ON CONDUCTOR OVERRINGS OF A VALUATION DOMAIN

By

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Introduction. It is well known that every overring of a valuation domain $V$ is of the form $V_P$ for some prime ideal $P$ of $V$. Hence, if $I$ is an ideal of a valuation domain $V$ with quotient field $K$, then the conductor overring $I: KI$ is of the form $V_P$ for some prime ideal $P$ of $V$. In case $I: KI = V_P$, is there any relation between $I$ and $P$? The main purpose of this paper is to investigate this relation.

In order to give a complete answer to the question stated above, we introduce the notion of "recurrent closure": If $I$ is an ideal of an integral domain $R$ with quotient field $K$, then the ideal $R:x(I:xI)$ of $R$ is called the "recurrent closure" of $I$ and is denoted by $l_x$. We prove, in Theorem 13, that if $I$ is an ideal of a valuation domain $V$ with quotient field $K$ such that $I: KI \neq V$, then $l_x$ is always a prime ideal of $V$ and if we set $I: KI = V_P$ for some prime ideal $P$ of $V$, then $P$ is equal to the recurrent closure $l_x$.

In general, our terminology and notation will be the same as [3] and [6]. Throughout the paper, $V$ denotes a valuation domain, with quotient field $K$.

**Theorem 1.** If $P$ is a proper prime ideal of $V$, then $P: KP = V_P$. In particular, if $M$ is the unique maximal ideal of $V$, then $M: KM = V$.

**Proof.** If $P=(0)$, then $(0): K(0)=K=V_{(0)}$ (cf. [9, Remark 1.2]) and hence our assertion is trivial. Thus we may assume that $P \neq (0)$. Then, by [3, Theorem 17.3], $P(x)=P$ for any $x \in V\setminus P$ and accordingly $1/xP \subseteq P$. Thus $1/x \in P: KP$ for any $x \in V\setminus P$. From this fact it follows that $V_P \subseteq P: KP$. Hence, if we put $P: KP = V_Q$ for some prime ideal $Q$ of $V$, then we have $V_P \subseteq P: KP = V_Q$ and so $Q \subseteq P$. Assume now that $Q \neq P$. Then $Q: KP$ is a nonmaximal prime ideal of $P: KP$ by [9, Corollary 2.4]. On the other hand, $Q=QV_Q$ is a maximal ideal of $V_Q$ by [3, Theorem 17.6]. Since $Q \subseteq Q: KP$, we have $Q=Q: KP$ and therefore $Q: KP$ is a maximal ideal of $P: KP$, a contradiction. Hence we must have $Q=P$, and accordingly $P: KP = V_P$ as desired. Thus our first assertion is proved. The second assertion follows immediately from the first one.

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Before proving the next theorem, we first establish the following lemma.

**Lemma 2.** Let $R$ be an integral domain with quotient field $K$ and let $I$ be a proper ideal of $R$. If $I:_{K}I=RP$ for some prime ideal $P$ of $R$, then we have $I\subseteq P$.

**Proof.** Assume the contrary. Then we can choose an element $a\in I\setminus P$. Then, by hypothesis, $1/a\in RP=I:_{K}I$ since $a\not\in P$. Therefore we have $1=a\cdot 1/a\in I:_{K}I\subseteq I$, which implies that $I=\emptyset$. This clearly contradicts our assumption.

**Theorem 3.** If $Q$ is a primary ideal of $V$, then $Q:\kappa Q=V,\sqrt{Q}$.

**Proof.** If $Q=(0)$, then $(0):_{K}(0)=K=V,\sqrt{Q}$ and hence our assertion is evident. Therefore we may assume that $Q\neq (0)$. If we set $Q:\kappa Q=VP$ with some prime ideal $P$ of $V$, then $Q\subseteq P$ and hence $\sqrt{Q}\subseteq P$. We shall next show that $P\subseteq \sqrt{Q}$. By [3, Theorem 17.3], $Q(x)=Q$ for any element $x\in V,\sqrt{Q}$, and accordingly $1/a\in Q:_{\kappa}Q$ for any $a\in V,\sqrt{Q}$. Thus we have $V,\sqrt{Q}\subseteq Q:_{\kappa}Q=V,\kappa$ and hence $P\subseteq \sqrt{Q}$, as required. This completes the proof.

**Corollary 4.** If $Q$ is a primary ideal of $V$, then $Q:_{\kappa}Q=\sqrt{Q},\sqrt{Q}$.

**Proof.** This follows immediately from Theorem 1 and Theorem 3.

**Definition 5.** Let $R$ be an integral domain with quotient field $K$ and let $I$ be a proper ideal of $R$. Then the ideal $R:_{I}I$ of $R$ is called the "recurrent closure" of $I$ and is denoted by $I_r$. An ideal $I$ of $R$ is said to be "recurrent" in case $I=I_r$.

**Remark 6.** If $I$ is a recurrent ideal of an integral domain $R$ with quotient field $K$, then $I:_{K}I=\emptyset$. For, if $I:_{K}I=R$, then $I=I_r=R:_{R}I=\emptyset:_{R}R=R$, a contradiction. Moreover, if $M$ is a maximal ideal of $R$, then the converse of the above statement also holds. In fact, if $M:_{K}M=\emptyset$, then $M\subseteq R:_{M}M=\emptyset$ and hence $M=\emptyset:_{M}M$, since $M$ is a maximal ideal of $R$. Therefore $M$ is a recurrent ideal of $R$ as required.

**Remark 7.** If $M$ is the unique maximal ideal of $V$, then $M$ is not recurrent. By Theorem 1, $M:_{K}M=V$ and therefore our assertion follows from Remark 6.

We first collect some facts about recurrent ideals that will be needed later.

**Lemma 8.** Let $R$ be an integral domain with quotient field $K$. If $I$ is an ideal of $R$ such that $I:_{K}I=\emptyset$, then $I\subseteq I_r$ and $I$, itself is recurrent.
Proof. By definition the containment $I \subseteq I_r$ is evident. Next, we shall establish the second assertion. First it should be noted that $I_r$ is an ideal of $I : kI$ (cf. [9, Lemma 1.1(2)]). It follows from this fact that if $x \in I : kI$ and $a \in I_r$, then $ax \in I_r$. Thus we have $I : kI \subseteq I_r$. Therefore $I_r = R : k(I : kI) \supseteq R : k(I_r : kI_r) \supseteq I_r$, whence $I_r = R : k(I_r : kI_r) = (I_r)_r$, completing the proof.

**Lemma 9.** Let $R$ be an integral domain with quotient field $K$ and let $I$ be a proper ideal of $R$. Then

1. If $P$ is a prime ideal of $R$ contained in $I$, then $I : kI \subseteq P : kP$.
2. If $I$ is a recurrent ideal of $R$, then, for any prime ideal $P$ of $R$, $P \subseteq I$ if and only if $I : kI \cap P : kP$.

**Proof.** (1) Let $x \in I : kI$ and $p \in P$. Since $x^p \in I : kI$ and $p \in I$, $x^p \in (I : kI) \subseteq I$, and accordingly $(xp)^p = (x^p)p \in I P \subseteq P$, which implies that $xp \in P$ because $x \in R$. Thus $(I : kI)P \subseteq P$ and hence $I : kI \subseteq P : kP$ as required.

(2) The "only if" half is proved in (1). Conversely, assume that $I : kI \subseteq P : kP$. Then $P$ is an ideal of $I : kI$, since $P(I : kI) \subseteq P(P : kP) \subseteq P$. Hence, by [9, Lemma 1.1 (4)], $P \subseteq R : k(I : kI) = I_r$. Then we have $P \subseteq I_r = I$ because $I$ is, by hypothesis, recurrent. This completes the proof.

**Remark 10.** The part (1) of Lemma 9 is also found in [1, Lemma 2.2] or in [2, Lemma 3.7].

**Lemma 11.** Let $R$ be an integral domain with quotient field $K$ and let $I$ be a proper ideal of $R$. If $P$ is a recurrent prime ideal of $R$ properly contained in $I$, then $I : kI \equiv P : kP$.

**Proof.** By part (1) of Lemma 9, we have $I : kI \subseteq P : kP$. Hence, it suffices to show that $I : kI \cap P : kP$. Assume that $I : kI = P : kP$. Then $I$ is an ideal of $P : kP$ and therefore, by [9, Lemma 1.1 (4)], $I \subseteq P_r$. By hypothesis, $P_r = P$ and hence $I \subseteq P$, the desired contradiction. This completes the proof.

In the proof of Lemma 8, we showed that if $I$ is an ideal of an integral domain $R$ with quotient field $K$, then $I : kI \subseteq I_r : kI_r$. If $P$ is a prime ideal of $R$, then it can be shown that $P : kP = P_r : kP_r$.

**Theorem 12.** Let $R$ be an integral domain with quotient field $K$. If $P$ is a prime ideal of $R$, then we have $P : kP = P_r : kP_r$.

**Proof.** We have already shown in Lemma 8 that $P : kP \subseteq P_r : kP_r$. Hence,
we need only prove the reverse containment $P_r : kP_r \subseteq P : kP$. If $P=Pr$, then there is nothing to prove. Therefore we may assume that $P \neq Pr$. If we choose $teP_r \setminus Pr$, then, for any $x \in P_r : kP_r$, we have $xt \in P_r \subseteq R$. Then we have $xtP \in P$ for any $p \in P$. But, since $xP \in P_r : kP_r \subseteq P_r \subseteq R$ and $teR \setminus P, (xp)tP$ implies that $xp \in P$. Thus $P_r : kP_r \subseteq P : kP$ as desired and our proof is complete.

We are now in a position to prove the main theorem of this paper.

**Theorem 13.** Let $V$ be a valuation domain with quotient field $K$. Then

1. Every nonmaximal prime ideal $P$ of $V$ is recurrent.
2. If $I$ is an ideal of $V$ such that $I : kI \neq V$, then $I_r$ is a prime ideal of $V$ and we have $I : kI = V/I_r$.
3. If $I$ is an ideal of $V$ such that $I : kI \neq V$, then $\sqrt{T} \subseteq I_r$.
4. If $Q$ is a primary ideal of $V$ such that $\sqrt{Q}$ is not the unique maximal ideal $M$ of $V$, then $\sqrt{Q} = Q_r$.

**Proof.** (1) First, by Theorem 1, $P : kP = V : V (P : kP) \neq V$. Indeed, if $P = V$ then $1 \in P_r$, and so $P : kP \subseteq V$, a contradiction. Thus we get $P \subseteq P_r \neq V$. Next, by [9, Lemma 1.1 (2)], $P_r$ is an ideal of $P : kP = V_r$ and therefore $P_r \subseteq PV_r = P$. Accordingly, $P = P_r$, which implies that $P$ is recurrent.

(2) By hypothesis, $I : kI$ is a proper overring of $V$ and so we can write $I : kI = V_r$ with some nonmaximal prime ideal $P$ of $V$. Since, by Theorem 1, $V_r = P : kP$, it follows that $I : kI = P : kP$. Then we have $I_r = V : V (I : kI) = V : V (P : kP) = P$, since $P$ is recurrent by (1). Thus, $I_r$ is a prime ideal of $V$ and moreover $I : kI = V/I_r$ as required.

(3) Since $I \subseteq I_r$, we always have $\sqrt{T} \subseteq \sqrt{T_r}$. If $I : kI \neq V$, then, by (2), $I_r$ is prime and therefore $\sqrt{T} \subseteq \sqrt{T_r} = I_r$ as wanted.

(4) First, by Theorem 3, $Q : kQ = V_\sqrt{Q}$. Moreover, $Q : kQ \neq V$, since $\sqrt{Q}$ is not maximal. Hence, by (2), $Q_r$ is prime and $Q : kQ = V_{\sqrt{Q_r}}$. Thus $V_{\sqrt{Q}} = V_{\sqrt{Q_r}}$, and accordingly $\sqrt{Q} = Q_r$, completing the proof.

**Remark 14.** Let $R$ be an integral domain with quotient field $K$ and let $P \subseteq I$ be ideals of $R$ with $P$ prime. Then we cannot in general expect that $P$ is also prime in $I : kI$. To show this, we shall give the following example.

**Example 15.** Let $R = \mathbb{Z}[2X, X^*, X^+]$ be the subdomain of $T = \mathbb{Z}[X]$, where $X$ is an indeterminate over $\mathbb{Z}$. Then $K = \mathbb{Q}(X)$ is the quotient field of $R$. If we set $M = 2\mathbb{Z}R + 2XR + X^*R + X^+R$, then $R/M = \mathbb{Z}/2\mathbb{Z}$ is a field and so $M$ is a maximal ideal of $R$. Moreover, it is easy to see that $M : kM = \mathbb{Z}[X]$. If we put $P = 2XR$
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+ $X^2R + X^3R$, then, since $R/P = \mathbb{Z}$, $P$ is a prime ideal of $R$ properly contained in $M$. But $P$ is not a prime ideal of $M:K$, because $3X \in \mathbb{Z}[X] \setminus P$, but $(3X)^3 \in P$.

**Corollary 16.** If $P \subset I$ are ideals of $V$ with $P$ prime, then $P$ is also prime in $I:K$ and $P = P:K$.

**Proof.** If $I:K = V$, then there is nothing to prove. Hence we may assume that $I:K \neq V$. Then, by Theorem 13 (2), $I:K = V_r$, and $I_r$ is a prime ideal of $V$. Hence, by [3, Theorem 17.6 (b)], $P = PV_r$ is a prime ideal of $V_r$, since $P \subset I \subseteq I_r$. Thus, $P$ is a prime ideal of $I:K$. Our second assertion follows then from [9, Corollary 1.5].

We close this paper with a characterization of primary ideals $Q$ of $V$ such that $Q:K \neq V$.

We first prepare the following two lemmas.

**Lemma 17.** Let $Q$ be a primary ideal of $V$. Then $Q:K \neq V$ if and only if $\sqrt{Q}$ is not the unique maximal ideal of $V$.

**Proof.** Let $M$ be the unique maximal ideal of $V$. First, suppose that $\sqrt{Q} = M$. Then, by Theorem 3, $Q:K = V, \sqrt{Q} = V_M = V$. Thus, the “only if” half is proved. Conversely, suppose that $Q:K = V$. Then, also by Theorem 3, $V = Q:K = V, \sqrt{Q}$, and so $\sqrt{Q} = M$. Hence, the “if” half is also proved.

**Lemma 18.** Let $I$ be a nonzero ideal of an integral domain $R$ with quotient field $K$. Then, for any $x \in I:K$, $x$ is a unit of $I:K$ if and only if $xI = I$.

**Proof.** First, assume that $x$ is a unit of $I:K$. Then there is an element $y \in I:K$ such that $xy = 1$. Then, $I = (xy)I = x(yI) \subseteq xI \subseteq I$, and so $I = xI$, as we required. Conversely, suppose that $I = xI$. Since $I \neq (0)$, $x$ is a nonzero element of $K$, and so $x^{-1} \in K$. Hence, by hypothesis, $x^{-1}I = x^{-1}(xI) = (x^{-1}x)I = I$, and so $x^{-1} \in I:K$, which implies that $x$ is a unit of $I:K$. This completes the proof.

**Theorem 19.** Let $I$ be an ideal of $V$ such that $I:K \neq V$. Then $I$ is a primary ideal of $V$ if and only if $\sqrt{I} = I_r$.

**Proof.** The “only if” half is proved in part (4) of Theorem 13. To prove the “if” half, suppose that $I$ is not a primary ideal of $V$. By part (2) of Theorem 13, $I:K = V_{r_I}$, and therefore, to prove that $\sqrt{I} \neq I_r$, it suffices to show that $I:K \neq V_{r_I}$. Now, since $I$ is not primary, there exist $a, b \in V$ such that $a \notin I, b \notin \sqrt{I}$,
but $ab \notin I$. Then $b \notin \sqrt{I}$ implies that $I \subseteq (b)$, since $V$ is a valuation domain. Then, since $(b)$ is invertible, there exists an ideal $J$ of $V$ such that $I = f(b)$. Therefore, by hypothesis, $ab \notin I = f(b)$, and so $a \notin I$. Since $a \in f \setminus J, I = f(b) \subseteq J$ and therefore $bI = (bI = f(b)I) \subseteq J(b) = J$. Thus, $bI \subseteq I$ and therefore it follows from Lemma 18 that $b$ is not a unit of $I : _K I$. On the other hand, $b$ is a unit of $V_{J_I}$, since $b \notin \sqrt{I}$. Therefore $I : _K I = V_{J_I}$, as we wanted and hence our proof is complete.

**Remark 20.** If $I$ is an ideal of $V$ such that $I : _K I \neq V$, then $\sqrt{I}$ is not maximal in $V$. For, if $\sqrt{I}$ is maximal, then, by part (3) of Theorem 13, $I_r$ is also maximal in $V$ and therefore, by part (2) of Theorem 13, $I : _K I = V_{I_r} = V$, a contradiction.

**Corollary 21.** Let $I$ be an ideal of $V$ such that $I : _K I \neq V$. Then $I$ is recurrent if and only if $I$ is prime.

**Proof.** First, assume that $I$ is prime in $V$. Then it follows from Theorem 1 that $I$ is not maximal in $V$, since $I : _K I \neq V$. Therefore the “if” half follows from part (1) of Theorem 13. Furthermore, the “only if” half follows immediately from part (2) of Theorem 13.

**References**