FIBER SHAPE THEORY

By

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Abstract. In this paper we develop a fiber shape theory for maps between metric spaces. Our approach is based on the Mardesić-Segal method and, instead of ANR's, their fiber preserving analogues are used. A fiber preserving version of Chapman's complement theorem is proved.


Key words and phrases: fiber shape, ANFR, shape fibration, movability, complement theorem.

§ 0. Introduction.

The purpose of this paper is to develop a fiber shape theory for maps between metric spaces. There are several approaches to the fiber shape theory for maps between compact metric spaces ([CM], [Ka₁, x]), which correspond to those to the shape theory ([Bₙ], [Ch₁], [MS]). The description of our fiber shape category is based on the general construction of shape categories in [MS].

In shape theory ([DS], [MS]), the shape of a space is represented by an ANR-system associated with the space. In our setting, the same role will be played by a fiber preserving version of ANR's (cf. [CM]). § 1 contains the definition and some examples of such fibered ANR's.

In § 2 we will give the description of our fiber shape category. It is proved that our approach is particularly useful to treat proper maps and many results in [Ka₁, x] have natural generalizations in our setting. For example, among proper maps, hereditary shape equivalences, shape fibrations or the notion of movability introduced in [Yₙ] are shown to be fiber shape invariant.

In § 3, we will prove a fiber preserving version of Chapman's complement theorem, which gives the fiber shape classification of proper maps over a separable metric base space. The same statement is also found in [CM], where the base space is restricted to ENR's.

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Finally, we will list some notations and conventions used throughout this paper. All spaces are metric spaces and in §3 they are assumed to be separable. ANFR's are ones for the metric spaces ([Hu]). $id_{x}$ denotes the identity map on the space $X$ and $\pi_{B}:B \times M \rightarrow B$, $\pi_{M}:B \times M \rightarrow M$ always denote the projections onto appropriate factors. Given maps $p:X \rightarrow B$ and $q:Y \rightarrow B$, a map $f:X \rightarrow Y$ is said to be fiber preserving (f.p.) if $qf=p$. Similarly a homotopy $f_{t}:X \rightarrow Y$ ($0 \leq t \leq 1$) is f.p. if $qf_{t}=p$ ($0 \leq t \leq 1$). In particular, a map $f:X \rightarrow B \times M$ is f.p. if $\pi_{B}f=p$. A map $p:X \rightarrow B$ is proper if $p^{-1}(K)$ is compact for each compact $K \subset B$.

For a subset $C \subset B$, $p_{C}$ denotes the restriction $p|_{p^{-1}(C)}:p^{-1}(C) \rightarrow C$. Let $CV$ be an open cover of a space $Y$. We say that the maps $f, g:X \rightarrow Y$ are $CV$-near, written $(f, g) \leq CV$, if each $x \in X$ admits a $V \in CV$ with $f(x), g(x) \in V$. A homotopy $F:X \times [0, 1] \rightarrow Y$ is a $CV$-homotopy if for each $x \in X$ there exists a $V \in CV$ with $F(\{x\} \times [0, 1]) \subset V$.

We refer to [DS] and [MS] for shape theory and related topics, and to [CM] and [Ka1,2] for fiber shape theory.

§1. Absolute neighborhood fiber retracts.

In this section we will define an f.p. version of ANR's and prove their elementary properties, which will be used in the next section to define a fiber shape category.

Let $B$ be a fixed space. A map $p:E \rightarrow B$ is said to be an absolute neighborhood fiber retract (ANFR) over $B$ provided for any map $q:X \rightarrow B$ and any f.p. closed embedding $i:E \rightarrow X$, there exist a neighborhood $U$ of $i(E)$ in $X$ and a map $r:U \rightarrow E$ such that $ri=id_{E}$ and $pr=q|_{U}$. In addition, if we can always take $U=X$, then we say $p$ is an absolute fiber retract (AFR) over $B$.

Similarly we may say a map $p:E \rightarrow B$ is an absolute neighborhood fiber extensor (ANFE) over $B$ provided for any map $q:X \rightarrow B$ and any map $f:A \rightarrow E$ from a closed subset $A$ of $X$ with $pf=q|_{A}$, there exists an extension $\bar{f}:U \rightarrow E$ of $f$ to a neighborhood $U$ of $A$ in $X$ with $p\bar{f}=q|_{U}$.

We will list some elementary properties of ANFR's, which are f.p. analogues of ones of ANR's ([Hu]).

1.1. PROPOSITION. (i) ([CM]) Let $M$ be an ANR and $U$ be an open set in $B \times M$. Then the projection $\pi_{B}|_{U}:U \rightarrow B$ is an ANFR. A map $p:E \rightarrow B$ is an ANFR iff $p$ is an f.p. retract of such a projection $\pi_{B}|_{U}$.

(ii) A map $p:E \rightarrow B$ is an ANFR iff $p$ is an ANFE.

(iii) Every f.p. neighborhood retract of an ANFR is also an ANFR.

(iv) (The f.p. homotopy extension property) Suppose $q:E \rightarrow B$ is an ANFR,
\( p: X \rightarrow B \) is a map and \( A \) is a closed subset of \( X \). Then for any \( f.p. \) map \( \phi: X \rightarrow E \) and any \( f.p. \) homotopy \( \phi_t: A \rightarrow E \) such that \( \phi_0 = \phi|_A \) and \( \phi_1 = p|_A \) \((0 \leq t \leq 1)\), there exists an \( f.p. \) homotopy \( \phi_t: X \rightarrow E \) such that \( \phi_0 = \phi \) and \( \phi_1|_A = \phi_1 \) \((0 \leq t \leq 1)\). Furthermore, if \( \phi_t \) is a \( \mathcal{U} \)-homotopy for an open cover \( \mathcal{U} \) of \( E \), then we can take \( \phi_t \) a \( \mathcal{U} \)-homotopy.

**PROOF.** (i)-(iii) follow from the following observations:

(a) \( \pi_B: B \times M \rightarrow B \) is an ANFE. (If \( M \) is an AR, then \( \pi_B \) is an AFR.)

(b) Every \( f.p. \) neighborhood retract of an ANFE is also an ANFE.

(c) Every map \( p: E \rightarrow B \) admits an \( f.p. \) closed embedding \( i: E \rightarrow B \times M \) for some ANR \( M \).

(iv) follows from the same argument as in [Hu, p. 116].

1.2. **COMMENTS AND EXAMPLES.** (i) *Every fiber of an ANFR (AFR) is an ANR (AR).*

(ii) *If an onto map \( p: E \rightarrow B \) is an ANFR, then \( p \) admits local sections (i.e., for each \( b \in B \) and each \( e \in p^{-1}(b) \), there exists a map \( s: V \rightarrow E \) from a neighborhood \( V \) of \( b \) such that \( ps = id_V \) and \( s(b) = e \). In particular, if \( E \) is an ANR then so is \( B \).*

(iii) *If \( p: E \rightarrow B \) is a proper ANFR, then \( p \) is a Hurewicz fibration. Conversely if \( p: E \rightarrow B \) is a Hurewicz fibration between ANR's then \( p \) is an ANFR.*

(iv) \([Fe_1, Y_2]\) *If \( p: E \rightarrow B \) is a proper strongly regular map with ANR fibers and \( \dim B < \infty \), then \( p \) is an ANFR.*

(v) *Every bundle map with ANR fibers is an ANFR.*

**PROOF.** (i) This follows from 1.1 (i). If \( p \) is an AFR, then \( p \) is an \( f.p. \) retract of a projection \( \pi_B: B \times M \rightarrow B \), with \( M \) an AR. Therefore each fiber \( p^{-1}(b) \) is a retract of the AR \( M \).

(ii) By 1.1. (i), \( p \) is an \( f.p. \) retract of some \( \pi_B|_U \) as in 1.1. (i). Since \( \pi_B|_U \) admits local sections, so does \( p \). The second assertion follows from [Hu, p. 98, Theorem 8.1].

(iii) Suppose \( p \) is an ANFR. Embed \( E \) into an ANR \( M \) as a closed subset and consider the \( f.p. \) embedding \( i: E \rightarrow B \times M \), \( i(e) = (p(e), e) \) \((e \in E)\). \((i(E) \) is the graph of \( p \).) By the definition there exists an \( f.p. \) retraction \( r: U \rightarrow E \) from some open neighborhood \( U \) of \( i(E) \). Since \( p \) is proper, each \( b_0 \in B \) admits neighborhoods \( W \) of \( b_0 \) in \( B \) and \( V \) of \( p^{-1}(b_0) \) in \( M \) such that \( W \times V \subset U \) and \( p^{-1}(W) \subset V \). Since \( r|_{W \times V} \) is an \( f.p. \) retraction from the projection \( \pi_W: W \times V \rightarrow W \) to \( p_W, p_W \) is a fibration. By [Du, p. 403], \( p \) is a fibration.

Conversely suppose \( p: E \rightarrow B \) is a fibration between ANR's. The ANR \( B \) admits a *local equiconnecting function* \( \lambda: V \times [0, 1] \rightarrow B \) ([Fo]), that is,
(a) $V$ is an open neighborhood of the diagonal $J(B):=\{(b, b'): b \in B \}$ in $B \times B$.

(b) $\lambda(b, b', 0)=b'$, $\lambda(b, b', 1)=b$ ($\langle b, b' \rangle \in V$) and $\lambda(b, b, t)=b$ ($b \in B$, $0 \leq t \leq 1$).

Let $U=(id_B \times p)^{-1}(V)$. Since $p$ is a regular fibration ([Du, p. 397]), there exists a homotopy $H: U \times [0, 1] \to E$ such that $pH(b, e, t)=\lambda(b, p(e), t)$, $H(b, e, 0)=e$ ($\langle b, e \rangle \in U$) and $H(p(e), e, t)=e$ ($e \in E$, $0 \leq t \leq 1$). Then $H_1: U \to E$ is an f.p. retraction and by 1.1. (i) $p$ is an ANFR.

(iv) See [Y2, Theorem 1.4] and also [Fe, Theorem 1].

(v) This follows from the next proposition.

1.3. **Proposition.** Let $p: E \to B$ be an onto map.

(i) If $p: E \to B$ is an ANFR and $C \subset B$ is a subset, then $p_C$ is an ANFR over $C$.

(ii) If $B=B_1 \cup B_2$, $B_1 \subset B$ closed and $p_{B_i}$ is an ANFR over $B$ ($i=1, 2$), then $p$ is an ANFR.

(iii) If each $b \in B$ admits a neighborhood $U$ for which $p_U$ is an ANFR over $U$ then $p$ is an ANFR.

**Proof.** (i) If $p$ is an f.p. retract of the projection $\pi_{B|E}$ as in 1.1. (i), then $p_C$ is an f.p. retract of $\pi_C|_{U \cup \cdots}$. Therefore (i) follows from 1.1. (i).

(ii) We may assume that $E$ is a closed subset in $B \times M$, $M$ is an ANR, and that $p=\pi_{B|E}$. Since $p_{B_1}$ is an ANFR, there exist an open neighborhood $U_i$ of $E|_{B_i}=E \cap B_i \times M$ in $B \times M$ and an f.p. retraction $s_i: U_i \to E|_{B_i}$. Similarly $E|_{B_2}$ is an f.p. retract of an open neighborhood $U_2$ in $B_2 \times M$. Since $M$ is an ANR, replacing $U_2$ by a smaller one, we have an f.p. deformation retraction

$$\phi: U_2 \times [0, 1] \to U_1|_{B_1 \cup B_2} \cup (B_2-B_1) \times M$$

such that $\phi_0=id$, $\phi_1(U_2) \subset E$ and $\phi_1|_{E|_{B_2}}=id$ ($0 \leq t \leq 1$). Since $p|_{U_1}$ is an ANFR, by 1.1 (iv) we can extend $\phi_1$ to an f.p. map $\tilde{\phi}_1: U=U_1|_{B_1 \cup U_2} \cup U_1|_{B_2 \cup B_2} \cup E$ such that $\tilde{\phi}_1|_E=id$. Define an f.p. retraction $r: U \to E$ by

$$r(b, m)=\begin{cases} 
\tilde{\phi}_1(b, m) & (b \in B_1) \\
\phi_1(b, m) & (b \in B_2).
\end{cases}$$

By 1.1. (i), $p$ is an ANFR.

(iii) This follows from (i), (ii) and [Mi, Theorem 5.5].

A map $p: X \to B$ is said to be movable ([Y2]) provided for some ANFR $q: E \to B$ and some f.p. closed embedding $i: X \to E$, the following holds:

For each neighborhood $U$ of $i(X)$ in $E$ there exists a neighborhood $V$ of $i(X)$ in $U$ such that for each neighborhood $W$ of $i(X)$ in $V$ there exists an
f.p. deformation \( \phi_t : V \to U \) such that \( \phi_0 = \text{id}, \phi_t(V) \subset W \) and \( q\phi_t = q|_V \) for \( 0 \leq t \leq 1 \).

In addition, if we can take \( \phi_t \) so that \( \phi_t|_Z = \text{id}_Z \) (0 \( \leq t \leq 1 \)) for some neighborhood \( Z \) of \( i(X) \), we say the map \( p \) is strongly movable.

For the definition of shape fibrations, see [MR1,2], [Ma] and also 2.6 (iii).

1.4. Proposition. Let \( p : E \to B \) be an ANFR. Then

(i) \( p \) is strongly movable.

(ii) If \( p \) is proper and \( B \) is separable then \( p \) is a shape fibration.

Proof. (i) This is obvious from the definition.

(ii) This follows from (i) and [Y, Theorem 1.1].

1.5. Proposition. (cf. [Hu, p. 43, Theorem 7.1]) A proper onto map \( p : E \to B \) is an AFR iff \( p \) is an ANFR and each fiber of \( p \) is contractible.

Proof. By 1.2 (i), every fiber of an AFR is contractible.

Conversely suppose \( p \) is an ANFR and \( p^{-1}(b) \equiv * \) for each \( b \in B \). Embed \( E \) into an AR \( M \) as a closed subset and define an f.p. closed embedding \( i : E \to B \times M \) by \( i(e) = (p(e), e) \). Let \( r : U \to E \) be an f.p. retraction from a neighborhood \( U \) of \( i(E) \) in \( B \times M \) given by the assumption.

First we will show that \( p \) is shrinkable ([Do]), that is, there exists a map \( f : B \to E \) and an f.p. homotopy \( \phi : E \times [0, 1] \to E \) such that \( pf = \text{id}_B, \phi_0 = \text{id}_E \) and \( \phi_1 = fp \). To see this, by [Do, 3.2] it suffices to show that each \( b \in B \) admits a neighborhood \( V \) in \( B \) such that \( p|_V \) is shrinkable over \( V \). Let \( b \in B \). Since \( p \) is proper and \( p^{-1}(b) \equiv * \) (hence cell-like), there exist neighborhoods \( V \) of \( b \) in \( B \) and \( W_1 \subset W_0 \) of \( p^{-1}(b) \) in \( M \) such that \( V \times W_0 \subset U \), \( p^{-1}(V) \subset W_1 \) and \( W_1 \equiv * \) in \( W_0 \) by a contraction \( \phi : W_1 \times [0, 1] \to W_0 \). Let \( \phi_0(W_k) = \{m_k\} \). Then the desired section \( f' : V \to p^{-1}(V) \) and the f.p. homotopy \( \phi' : p^{-1}(V) \times [0, 1] \to p^{-1}(V) \) are defined by \( f'(b) = r(b, m_1) \) and \( \phi'(e, t) = r(p(e), \phi(e, t)) \). This completes the proof of the shrinkability of \( p \).

Now let \( f \) and \( \phi \) be as above. Since \( i^{-1} \) is f.p. homotopic to \( \phi_i^{-1} : i(E) \to E \) and \( \phi_i^{-1} \) admits an extension \( \bar{\phi}_i : B \times M \to E \) defined by \( \bar{\phi}_i(b, m) = f(b) \), by 1.1 (iv) we have an f.p. retraction \( r : B \times M \to E \) (i.e., an extension of \( i^{-1} \)). Since \( \pi_B \) is an AFR, so is \( p \).

§ 2. Fiber shape category.

The purpose of this section is to describe a fiber shape category. Our construction is based on [MS, Ch I, §§1, 2], to which we refer for definitions of
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basic terms (pro-category, expansion, etc.).

Fix a space $B$. $\mathcal{H}_B$ will denote the usual fiber homotopy category over $B$, whose objects are maps from metric spaces to $B$. By $\mathcal{A}_B$ we denote the full subcategory of $\mathcal{H}_B$ consisting of maps which are fiber homotopy dominated by some ANFR's over $B$. Below [*] denotes a fiber homotopy class of an appropriate f.p. map.

2.1. PROPOSITION. Every map $p:X \to B$ admits an $\mathcal{A}_B$-expansion $i:p \to p$ in pro-$\mathcal{H}_B$.

PROOF. Take an ANFR $q:E \to B$ and an f.p. closed embedding $i:X \to E$ and let $\{U_\lambda\}_{\lambda \in \Lambda}$ be an open neighborhood base of $i(X)$ in $E$. For each $\lambda \in \Lambda$, let $i_\lambda = i:X \to U_\lambda$, $p_\lambda = q|U_\lambda : U_\lambda \to B$ and for each $\lambda \leq \lambda'$ (defined by $U_\lambda \cap U_{\lambda'}$) let $i_{\lambda'}:U_{\lambda'} \subset U_\lambda$ be the inclusion. By the same argument as in [MS, p. 50, Theorem 4], it is easily verified that $\mathcal{F} = \{[i_\lambda]: p \to p = \{p_\lambda, [i_\lambda], A\}$ satisfies the condition required in [MS, p. 20, Theorem 1].

By [MS, Ch I, §2] we obtain a shape category $sh(\mathcal{H}_B, \mathcal{A}_B)$, which we will denote by $Sh_B$ and call the fiber shape category over $B$. Let $S: \mathcal{H}_B \to Sh_B$ be the associated shape functor. The next proposition justifies the definition.

Assume $B$ is a compactum (a compact metric space) and let $Sh_B^c$ denote the full subcategory of $Sh_B$ consisting of all maps from compacta to $B$. $M_B$ will denote the fiber shape category over $B$ defined in [Ka1, §3].

2.2. PROPOSITION. (cf. [MS, Appendix 2]) There exists an isomorphism $\mathcal{Q}: M_B \to Sh_B^c$ which commutes with the shape functors.

PROOF. The proof is just an f.p. analogue of [MS, p. 332, Theorem 1]. For the sake of completeness, we will give the definition of the functor $\mathcal{Q}$.

Let $Q$ denote the Hilbert cube, $[0,1]^\omega$. By $\pi_1, \pi_2 : Q \times Q \to Q$, we denote the projections onto the first and second factor resp. Let $d$ be a fixed metric on $Q$. Fix an embedding $B \subset Q$.

Let $p:X \to B$ and $q:Y \to B$ be maps from compacta and $\phi : p \to q$ be a morphism in $M_B$. The corresponding morphism $\mathcal{Q}(\phi) : p \to q$ in $Sh_B$ is defined as follows.

Take f.p. embeddings $i$ of $X$ and $j$ of $Y$ into $Q \times Q$ (i.e., $\pi_1i = p$ and $\pi_2j = q$). Since $\pi_1$ can be regarded as an extension of both $p$ and $q$, by the definition of $M_B$, $\phi$ is represented by a fiber fundamental sequence $\phi_n : Q \times Q \to Q \times Q$ ($n \geq 1$) ([Ka1, §3]). By the definition of a fiber fundamental sequence, $\{\phi_n\}$ satisfies the following: 
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(*) For each neighborhood $V$ of $j(Y)$ in $Q \times Q$ and each $\epsilon > 0$ there exist a neighborhood $U$ of $i(X)$ in $Q \times Q$ and $n_0 \geq 1$ such that for each $n \geq n_0$ there exists a homotopy $F: U \times [0, 1] \to V$ such that $F_0 = \phi_n, F_1 = \phi_n$ and $d(\pi_i F(y, x, t), y) < \epsilon ((y, x) \in U, 0 \leq t \leq 1)$.

Define $\tilde{\phi}_n: Q \times Q \to Q \times Q$ by $\tilde{\phi}_n(y, x) = (y, \pi_2 \phi_n(y, x))$ and let $\phi_n = \tilde{\phi}_n |_{B \times Q}$. Then $\{\tilde{\phi}_n\}$ is also a fiber fundamental sequence which is fiber homotopic to $\{\phi_n\}$, and $\{\phi_n\}$ satisfies the following:

(**) For each neighborhood $V$ of $j(Y)$ in $B \times Q$ there exist a neighborhood $U$ of $i(X)$ in $B \times Q$ and $n_0 \geq 1$ such that for each $n \geq n_0, \phi_n, \phi_n: U \to V$ are fiber homotopic (w. r. t. $\pi_1 |_U$).

Therefore for any decreasing open neighborhood base $\{V_n\}_{n \geq 1}$ of $j(Y)$ in $B \times Q$ a there exist a decreasing open neighborhood base $\{U_n\}_{n \geq 1}$ of $i(X)$ in $B \times Q$ and a strictly increasing sequence $\{m_n\}_{n \geq 1}$ of positive integers such that for each $n \geq 1$ and each $m \geq m_n$, maps $\psi_m, \psi_{m_n}: U_n \to V_n$ are fiber homotopic.

By 2.1, $\{\pi_1 |_{U_n}\}$ and $\{\pi_1 |_{V_n}\}$ induce $\mathcal{F}_B$-expansion of $p$ and $q$ resp. Define $\mathcal{Q}(\phi)$ as the morphism in $\mathcal{S}_{B}$ represented by a level morphism $[\psi_{m_n}] : \{\pi_1 |_{U_n}\} \to \{\pi_1 |_{V_n}\}$ in pro-$\mathcal{F}_B$. One can proceed in the same way as in [DS, Ch 3, § 4] or [MS, Appendix 2] to show that $\mathcal{Q}$ is well defined and is an isomorphism of categories.

The next proposition follows from [MS, p. 27, Theorem 4, Corollary 2] and implies that for ANFR's the fiber shape theory coincides with the fiber homotopy theory. $[\cdot, \cdot]_e$ will denote the set of morphisms in the appropriate category.

2.3. PROPOSITION. Let $p: X \to B$ and $q: Y \to B$ be maps.

(i) If $q$ is an ANFR then the function

$$S: [p, q]_{\mathcal{S}_{B}} \to [p, q]_{\mathcal{S}_{B}}$$

is bijective.

(ii) If both $p$ and $q$ are ANFR's then an f.p. map $f: X \to Y$ is a fiber homotopy equivalence iff $S[\cdot]$ is an isomorphism in $\mathcal{S}_{B}$.

We will call any isomorphism in $\mathcal{S}_{B}$ a fiber shape equivalence and say that two maps $p$ and $q$ to $B$ are fiber shape equivalent if there exists an isomorphism of $p$ to $q$ in $\mathcal{S}_{B}$. Next we will list some properties of maps which are fiber shape invariant.

2.4. PROPOSITION. ([Ka$^1$]) A proper onto map $p: X \to B$ is a hereditary shape equivalence iff $p$ is fiber shape equivalent to $id_B$. 
PROOF. Take an ANFR $q: E \to B$ and an f.p. closed embedding $i: X \to E$ and define $p=\{p_\lambda, [i\lambda], \lambda\}$ as in 2.1. By [A, Theorem 4.5], $p$ is a hereditary shape equivalence iff for each neighborhood $U$ of $i(X)$ in $E$ there exists a neighborhood $V$ of $i(X)$ in $U$, a map $g: B \to U$ and an f.p. homotopy $\phi: V \times [0, 1] \to U$ such that $qg=id_B, \phi_0=id_V$ and $\phi_1(V)=g(B)$.

The latter condition can be translated as follows:

(*) For each $\lambda \in \Lambda$ there exists $\lambda' \geq \lambda$ such that $[i, l\lambda]$ is factored through $id_B$ (i.e., $[i, l\lambda]=[g_\lambda][p_{\lambda}]$ for some f.p. map $g_\lambda: B \to U_\lambda$).

Observing that any map $q: Y \to B$ admits a unique morphism to $id_B$ in $\mathcal{F}.K_B$ (i.e., $[q]$), the above (*) is equivalent to the assertion that $p$ is isomorphic to $id_B$ in pro-$\mathcal{F}.K_B$ (cf. [MS, p. 116. Theorem 7]), which implies the conclusion.

2.5. PROPOSITION. Let $p: X \to B$ and $q: Y \to B$ be two proper maps.

(i) If there exists a morphism from $p$ to $q$ in $\text{Sh}_B$ and $p$ is approximately invertible, then so is $q$.

(ii) If there exists an epimorphism from $p$ to $q$ in $\text{Sh}_B$ and $p$ is a hereditary shape equivalence, then so is $q$.

(iii) ([Kan]) If $p$ weakly dominates $q$ and $p$ is a shape fibration (or $p$ has the approximate section extension property (ASEP)), then so is $q$.

PROOF. (ii) By 2.4 there exists an epimorphism $\phi: id_B \to q$. As noted in the proof of 2.4, every map $r: Z \to B$ admits a unique morphism to $id_B$ in $\mathcal{F}.K_B$ and hence in $\text{Sh}_B$ (see 2.3 (i)). Therefore $S[q]\phi=1_{id_B}$. Since $\phi$ is an epimorphism and $\phi S[q]=\phi$, $\phi S[q]=1_q$. Hence $\phi$ is an isomorphism and by 2.4, $q$ is a hereditary shape equivalence.

For the proof of (i) and (iii), we need a lemma.

2.6. LEMMA. (i) Let $p: X \to B$ be a proper map, $\tilde{p}: E \to B$ be an ANFR and $i: X \to E$ be an f.p. closed embedding.

(i) ([A]) $p$ is approximately invertible iff each neighborhood $U$ of $i(X)$ in $E$ admits a map $s: B \to U$ with $\tilde{p}s=id_B$.

(ii) ([Y, Proposition 1.3]) $p$ has the ASEP iff for each neighborhood $U$ of $i(X)$ in $E$, there exists a neighborhood $U_1$ of $i(X)$ in $U$ such that any map $s: C \to U_1$ from a closed subset of $B$ to $U_1$ with $\tilde{p}s=id_C$ admits an extension $\tilde{s}: B \to U$ with $\tilde{s}s=id_B$.

(ii) Let $p: X \to B$ be a proper map, $\tilde{p}: M \to L$ an ANFR between ANR's and $i: X \to M, j: B \to L$ be closed embeddings such that $\tilde{p}i=jp$.

(iii) $p$ is a shape fibration iff the following holds:
(*) For each neighborhood \( U' \) of \( i(X) \) in \( M \) there exist neighborhoods \( U'_i \) of \( i(X) \) in \( U' \) and \( W \) of \( B \) in \( L \) such that for any maps \( h:Z \to U'_i \) and \( H:Z \times [0, 1] \to W \) with \( \bar{p}h = H_0 \), there exists a map \( H':Z \times [0, 1] \to U' \) with \( H'_0 = h \) and \( \bar{p}H' = H \).

**Proof** of 2.6. (iii) By [Ma] and [MR1, Proposition 2] \( p \) is a shape fibration if and only if the following holds:

(\( ** \)) For each neighborhood \( U'' \) of \( i(X) \) in \( M \) and each open cover \( \mathcal{U} \) of \( L \) there exist neighborhoods \( U'_i \) of \( i(X) \) in \( U'' \) and \( W \) of \( B \) in \( L \) such that for any maps \( h:Z \to U'_i \) and \( H:Z \times [0, 1] \to W \) with \( ph = H_0 \), there exists a map \( H':Z \times [0, 1] \to U'' \) such that \( H'_0 = h \) and \( \bar{p}H' = H \).

We must show \( (**) \to (*) \). First consider the special case that \( q \) is the projection \( \pi_L:L \times M \to L \), where \( M \) and \( L \) are ANR's containing \( X \) and \( B \) as a closed subset resp. Let \( U' \) be given. Since \( p \) is proper, if we choose \( U'' \) so small and \( \mathcal{U} \) so fine, then we can adjust the map \( H':Z \times [0, 1] \to U'' \) given by \( (** \) to the desired \( H':Z \times [0, 1] \to U' \) by defining

\[
H'(z, t) = (H(z, t), \pi_M H''(z, t)).
\]

We have shown that for any ANR \( L \) and some ANFR \( q:M \to L \) between ANR's, the shape fibration \( p \) satisfies \( (*) \). It remains to show that if \( p \) satisfies \( (*) \) for some ANFR \( \bar{p}:M \to L \), then so does \( p \) for any such ANFR over \( L \). This follows from the proof of 2.5 (iii) (see below), considering the identity fiber shape morphism on \( \bar{p} \).

(i) and (ii) are also known in the special case that \( q \) is the projection \( \pi_B:B \times M \to B \), with \( M \) an ANR. The general case follows from the proof of 2.5.

We return to the proof of 2.5.

(i) Take ANFR's \( E \xrightarrow{i} B \xrightarrow{\bar{p}} F \) and f. p. closed embeddings \( X \xrightarrow{i} E \) and \( Y \xrightarrow{j} F \) (i.e., \( \bar{p}i = p \) and \( \bar{q}j = q \)). The existence of a morphism from \( p \) to \( q \) implies that for each neighborhood \( V \) of \( j(Y) \) in \( F \) there exist a neighborhood \( U \) of \( i(X) \) in \( E \) and an f. p. map \( f:U \to V \) (i.e., \( \bar{q}f = \bar{q}|_B \)). Then for a section \( s:B \to U \), \( fs \) gives the section required in 2.6 (i).

(iii) Take maps \( X \xrightarrow{i} M \) and \( Y \xrightarrow{j} N \), where \( \bar{p}, \bar{q} \) are ANFR's between ANR's, \( L \) contains \( B \) as a closed subset and \( i, j \) are f. p. closed embeddings. Let \( M|_B = \bar{p}^{-1}(B) \) and \( N|_B = \bar{q}^{-1}(B) \). By 1.3. (i) the restrictions \( M|_B \xrightarrow{\bar{p}} B \xleftarrow{\bar{q}} N|_B \)
are also ANFR's.

The weak domination condition implies the following (see [Dy$_1$, § 2]):

(a) For each open neighborhood $V$ of $j(Y)$ in $N|B$ there exist a neighborhood $U$ of $i(X)$ in $M|B$ and an f.p. map $f: U \rightarrow V$ such that for any neighborhood $U_i$ of $i(X)$ in $U$ there exist a neighborhood $V_i$ of $j(Y)$ in $V$ and an f.p. map $g: V_i \rightarrow U_i$ such that $fg: V_i \rightarrow V$ is f.p. homotopic to the inclusion $V_i \subset V$.

Suppose $p$ is a shape fibration. To see that $q$ is a shape fibration, let $V'$ be any open neighborhood of $j(Y)$ in $N$. We must find neighborhoods $V'_1$ of $j(Y)$ in $V'$ and $W$ of $B$ in $L$ as in 2.6 (iii) for $V'$. By (a) and 1.1. (ii) we have:

(b) a neighborhood $U'$ of $i(X)$ in $M$ and an f.p. map $f': U' \rightarrow V'$,
(c) neighborhoods $U'_1$ of $i(X)$ in $U'$ and $W$ of $B$ in $L$ as in 2.6 (iii) for $U'$,
(d) a neighborhood $V'_1$ of $j(Y)$ in $V'$ and an f.p. map $g': V'_1 \rightarrow U'_1$ such that $f'g': V'_1 \rightarrow V'$ is f.p. homotopic to the inclusion $V'_1 \subset V'$.

To see that $V'_1$ and $W$ satisfy the required condition, let $h: Z \rightarrow V'_1$ and $H: Z \times [0, 1] \rightarrow W$ be maps with $H_0 = \tilde{h}$. By (c) we have a map $G: Z \times [0, 1] \rightarrow U'$ with $\tilde{p}G = H$ and $G_0 = g'h$. Define $H' = f'G$. Then $\tilde{q}H' = H$ and $H'_0 = f'g'h$ is f.p. homotopic to $h$. Using 1.1. (iv), $H'$ can be adjusted so that $H'_0 = h$.

Using 2.6 (ii), the same argument shows that the ASEP is preserved by any weak domination.

Finally, we will be concerned with inverse limits (cf. [MS, Ch I, § 5]). Let $p: X \rightarrow B$ be a map between compacta. Suppose that $X$ is the inverse limit of an inverse sequence $X = \{X_i, f_{ij}\}$ of compacta, together with the projections $f_i: X \rightarrow X_i$ ($i \geq 1$) ($f_if_j = f_k$, $i \leq j$) and that $p$ is induced from a level map $p = \{p_i: X_i \rightarrow B\}$, that is, $p_if_{ij} = p_j$ and $p_if_i = p$ ($j \geq i \geq 1$). The following proposition shows that the level map $p$ reflects the fiber shape of the inverse limit $p$.

2.7. Proposition. (cf. [MS, p. 65, Theorem 9]) Under the above notations, the induced morphism $\tilde{f} = \{[f_i]\}: p \rightarrow p = \{p_i, [f_{ij}]\}$ in pro-$\mathcal{M}_B$ is an $\mathcal{M}_B$-expansion of $p$.

Proof. Let $q: E \rightarrow B$ be an ANFR. We must show the followings ([MS, p. 20, Theorem 1]):
(i) For each f.p. map $g : X \to E$ there exist $i \geq 1$ and an f.p. map $g_i : X_i \to E$ such that $g_i f_i$ is f.p. homotopic to $g$.

(ii) for each $i \geq 1$ and any f.p. maps $g_0, g_1 : X_i \to E$ such that $g_0 f_i$ and $g_1 f_i$ are f.p. homotopic, there exists $j \geq i$ such that $g_0 f_{ij}$ and $g_1 f_{ij}$ are f.p. homotopic.

The simplest way to verify (i) and (ii) may be an f.p. analogue of [DS, Ch 4, §1]. Let $\tilde{X}$ be a compactum defined as follows: The underlying set of $\tilde{X}$ is the disjoint union of $\{X_i\}_{i \geq 1}$ and $X$. The topology of $\tilde{X}$ is given by the open basis consisting of all subsets of the form $U_i$ or $f^i(U_i) \cup \cup \{f^i_j(U_j) : j \geq i\}$, where $i \geq 1$ and $U_i$ is an open set of $X_i$. Note that each neighborhood $U$ of $X$ contains almost all $X_i$ (finitely many exceptions). Define $\tilde{p} : \tilde{X} \to B$ by $\tilde{p} | X_i = p_i$ and $\tilde{p} | X = p$ ($i \geq 1$).

Now (i) and (ii) are verified as follows.

(i) By the f.p. neighborhood extension property, $g$ admits an extension $\tilde{g} : U \to E$ to a neighborhood $U$ of $X$ in $\tilde{X}$ with $q \tilde{g} = \tilde{p} | U$. If we choose $i \geq 1$ sufficiently large, then $X_i \subset U$ and $\tilde{g} f_i$, $g$ are so close that they are f.p. homotopic (recall 1.1 (i)). Define $g_i = \tilde{g} | X_i$.

(ii) Let $X' = X \times [0, 1] \cup (\cup \{X_j : j \geq i\} \times [0, 1]) \subset \tilde{X} \times [0, 1]$. Then $\tilde{p}$ extends to a map $\tilde{G} : \tilde{X} \times [0, 1] \to B$ with $q \tilde{G} = \tilde{p}$. Define $G : X' \to E$ by $G | X \times [0, 1] = \tilde{G}$ and $G | X_j \times [0, 1] = g_j f_{ij}$ ($j \geq i$, $k = 0, 1$).

§ 3. Complements of maps.

In this section we will prove Chapman's complement theorem in the fiber shape theory and give some applications.

All spaces below are assumed to be separable. $Q = [0, 1]^m$ (the Hilbert cube). A closed set $X$ of $B \times Q$ is a sliced $Z$-set ([Fe2]) if for each open cover $U$ of $B \times Q$ there exists an f.p. map $f : B \times Q \to B \times Q - X$ with $(f, id_{B \times Q}) \leq U$, where f.p. means that $\pi_B f = \pi_B$.

3.1. COMPLEMENT THEOREM. Let $X$ and $Y$ be sliced $Z$-sets in $B \times Q$. Then the projections $\pi_B | X$ and $\pi_B | Y$ are fiber shape equivalent iff there exists an f.p. homeomorphism

$$h : B \times Q - X \to B \times Q - Y.$$
the one in [DS, Ch 3, §5]). First we will recall some results on Q-manifold bundles. Note that every proper map $p : X \to B$ admits an f.p. closed embedding $i : X \to B \times Q$ since $X$ is separable.

3.2. **Lemma.** ([Fe8], [Sa]) Let $p : X \to B$ be a proper map.

(i) Every f.p. map $f : X \to B \times Q$ can be approximated arbitrarily closely by a sliced Z-embedding (i.e., an f.p. embedding whose image is a sliced Z-set) which is f.p. homotopic to $f$ by a small homotopy.

(ii) If maps $f, g : X \to B \times Q$ are sliced Z-embeddings and f.p. homotopic in an open subset $U$ of $B \times Q$, then there exists an f.p. ambient isotopy $f_t : B \times Q \to B \times Q$ ($0 \leq t \leq 1$) such that $f_0 = id$, $f_1 = g$ and $f_t|_{B \times Q-V} = id$ ($0 \leq t \leq 1$).

The next lemma is an f.p. analogue of the main part of the proof of the Complement theorem.

Let $U$ be an open set in $B \times Q$ and let $X$ and $Y$ be sliced Z-sets in $B \times Q$ contained in $U$. Suppose there exists an isomorphism $\phi : \pi_B|_X \to \pi_B|_Y$ in $\Sh_B$ such that $S[i(Y, U)] \phi = S[i(X, U)]$, where $i(X, U)$ denotes the inclusion $X \subseteq U$ and $S[i(X, U)]$ is the morphism in $\Sh_B$ induced from $[i(X, U)] : \pi_B|_X \to \pi_B|_U$. $S[i(Y, U)]$ is defined similarly. In this case we say that $\pi_B|_X$ and $\pi_B|_Y$ are fiber shape equivalent in $U$.

3.3. **Lemma.** (cf. [DS, 3.5.6, Claim 1]) Under the above notations, for each neighborhood $V$ of $Y$ in $U$ there exists a neighborhood $U_0$ of $X$ in $U$ such that for each neighborhood $U_1$ of $X$ in $U_0$ there exists an f.p. ambient isotopy $h_1 : B \times Q \to B \times Q$ such that $h_0 = id$, $h_1(U_0) \subseteq V$, $h_1(U_1) \supseteq Y$, $h_1|_{B \times Q-V} = id$ ($0 \leq t \leq 1$) and $\pi_B|_{h_1(U_0)}$ are fiber shape equivalent in $h_1(U_1)$.

**Proof.** Since $\pi_B|_Y$ is an ANFR, by 2.3 (i), there exists an f.p. map $f : X \to V$ such that $S[f] = S[i(Y, U)] \phi$. By 3.2 (i) we may assume $f$ is a sliced Z-embedding. Since $S[i(V, U)] S[f] = S[i(X, U)]$, by 2.3 (i) $i(V, U)f$ is f.p. homotopic to $i(X, U)$. By 3.2 (ii) there exists an f.p. ambient isotopy $f_t : B \times Q \to B \times Q$ such that $f_0 = id_{B \times Q}$, $f_1|_X = f$ and $f_t|_{B \times Q-V} = id$ ($0 \leq t \leq 1$). Take a neighborhood $U_0$ of $X$ such that $f_1(U_0) \subseteq V$.

Let $U_1$ be any neighborhood of $X$ in $U_0$. Applying the same argument to the fiber shape equivalence $S[f](\phi)^{-1} : \pi_B|_Y \to \pi_B|_X$ in $V$ and the neighborhood $f_1(U_0)$ of $f(X)$, we obtain an f.p. ambient isotopy $g_t : B \times Q \to B \times Q$ such that $g_0 = id_{B \times Q}$, $g_1(Y) \subseteq f_1(U_1)$ and $g_t|_{B \times Q-V} = id$ ($0 \leq t \leq 1$). Define $h_1 = g_1^{-1}f_1$ ($0 \leq t \leq 1$).

**Proof of 3.1.** Suppose there exists an isomorphism $\phi : \pi_B|_X \to \pi_B|_Y$ in $\Sh_B$. 
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Note that $S[i(Y, B \times Q)] = S[i(X, B \times Q)]$, since $\pi_B$ is isomorphic to $id_B$ in $Sh_B$. Applying 3.3 inductively, we can find:

(i) open neighborhoods $U_i (i \geq 1)$ of $X$ and $V_i (i \geq 1)$ of $Y$ in $B \times Q$ such that $U_{i+1} \subset U_i \subset N(X, 1/i)$ (the $1/i$-neighborhood of $X$ in $B \times Q$) and $V_{i+1} \subset V_i \subset N(Y, 1/i)$ ($i \geq 1$).

(ii) f.p. homeomorphisms $h_i : B \times Q \to B \times Q$ ($i \geq 1$) such that $V_i \supset h_i \cdots h_1(U_i) \supset V_{i+1}$ and $h_{i+1} \mid B \times Q - h_{i+1} \triangledown \subset (i \geq 1)$.

The desired f.p. homeomorphism $h : B \times Q - X \to B \times Q - Y$ is defined by $h \mid B \times Q - V_i = h_i \cdots h_1 \mid B \times Q - V_i$ ($i \geq 1$).

Conversely suppose there exists an f.p. homeomorphism $h$ as above. Let \{U_j\}_{\lambda \in A} be an open neighborhood base of $X$ in $B \times Q$. Define $V_j = h(U_j - X) \cup Y$ ($\lambda \in A$). Then $V_2$ is open in $B \times Q$. To see this, let $(b, q) \in V_2$. Since $\{b\} \times Q - V_2 = h(\{b\} \times Q - U_j)$ is compact, there exist open neighborhoods $U$ of $q$ and $V$ of $\pi_B(\{b\} \times Q - V_2)$ in $Q$ such that $U \cap V = \emptyset$. Note that $\pi_B \mid B \times Q - V_2$ is a closed map since $\pi_B \mid B \times Q - V_2$ is a closed map and $\pi_B \mid B \times Q - V_2 = \pi_B \mid B \times Q - V_2(h^{-1})\mid B \times Q - V_2$. Therefore there exists a neighborhood $W$ of $b$ in $B$ such that $W \times Q - V_2 \subset B \times V$. Then $W \times U$ is a neighborhood of $(b, q)$ in $B \times Q$ contained in $V_2$. Therefore $V_2$ is open and \{V_j\} is an open neighborhood base of $Y$ in $B \times Q$.

To see that $\pi_B \mid X$ and $\pi_B \mid Y$ are isomorphic in $Sh_B$, by 2.1, it suffices to show that the ANFR-neighborhood systems \{\pi_B \mid U_j, [i_{j_1} \cdots j_{j_2}]\} and \{\pi_B \mid V_j, [j_{j_2} \cdots i_{j_1}]\} are isomorphic in pro-$\mathcal{F}_{\mathcal{A}_B}$. Note that $i_1 : U_1 - X \subset U_1$ is a fiber homotopy equivalence. In fact, since $X$ is a sliced $Z$-set, by [Fe, §4] there exists an f.p. homotopy $f_1 : B \times Q - B \times Q (0 \leq t \leq 1)$ such that $f_0 = id, f_1(B \times Q) \subset B \times Q - X (0 < t \leq 1)$ and $f_t(U_1) \subset U_1 (0 \leq t \leq 1)$. Then $f_1 : U_1 - X \to U_1 - X$ is a fiber homotopy inverse of $i_2$ since $f_1 : U_1 - X \to U_1 - X : id = f_1 f_1$ and $f_1 : U_1 - U_1 : id = i_1 f_1$. Similarly the inclusion $j_2 : V_1 - V \subset V$ is a fiber homotopy equivalence. Therefore we have isomorphisms:

$$\{\pi_B \mid U_j\} \xrightarrow{\sim} \{\pi_B \mid V_2 - X\} \xrightarrow{\sim} \{\pi_B \mid V_2 - Y\} \xrightarrow{\sim} \{\pi_B \mid V_j\}$$

This completes the proof of 3.1.

By the construction of $\mathcal{F}_{\mathcal{A}_B}$-expansions in 2.1, one can easily show that the notion of movability defined in [Y_2] (see the definition before 1.4) coincides with the one in the shape category $Sh_B$ ([MS, Ch II, §6]). Therefore the movability of maps is preserved by any weak domination. Once we have obtained the Complement theorem 3.1, by the same argument as in [Dy, Lemma 2], we can show that the strong movability is also a fiber shape invariant.
3.4. **Corollary.** If proper maps \( p : X \to B \) and \( q : Y \to B \) are fiber shape equivalent and \( p \) is strongly movable then so is \( q \).

Finally we will characterize hereditary shape equivalences and approximate fibrations by their complements. Below \( p : X \to B \) will denote a proper onto map.

3.5. **Corollary.** Let \( i : X \to B \times Q \) be a sliced \( Z \)-embedding. Then the map \( p \) is a hereditary shape equivalence iff the projection \( \pi_B : B \times Q \to i(X) \to B \) is \( f.p. \) homeomorphic to the projection \( B \times Q \times [0,1] \to B \).

**Proof.** Consider the sliced \( Z \)-embedding \( B \approx B \times \{ q \} \subset B \times Q \), where \( q \in Q \) is fixed. Note that \( Q - \{ q \} \approx Q \times [0,1] \) ([Ch, 12.2]). Then 3.5 follows from 2.4 and 3.1.

The map \( p \) is said to be **locally shape trivial** provided each \( b \in B \) admits a closed neighborhood \( V \) for which \( p_V \) is fiber shape equivalent to the projection \( \pi_V : V \times p^{-1}(b) \to V \). The space \( B \) is said to be **semi-locally contractible** if each \( b \in B \) admits a neighborhood \( V \) which contracts in \( B \).

3.6. **Proposition.** Suppose \( B \) is locally compact and semi-locally contractible and that each fiber of \( p \) is an FANR. Then the following assertions are equivalent:

(i) \( p \) is a shape fibration
(ii) \( p \) is locally shape trivial
(iii) \( p \) is strongly movable.

Moreover if \( B \) is finite dimensional, then (i)-(iii) is equivalent to the following:
(iv) \( p \) is completely movable.

**Proof.** (i)\(\to\) (ii). Let \( b \in B \) and let \( K \) be a compact neighborhood which contracts in \( B \). By the same argument as in [Kas, Proposition 1.3] (cf. [Sp, p. 102, Theorem 14]) it is seen that \( p_K \) is fiber shape equivalent to the projection \( K \times p^{-1}(b) \to K \).

(ii)\(\to\) (iii). Let \( b \in B \) and let \( V \) be a neighborhood of \( b \) for which \( p_V \) is fiber shape equivalent to \( \pi_V : V \times p^{-1}(b) \to V \). Since \( p^{-1}(b) \) is an FANR, by [Ys, Example 3.4, (3)], \( \pi_V \) is strongly movable. Then by 3.4, so is \( p_V \). By [Ys, Proposition 3.5], \( p \) is strongly movable.

(iii)\(\to\) (i). This follows from [Ys, Theorem 1.1].

As for (iii)\(\leftrightarrow\) (iv) under the assumption \( \dim B < \infty \), see [Ys, Remark 5.3, Theorem 1.3].
3.7. **Corollary.** Suppose $B$ is locally compact and locally contractible. If $p$ is a local shape fibration (i.e., each $b \in B$ admits a (closed) neighborhood $V$ for which $p_V$ is a shape fibration) and each fiber of $p$ is an FANR, then $p$ is a shape fibration.

**Proof.** Let $b \in B$. Take compact neighborhoods $K \subseteq L$ of $b$ such that $p_L$ is a shape fibration and $K \approx*$ in $L$. As in the proof of 3.6 (i)→(ii), $p_K$ is fiber shape equivalent to the projection $\pi_K: K \times p^{-1}(b) \to K$. Therefore $p$ is locally shape trivial and then by 3.6 $p$ is a shape fibration.

3.8. **Corollary.** Let $i: X \to B \times Q$ be a sliced $Z$-embedding.

(i) $p$ is locally shape trivial iff the projection $\pi_B: B \times Q - i(X) \to B$ is a bundle map.

(ii) Suppose $B$ is a locally compact ANR and each fiber of $p$ is an FANR. Then $p$ is a shape fibration iff $\pi_B: B \times Q - i(X) \to B$ is a bundle map.

(iii) Suppose $B$ and $X$ are locally compact ANR's. Then $p$ is an approximate fibration iff $\pi_B: B \times Q - i(X) \to B$ is a bundle map.

**Proof.** (i) Let $b \in B$ and $V$ be a neighborhood of $b$ in $B$. We may assume $p^{-1}(b)$ is $Z$-embedded into $Q$. If $p_V$ is fiber shape equivalent to $\pi_V: V \times p^{-1}(b) \to V$, then by 3.1, $\pi_B'(V) = V \times Q - i(p^{-1}(V))$ is f.p. homeomorphic to $V \times (Q - p^{-1}(b))$. This implies $\pi_B$ is trivial over $V$.

Conversely, if $\pi_B'(V)$ is f.p. homeomorphic to a product $V \times F$, then since $F = Q - p^{-1}(b)$, by 3.1 $p_V$ is fiber shape equivalent to $\pi_V$.

(ii) This follows from (i) and 3.6.

(iii) By [Ka, Theorem 1.4], $p$ is an approximate fibration iff $p$ is locally shape trivial. Then (iii) follows from (i).

3.9. **Remark.** (i) In 3.6, in general, (iv) does not imply (i), since the Taylor map ([T]) is not a shape fibration ([MR, Example 6]).

(ii) In 3.6, if each fiber of $p$ is cell-like, then by [Y, Theorem 1.2], the conditions (i)-(iii) are equivalent to the condition that $p$ is a hereditary shape equivalence (cf. [Ka, Theorem 2.5]).

(iii) In 3.7 we cannot omit the assumption that each fiber of $p$ is an FANR (even if each fiber is movable). See [Ru, Example 1].
References


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