal{M}^C\) is equivalent to a quotient category of \(gr^C\).

Proof. A consequence of Theorem 1.6, Theorem 2.4 and Proposition 2.2.

Recall that a \(G\)-graded coalgebra \(C = \bigoplus_{u \in G} C_u\) is said to be strongly graded if the canonical \(k\)-linear map \(\gamma_{u,v}: C_{u \cdot v} \otimes C_v \to C_{u} \otimes \sum \pi_u(c_1) \otimes \pi_v(c_2),\) is injective for all \(u, v \in G\) (see [NT]). The next result establishes that strongly graded coalgebras may be characterized using the Morita-Takeuchi context from Theorem 2.4 just like in the case of group-graded rings (see [CM]).

2.6. Corollary. Let \(C = \bigoplus_{u \in G} C_u\) be a \(G\)-graded coalgebra, then the following assertions are equivalent:

1. \(C\) is strongly \(G\)-graded
2. The context given in Theorem 2.4 is strict
3. \(C\) is faithfully coflat as a left \(C \times kG\)-comodule.
Proof. 2.⇒1. Take \( u, v \in G \) and \( c \in C_{uv} \) such that we have: \( g_u \cdot (c) = \sum \pi_u(c_1) \otimes \pi_u(c_2) = 0 \). Then \( g(c \times v^{-1}) = \sum c_1 \otimes \pi(c_2) = \sum \pi_u(c_1) \otimes \pi_u(c_2) = 0 \), hence \( c \times v^{-1} = 0 \) and \( c = 0 \).

1.⇒2. Let \( \alpha = \sum c_i \times x_i \in C \times kG \) with \( c_i \) homogeneous of degree \( \sigma_i \). Suppose that for \( i \neq j \) we have \( (\sigma_i, x_i) \neq (\sigma_j, x_j) \). If \( g(\alpha) = 0 \) then \( \sum_i c_i \otimes \pi_{\sigma_i^{-1}}((c_i)_2) = 0 \), therefore \( \sum_i c_i \otimes \pi_{\sigma_i^{-1}}((c_i)_2) = 0 \). On the other hand: \( \pi_{\sigma_i^{-1}}((c_i)_2) \otimes \pi_{\sigma_i^{-1}}((c_i)_2) = C_{\sigma_i} \otimes C_{\sigma_i} \). Since \( C \otimes C = \bigoplus_{u, v \in G} C_u \otimes C_v \) we obtain for fixed \( i \), the relation: \( \sum_i c_i \otimes \pi_{\sigma_i^{-1}}((c_i)_2) \otimes \pi_{\sigma_i^{-1}}((c_i)_2) = 0 \). The latter yields \( g_{\sigma_i, x_i} \otimes \pi_{\sigma_i^{-1}}((c_i)_2) = 0 \) and therefore \( c_i = 0 \) for every choice of \( i \), i.e. \( \alpha = 0 \) follows.

2.⇔3. Follows from Corollary 2.3.

As a further application we reobtain Theorem 5.3 of [NT] which is a co-algebra version of a well-known result of E. Dade.

2.7. Corollary. The graded coalgebra \( C \) is strongly graded if and only if the induced functor \( - \otimes C : \mathcal{H}^f \rightarrow \text{gr} \) is an equivalence of categories.

2.8. Remark. The functor \( (-)_1 : \text{gr} \rightarrow \mathcal{H}^f \), \( M \rightarrow M_1 \), is naturally isomorphic to the functor \( - \otimes_{\mathcal{C} \times G} \) since they are both left adjoints of the induced functor \( - \otimes C \) (see [NT] Proposition 4.1, [T] Remark 2.4). Therefore the localizing category implicit in Corollary 2.5 is just \( \text{Ker}(-)_1 = \text{Ker}(- \otimes_{\mathcal{C} \times G} C) \).

As a final application of these techniques let us include a short proof of Corollary 6.4 in [NT].

2.9. Corollary. If \( C \) is a strongly graded coalgebra for the group \( G \) then \( G \) is a finite group.

Proof. If \( G \) is infinite we could select a non-zero homogeneous \( c \in C \) and \( x \in G \) such that \( x \neq \text{deg}(c) \) for all \( c \). Then \( g(c \times x) = 0 \), but that would contradict injectivity of \( g \).

3. Duality.

For a quasi-finite right \( C \)-comodule \( M \), the so-called coalgebra of "co-endomorphisms" of \( M \) has been defined in [T., 1.17] and it is denoted by \( e_c(M) \). Unfortunately this coalgebra is not easy to use because of the rather complex comultiplication, so it will be useful to give a nicer description of \( e_c(M) \) in some particular situation, e.g. in case \( M \) is a finitely cogenerated free-comodule (that is, \( M \cong X \otimes C \), for some finite dimensional \( k \)-vectorspace \( X \), with the obvious
comodule structure).

Let $C$ be a coalgebra, $X$ an $n$-dimensional $k$-space with basis $\{x_1, \ldots, x_n\}$. Consider the $n \times n$ comatrix coalgebra $M^e(n, k)$ which is a $k$-space with basis $\{x_{ij}, 1 \leq i, j \leq n\}$ and $\Delta, \varepsilon$ given as follows: $\Delta(x_{ij}) = \sum_{p} x_{ip} \otimes x_{pj}$, $\varepsilon(x_{ij}) = \delta_{ij}$.

The $n \times n$ comatrix coalgebras over $C$, denoted by $M^e(n, C)$ is defined to be the tensor product of coalgebra $C \otimes M^e(n, k)$. We endow $C \otimes X$ with a left $C$- and a right $M^e(n, C)$-bicomodule structure as follows. The left $C$-comodule structure is given by by the map: $\rho_1 : C \otimes X \to C \otimes C \otimes X$, $c \otimes x \mapsto \sum c_i \otimes c_j \otimes x_i$.

The right $M^e(n, C)$-comodule structure is given by the map: $\rho_1^T : C \otimes X \to C \otimes X \otimes M^e(n, C)$, $c \otimes x_i \mapsto \sum p c_i \otimes c_j \otimes x_j \otimes x_{pi}$.

In a similar way $C \otimes X$ is a left $M^e(n, C)$-right $C$-bicomodule via the structure maps:

$$
\rho_1^T : C \otimes X \to C \otimes M^e(n, C) \otimes C \otimes X,
$$

$$
\rho_1 : C \otimes X \to M^e(n, C) \otimes C \otimes X,
$$

Define $f : C \to (C \otimes X) \otimes M^e(n, C)(C \otimes X)$, $c \mapsto \sum_{i, (c_i) (c_i x_i) \otimes (c_j x_j)}$, which is obviously injective and $C$-bilinear. Define $g : M^e(n, C) \to (C \otimes X) \otimes C(C \otimes X)$, $c \otimes x_i \mapsto \sum (c_i \otimes x_i) \otimes (c_j \otimes x_j)$ which is also injective and $M^e(n, C)$-bilinear. One easily verifies the following relations:

$$(I \otimes f)\rho_1^T(c \otimes x_i) = (g \otimes I)\rho_1^T(c \otimes x_i) = \sum_p c_i \otimes c_j \otimes x_p \otimes c_i \otimes x_p$$

$$(f \otimes I)\rho_1^T(c \otimes x_i) = (I \otimes g)\rho_1^T(c \otimes x_i) = \sum_p c_i \otimes c_j \otimes x_p \otimes c_j \otimes x_i$$

According to results of [T] we immediately obtain:

3.1. Proposition. $(C, M^e(n, C), C \otimes X, C \otimes X, f, g)$ is a strict Morita-Takeuchi context. In particular we have coalgebra isomorphisms:

$$e_C(C \otimes X) \cong M^e(n, C) \cong e_C(C \otimes X)$$

3.2. Theorem. Let $G$ be a finite group acting on the coalgebra $D$, then $D \rtimes kG^*$ is a strongly graded coalgebra and there exist coalgebra isomorphisms:

$$D \rtimes kG^* \rtimes kG \cong e_{D,-}(D \rtimes kG^*) \cong M^e(n, D)$$

where $n = |G|$.

Proof. The map $\rho : D \otimes kG^*$, $d \mapsto \sum_{x} (g \cdot d) \otimes p_u$, makes $D$ into a $kG^*$-comodule. The comultiplication of $D \rtimes kG^*$ is given by $\Delta(d \rtimes p) = \sum_{u \in x} (d \rtimes p_u) \otimes \nu_{d_u} \rtimes p_u$. This establishes that $D \rtimes kG^*$ is a graded coalgebra of type $G$ with grading given by $(D \rtimes kG^*)_z = D \rtimes p_{z-1}$. The canonical morphism $D \rtimes p_1 \to$
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$(D \times p_{\sigma^{-1}}) \otimes (D \times p_{\alpha})$, $d \times p_1 \rightarrow \sum (d_1 \times p_{p_{\sigma^{-1}}}) \otimes (\sigma^{-1} d_2 \times p_{\alpha})$, is clearly injective. Thus $D \times kG^*$ is a strongly graded coalgebra, and $(D \times kG^*)_1 = D \times p_1 \simeq D$. Applying the Morita-Takeuchi context (constructed in Section 2) to $D \times kG^*$, we have a strict context and so it provides us with coalgebra isomorphisms:

$\left(D \times kG^* \right) \otimes kG \cong e_{(D \times p_{\sigma^{-1}})} \otimes (D \times kG^*) \cong e_{D \times (D \times kG^*)}$.

The left $(D \times p_1)$-structure of $D \times kG^*$ is given by $d \times p_2 \rightarrow \sum (d_1 \times p_{\sigma^{-1}}) \otimes (d_2 \times p_\alpha)$, and this yields exactly the left $D$-comodule structure of $D \otimes X$ where $X = kG^*$ is a $k$-space of dimension $n$. Proposition 3.1 yields the second isomorphism. □

A similar result holds for graded coalgebras (or coactions).

3.3. Theorem. Let $C$ be a coalgebra graded by the finite group $G$. Then $G$ acts on the coalgebra $C \times kG$ and there are coalgebra isomorphisms:

$\left(C \times kG \right) \otimes kG^* \cong e_{C \otimes (C \times kG^*)} \cong M^e(n, C)$

Proof. An action of $G$ on the coalgebra $C \times kG$ is given by $h \cdot (c \times g) = c \times gh^{-1}$, $g, h \in G$ and $c \in C$. Thus $C \times kG$ becomes a $kG^*$-comodule coalgebra via the map:

$c \times g \rightarrow \sum y \cdot (c \times g) \otimes p_y = \sum (c \times g y^{-1}) \otimes p_y$.

The comultiplication of $(C \times kG) \otimes kG^*$ is given by

$\Delta((c \times x) \otimes p_g) = \sum u \otimes (c_1 \otimes x \otimes c_2 \times p_u) \otimes (c_2 \otimes x \otimes p_u)$

for any $x, g \in G$ and homogeneous $c \in C$. Now let $\{e_x, y, x, y \in G\}$ be a basis for $M^e(n, k)$. Define a map $F : (C \times kG) \otimes kG^* \rightarrow M^e(n, C)$, $(c \times x) \otimes p_g \mapsto c \otimes e_{\alpha, \beta}$ where $\alpha = \deg c \times x$, $\beta = xg^{-1}$ for $x, g \in G$ and homogeneous $c \in C$. Let us check that $F$ is a coalgebra morphism. Indeed,

$\Delta(F((c \times x) \otimes p_g)) = \Delta(c \otimes e_{\alpha, \beta})$

$= \sum (c_1 \otimes e_{\alpha, \beta}) \otimes (c_2 \otimes e_{\alpha, \beta})$

and also

$(F \otimes F) \Delta((c \times x) \otimes p_g) = \sum u \otimes (c_1 \otimes c_2 \otimes c_2 \otimes x \otimes v^{-1}) \otimes (c_3 \otimes c_4 \otimes x \otimes v^{-1}, u^{-1}, v^{-1})$

$= \sum (c_1 \otimes c_2 \otimes c_2 \otimes c_2 \otimes x \otimes v^{-1}, u^{-1}, v^{-1}, \beta)$.

Since $\{\deg c_2 \otimes x \otimes v^{-1}, v \in G\} = G$, both sums are equal. Now, consider $(c \times x) \otimes p_g \in (C \times kG) \otimes kG^*$ for $x, g \in G$ and $c$ homogeneous. Write $\varepsilon$ for the co-unit of $(C \times kG) \otimes kG^*$ and $\varepsilon'$ for the co-unit of $M^e(n, C)$. Then we have:
Therefore $F$ is a coalgebra map as claimed. Now define $H: M^e(n, C) \rtimes (C \times kG) \rtimes kG^*$ by putting $H(c(g)cu,v) = (c \otimes (\deg c)^{-1}u) \rtimes p_{v^{-1}(\deg c)^{-1}u}$, for $u, v \in G$ and homogeneous $c \in C$. Again $H$ is a coalgebra morphism because:

\[
\Delta(H(c \otimes e_{u,v})) = \sum_{z \in v^{-1}(\deg c)^{-1}u} ((c \otimes \deg c \otimes \deg c)^{-1}u) \rtimes p_z \otimes ((c \otimes \deg c)^{-1}u) \rtimes p_z
\]

For fixed $c$, and $u$ we have that $\{ h^{-1}(\deg c)^{-1}u, h \in G \} = G$ and if we write $t = h^{-1}(\deg c)^{-1}u$, $z = v^{-1}(\deg c)^{-1}h$, then the above sums are clearly equal as desired. The fact that $H$ preserves the co-unit too is obvious. Finally it is clear that $F \cdot H$ and $H \cdot F$ are the identities so that we do arrive at a coalgebra isomorphism. The isomorphism involving $e_{c_1}(C \rtimes kG)$ is obvious because of Proposition 3.1 (the left $C$-comodule structure of $C \times kG$ is given by $c \otimes g \mapsto \sum c_1 \otimes (c_2 \otimes g)$).

3.4. COROLLARY. There exists a strict Morita-Tekeuchi context connecting $C$ and $(C \times kG) \rtimes kG^*$.

PROOF. $C \times kG$ is a left $C$-comodule that is a quasi-finite injective cogenerator (in view of Proposition 3.1 and [T]). Moreover $C \times kG$ is a right $(C \times kG) \rtimes kG^*$-comodule via $c \times g \mapsto \sum u(c_1 \otimes \deg c_2gu) \otimes (c_3 \otimes gu) \rtimes p_{v^{-1}}$, for $g \in G$ and homogeneous $c \in C$. Hence $C \times kG$ is a $C - (C \times kG) \rtimes kG^*$-bicomodule. The assertion now follows from [T, Theorem 3.5 iv].

3.5. REMARKS. The Morita-Tekeuchi context of the above corollary may be given in detail. This may have an independent interest because it provides another proof of Theorem 3.3 and provides a hint for establishing a more general duality result we do not dwell upon here. The second bicomodule is also $C \times kG$ with right $C$-comodule structure given by the map: $c \otimes g \mapsto \sum (c_1 \otimes \deg c_2g) \otimes c_3$ (for homogeneous $c$) and left $(C \times kG) \rtimes kG^*$-comodule struc-
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ture given by: $c \times g \rightarrow \sum \gamma (c_1 \times \deg c_2 g) \rtimes p_h \otimes (c_3 \times g h)$ (for homogeneous $c$) we have $f : (C \times k G) \square (C \times k G, k G) \rightarrow \sum \gamma (c_1 \times \deg c_2 h) \otimes (c_3 \times h_3)$ for homogeneous $c \in C$, $g : (C \times k G) \times k G \rightarrow (C \times k G) \square (C \times k G)$, $g((c \times g) \rtimes p_h) = \sum (c_1 \times \deg c_2 g) \otimes (c_3 \times g h)$, for homogeneous $c \in C$. It is also easily seen that $f$ and $g$ are injective maps.

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Added in proof. A general duality result for crossed coproducts was proved by S. Dăscălescu, S. Raianu, Y. Zhang in “Finite Hopf-Galois coextensions, crossed coproducts and duality”, to appear in J. Algelm.