ON A CLASSIFICATION OF AROSZAJN TREES

By

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§ 1. Introduction.

In their paper [5], Devlin and Shelah introduced new concepts on $\omega_1$-trees to characterize certain topological concepts on $\omega_1$-trees. These are almost-Souslin trees and $\omega_1$-trees with property $\gamma$. The concept of $\omega_1$-trees with club (closed unbounded) antichains also has been stated there. Souslin trees and special Aronszajn trees are famous concepts on octrees. If we assume $V=L$, every concept on $\omega_1$-trees treated in [5] is equivalent to one of the above in the sequel as shown there (In particular, an $\omega_1$-tree $T$ is a normal space under the tree topology iff it has property $\gamma$, if $V=L$ holds. In this connection, if we would study a classification of Aronszajn trees without assuming $V=L$, it might be natural to consider the normality instead of property $\gamma$, since the class of $\omega_1$-trees with property $\gamma$ is just the intersection of the class of almost-Souslin trees and the class of normal $\omega_1$-trees; see the proof of Theorem 4.2 there). Recalling that the special Aronszajn trees are just the $Q$-embeddable $\omega_1$-trees, we deal with also the concept of $R$-embeddable $\omega_1$-trees in this paper.

Now by these concepts we can classify Aronszajn trees into several categories. Our purpose is to check the non-voidness of each of these categories assuming $V=L$.

The particular knowledge on $\omega_1$-trees could be obtained from [2], [4] or [5]. The familiarity of the properties of $\diamondsuit$ and $\diamondsuit^*$ given in [3] and [4] may be helpful.

§ 2. Preliminaries.

We adopt the notations and conventions of current set theory. In particular an ordinal is the set of its predecessors, lower case Greek letters are used to denote ordinals, cardinals are initial ordinals, $\omega$ is the first infinite ordinal and hence the set of natural numbers and $\omega_1$ is the first uncountable ordinal. The cardinality of $X$ is denoted by $|X|$. A set $E \subseteq \omega_1$ is stationary iff $E \cap C \neq \emptyset$ for every closed unbounded (club) set $C \subseteq \omega_1$. When $X$ is a subset of the domain of

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a function \( f \in B^A (\subset \{ R : R \subseteq B \times A \}) \), we denote \( \{ f(x) : x \in X \} \) by \( f[X] \).

A tree is a poset \( T, \prec \) such that for every \( x \in T \), the set \( \check{x} = \{ y \in T : y \prec x \} \) is well-ordered by \( \prec \). The order-type of \( \check{x} \) under \( \prec \) is the height of \( x \) in \( T \), \( \text{ht}(x) \). The \( \alpha \)-th level of \( T \) is the set \( T_\alpha = \{ x \in T : \text{ht}(x) = \alpha \} \). We set \( T \upharpoonright \alpha = \bigcup_{\beta < \alpha} T_\beta \), which is also a tree with the ordering \( \prec \). Define \( T \upharpoonright C = \bigcup_{\alpha \in C} T_\alpha \). If \( X \subseteq T, X \upharpoonright \alpha \) stands for \( X \cap T \upharpoonright \alpha \). The height of \( T \), \( \text{ht}(T) \), is the least \( \alpha \) such that \( T_\alpha = 0 \).

A branch of \( T \) is a maximal chain of \( T \). If a branch has order-type \( \alpha \), it is called an \( \alpha \)-branch. A subset \( X \) of \( T \) will be called unbounded in \( T \) iff \( \{ \text{ht}(x) : x \in X \} \) is unbounded in \( \text{ht}(T) \). An antichain of \( T \) is a pairwise incomparable subset of \( T \). A subset \( T' \) of \( T \) is called a subtree of \( T \) if \( x \leq T' \) for all \( x \in T' \).

An \( \omega_i \)-tree is a tree \( T \) such that:

1. \( \text{ht}(T) = \omega_i \);
2. \( \forall \alpha < \omega_i \lfloor | T_\alpha | \leq \omega \) ;
3. \( \forall \alpha < \beta < \omega_i \)(\( \forall x \in T_\alpha \exists y_1, y_2 \in T_\beta \)[\( y_1 \neq y_2 \) & \( x \prec \tau y_1 \) & \( x \prec \tau y_2 \)];
4. \( \forall \alpha < \omega_i \)(\( \forall x, y \in T_\alpha \)[\( \lim(\alpha) \rightarrow [x = y \leftrightarrow \check{x} = \check{y}] \) .

A subset \( X \) of an \( \omega_i \)-tree will be called stationary (resp. club) iff \( \{ \text{ht}(x) : x \in X \} \) is stationary (resp. club) in \( \omega_i \).

Let \( T \) be an \( \omega_i \)-tree. If \( a, b \in T, a \prec \tau b \), we set:

\( [a, b] = \{ x \in T : a \leq \tau x \leq \tau b \} \) (a closed interval);
\( (a, b) = \{ x \in T : a \prec \tau x \prec \tau b \} \) ;
\( (a, b) = \{ x \in T : a < \tau x < \tau b \} \) (an open interval).

We make \( T \) into a topological space by taking as an open basis all sets of the form \( (a, b) \) for \( a \prec \tau b \) and all sets of the form \( [0, a) \) for \( a \in T \). This topology is the tree topology on \( T \). Notice that \( (a, b) \) is an open set in this topology.

We shall also use the following constants:
\( \mathcal{Q} = \{ \alpha < \omega_i : \lim(\alpha) \} \);
\( \mathcal{Q} = \text{the set of rational numbers} \);
\( \mathcal{R} = \text{the set of real numbers} \);
\( \mathcal{C} = \text{the set of all stationary subsets of } \omega_i \);
\( \mathcal{C} = \text{the set of all club subsets of } \omega_i \);
\( \mathcal{X} = \bigcup_{\alpha < \omega_i} \mathcal{R}^{\alpha+1} \).

We shall use the set \( \mathcal{X} \) as a tree by defining \( x <_\tau y \) by \( x \subseteq y \). \( \{0, 0\} \) is one of the minimal elements of the tree \( \mathcal{X} \). We denote it by \( 0_\tau \).

An Aronszajn tree is an \( \omega_i \)-tree with no \( \omega_i \)-branch. A Souslin tree is an \( \omega_i \)-tree with no uncountable antichain. An Aronszajn tree is special if it is the union of a countable collection of antichains. An \( \omega_i \)-tree is an almost-Souslin tree iff it has no stationary antichain. An \( \omega_i \)-tree \( T \) has property \( \tau \) iff, when-
ever \( X \) is an antichain of \( T \), there is a club set \( C \subseteq \omega_1 \) such that \( T \setminus T \upharpoonright C \) contains a closed neighborhood of \( X \). A tree \( T \) is called \( R \)-embeddable iff there is \( e : T \to R \) such that \( x <_T y \iff e(x) < e(y) \). \( Q \)-embeddability is defined similarly.

We set:

- \( ST \) = the class of Souslin trees;
- \( \gamma ST \) = the class of \( \omega_1 \)-trees with property \( \gamma \);
- \( AST \) = the class of almost-Souslin trees;
- \( NCA \) = the class of \( \omega_1 \)-trees with no club antichain;
- \( SAT \) = the class of special Aronszajn trees;
- \( RE \) = the class of \( R \)-embeddable \( \omega_1 \)-trees;
- \( AT \) = the class of Aronszajn trees.

The following are clear and/or famous facts:

**Proposition 1.** (i) \( ST \subseteq \gamma ST \subseteq AST \subseteq NCA \),
(ii) \( SAT \subseteq RE \subseteq AT \),
(iii) \( ST \subseteq AT \),
(iv) \( AST \cap SAT = \emptyset \),
(v) \( ST \cap RE = \emptyset \).

Thus we obtain a classification of \( AT \) as shown by the following diagram:

```
   NCA
  /   \
AST /     \
\gamma ST /       \
   2
  /   \
ST /   9   10   11

\[ 6 \quad 3 \quad 7 \quad 4 \quad 8 \quad 5 \]
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The purpose of this paper is to check that these eleven categories are all nonvoid under the hypothesis \( V=L \). It will be done in the next section.

It needs some more preparations for the next section. First we expand the principles \( \Diamond \) and \( \Diamond^* \).

\( \Diamond \) is the principle that asserts the existence of a sequence \( \langle Z_\alpha : \alpha < \omega_1 \rangle \) such that:

1. whenever \( T \) is an \( \omega_1 \)-tree which is a subtree of \( \mathcal{I} \) (an \( \omega_1 \)-subtree of \( \mathcal{I} \), for short) and \( X \subseteq T, \alpha < \omega_1 : X \upharpoonright \alpha = Z_\alpha \in \mathcal{E} \);
2. whenever \( T \) is an \( \omega_1 \)-subtree of \( \mathcal{I} \) and \( e \) is a function: \( T \to R, \alpha < \omega_1 : e \upharpoonright (T \upharpoonright \alpha) = Z_\alpha \in \mathcal{E} \);
3. \( (\forall S \subseteq \omega_1)[(\exists \alpha < \omega_1 : S \cap \alpha = Z_\alpha) \in \mathcal{E}] \).
PROPOSITION 2. $\diamondsuit \leftrightarrow \diamondsuit_{2}$.

PROOF. We shall show the non-trivial arrow. Recall that $\diamondsuit$ asserts that there is a sequence $\langle S_{n}: \alpha \prec \omega_{1} \rangle$ such that $S_{n} \subseteq \alpha$ and $(\forall S \subseteq \omega_{1})[\{\alpha : S \cap \alpha = S_{n}\} \in \mathcal{G}]$. Assume $\diamondsuit$. Since $\diamondsuit$ implies $CH$, $|\mathcal{M}| = |\bigcup_{\omega \in \omega_{1}}R^{*+1}| = |\omega_{1} \times (2^{\omega})^{+}| = \omega_{1}$ and also $|R \times \mathcal{M}| = \omega_{1}$. Hence there exists a bijection $b : \mathcal{M} \cup (R \times \mathcal{M}) \cup \omega_{1} \rightarrow \omega_{1}$. Let $\langle S_{n} : \alpha \prec \omega_{1} \rangle$ be a $\diamondsuit$-sequence. Put $Z_{n} = b^{-1}[S_{n}]$. We show that $\langle Z_{n} : \alpha \prec \omega_{1} \rangle$ is a $\diamondsuit_{2}$-sequence.

To check (1), let $T$ be an $\omega_{1}$-subtree of $\mathcal{M}$ and let $X \subseteq T$. Put $C_{n} = \{\lambda \in \Omega : (\forall X \subseteq T \cap \lambda)(b(x) < \lambda \rightarrow x \in T \cap \lambda)\}$. Clearly $C_{n} \in \mathcal{G}$. Put $E_{\alpha} = C_{\alpha} \cap \{\alpha : b[X] \cap \alpha = S_{\alpha}\} \in \mathcal{G}$. If $\lambda \in E_{\alpha}$, then $b[X \cap \lambda] \subseteq \lambda \cap b[X] = S_{\alpha}$ and $b^{-1}[S_{\alpha}] = b^{-1}[\lambda \cap b[X]] \subseteq X \cap T \cap \lambda = X \cap \lambda$. Thus $\{\alpha : \omega_{1} : X \cap \lambda = Z_{\alpha}\}$ contains the stationary set $E_{\alpha}$. To check (2), let $e \in R^{T}$. Put $C_{e} = \{\lambda \in \Omega : (\forall x \in T \cap \lambda)(b(\langle e(x), x \rangle) < x)\} \in \mathcal{G}$. Put $C_{e} = \{\lambda \in \Omega : (\forall x \in T)(b(\langle e(x), x \rangle) < \lambda \rightarrow x \in T \cap \lambda)\} \in \mathcal{G}$. Then it can be shown similarly that the set $\{\alpha \prec \omega_{1} : e \cap (T \cap \alpha) = Z_{\alpha}\}$ contains the stationary set $C_{e} \setminus C_{\alpha} \cap \{\alpha : b[e] \cap \alpha = S_{\alpha}\}$. We can check (3) similarly.

Let $\diamondsuit_{2}$ be the principle which asserts that there is a sequence $\langle \{W_{i} : i \prec \omega \} : \alpha \prec \omega_{1} \rangle$ such that:

1. whenever $T$ is an $\omega_{1}$-subtree of $\mathcal{M}$ and $X \subseteq T$, $\{\alpha \prec \omega_{1} : X \cap \alpha = W_{i} \}$ for some $i$ contains a club set;
2. whenever $T$ is an $\omega_{1}$-subtree of $\mathcal{M}$ and $e \in R^{T}$, $\{\alpha \prec \omega_{1} : e \cap (T \cap \alpha) = W_{i} \}$ for some $i$ contains a club set;
3. $(\forall S \subseteq \omega_{1})[\{\alpha \prec \omega_{1} : S \cap \alpha = W_{i} \}$ for some $i \prec \omega \}$ contains a club set].

Then we obtain the following by the same argument:

PROPOSITION 3. $\diamondsuit_{2} \leftrightarrow \diamondsuit_{2}$.

Next we define a subtree $\mathcal{M}_{R}$ of $\mathcal{M}$ by the following:

$\mathcal{M}_{R} = \{x \in \mathcal{M} : (\forall \alpha, \beta \in \text{dom}(x))[(\alpha < \beta \rightarrow x(\alpha) < x(\beta))]\}$.

Note that the ordinal $\text{ht}(x)$ is the maximum element of $\text{dom}(x)$ for any $x \in \mathcal{M}$. For $x \in \mathcal{M}_{R}$, we denote $x(\text{ht}(x))$ by $m(x)$. This function $m$ clearly embeds $\mathcal{M}_{R}$ in $R$. So $\mathcal{M}_{R}$ is an $R$-embeddable tree and hence the following holds:

PROPOSITION 4. If $T$ is an $\omega_{1}$-subtree of $\mathcal{M}_{R}$, then $T \subseteq R^{E}$.  

§ 3. Non-voidness of each of the categories.

The existence of a Souslin tree under $V=L$ is first proved by R.B. Jensen. The proof of the famous following theorem can be seen in [4].
Theorem 1 (◊). There is a Souslin tree.

Devlin and Shelah virtually showed the following (See Lemma 4.3 and its proof in [5]):

Theorem 2 (◊). There is an Aronszajn tree with property \( \gamma \) which is neither Souslin nor \( R \)-embeddable.

Devlin and Shelah virtually showed the following (See Theorem 4.4 and its proof in [5]):

Theorem 3 (◊*). There is an \( R \)-embeddable, almost-Souslin tree which has not property \( \gamma \).

The following is also pointed out by Devlin and Shelah (See § 5 of [5]):

Theorem 4 (◊). There is a special Aronszajn tree with no club antichain.

Theorem 5. There is a special Aronszajn tree with club antichains.

Proof. This is essentially a well-known construction of \( T \in SAT \). We need only a trivial care to guarantee \( T \in NCA \) here. We define an \( \omega \)-subtree \( T \) of \( \mathbb{X}_R \) by induction on levels. The construction is carried out to ensure that \( T \) satisfies the following condition:

(1) if \( \alpha < \beta < \omega \) and \( x \in T_\alpha \) and \( 0 < q < Q \), there is a \( y \in T_\beta \) such that \( x < T y \) and \( m(y) = m(x) + q \).

We set:

\[
T_0 = \{0\};
\]

\[
T_{n+1} = \{x \cup \langle q, \alpha + 1 \rangle : x \in T_\alpha \land m(x) < q \in Q\}.
\]

Let \( \lambda \) be a limit ordinal. To define \( T_\lambda \), with each \( x \in T \upharpoonright \lambda \) and each \( q \in Q \cap (0, \infty) \), associate an increasing sequence \( x < T x_0 < T x_1 < T \cdots < T x_n < T \cdots \) such that \( x_n \in T \upharpoonright \lambda \) and \( \lim_{n<\omega} \text{ht}(x_n) = \lambda \) and \( \lim_{n<\omega} m(x_n) = m(x) + q \). (This is possible by (1)). Set:

\[
t_\lambda(x, q) = \bigcup_{x < \omega} x \cup \{m(x) + q, \lambda\};
\]

\[
T_\lambda = \{t_\lambda(x, q) : x \in T \upharpoonright \lambda \land 0 < q \in Q\}.
\]

Then \( T = \bigcup_{\lambda < \omega} T_\lambda \) is a \( Q \)-embeddable \( \omega \)-tree and hence \( T \in SAT \) (it is known that \( T \in SAT \) iff \( T \) is \( Q \)-embeddable for an \( \omega \)-tree \( T \); see e.g. [4]). Besides the set \( \{t_\lambda(0, 1) : \lambda \in Q\} \) is a club antichain, q.e.d.
**Theorem 6 (\(\Diamond\)).** There is an \(R\)-embeddable \(\omega_1\)-tree with property \(\gamma\).

**Proof.** We define an \(\omega_1\)-subtree \(T\) of \(\mathcal{T}_R\) by induction on the levels. The construction is carried out to ensure that \(T\) satisfies the following condition:

(1) if \(\alpha < \beta < \omega_1\) and \(x \in T_\alpha\) and \(q > 0\), then there is a \(y \in T_\beta\) such that \(x <_T y\) and \(m(y) < m(x) + q\).

Let \(\langle W_n^\alpha : n < \omega \rangle : \alpha < \omega_1 \rangle\) be a \(\Diamond\)-sequence.

We set:

\[ T_0 = \{0_T\} ; \]
\[ T_{\alpha + 1} = \{x \cup \{q, \alpha + 1\} : x \in T_\alpha \land q \in Q \cap (m(x), 1)\} . \]

Suppose \(\lim(\lambda)\). We shall define \(T_\lambda\). Fix an increasing sequence \(\langle \lambda_n : n < \omega \rangle\) confinal in \(\lambda\). For each \(x \in T \uparrow \lambda\) and each \(q \in Q \cap (m(x), 1)\), we pick inductively \(x_n, x_n^* \in T \uparrow \lambda\) and \(q_n \in R\) for \(n < \omega\) so that the following hold:

- (a) \(x_0 = x \land q_0 = q\);
- (b) if \(W_n^f\) is an antichain of \(T \uparrow \lambda\) and \(\langle y \in T \uparrow \lambda \rangle [x_n <_T y \in W_n^f \land m(y) < q_n]\), then \(x_n <_T x_n^* \in W_n^f\) and \(m(x_n^*) < q_n\) \& \(q_{n+1} = (m(x_n^*) + q_n)/2\);
- otherwise, \(x_n <_T x_n^* \land m(x_n^*) < (m(x_n) + q_n)/2\) \& \(q_{n+1} = (m(x_n) + q_n)/2\);
- (c) \(x_n^* <_T x_{n+1}\) and \(\lambda_n < \text{ht}(x_{n+1})\) and \(m(x_{n+1}) < q_{n+1}\).

And put:

\[ t_\lambda(x, q) = \bigcup_{n < \omega} x_n \cup \{\langle \sup_{n < \omega} m(x_n), \lambda \rangle\} . \]

We set:

\[ T_\lambda = \{t_\lambda(x, q) : x \in T \uparrow \lambda, q \in Q \cap (m(x), 1)\} . \]

The following are immediate from the definition of \(t_\lambda(x, q)\):

- (d) \(m(t_\lambda(x, q)) \leq q_{n+1} < q_n \leq q\);
- (e) if \(W_n^f\) is an antichain of \(T \uparrow \lambda\) and \(\langle y \in T \uparrow \lambda \rangle [x_n <_T y \in W_n^f \land m(y) < q_n]\), then \(\langle y \in T \uparrow \lambda \rangle [y \in W_n^f]\);
- (f) if \(W_n^f\) is an antichain of \(T \uparrow \lambda\) and \(x_n^* \in W_n^f\), then

\[ (\forall y \in W_n^f) [x_n <_T y \rightarrow m(y) \leq m(x_n)] \land m(t_\lambda(x, q)) < (m(x_n) + q_n)/2 . \]

Now we shall show that \(T = \bigcup_{\alpha < \omega} T_\alpha\) is as required. \(T\) is obviously an \(R\)-embeddable \(\omega_1\)-tree. To show \(T \in \gamma \mathcal{ST}\), let \(X\) be an antichain of \(T\). Define \(C_\emptyset \subseteq \mathcal{C}\) by the following:

\[ C_\emptyset = \{\lambda \in \Omega : (\forall x \in T \uparrow \lambda)(\forall q \in Q \cap (m(x), 1))[(\exists y \in X)[x <_T y \land m(y) < q]] \}
\]

Take \(C_1 \subseteq \mathcal{C}\) so that:
On a classification of aronszajn trees

\[ C_i = \{ \lambda \in \Omega : X \upharpoonright \lambda = W_\lambda^i \text{ for some } n < \omega \} \]

Put \( C = C_0 \cap C_1 \in \mathbb{C} \). Note that \( X \cap T \upharpoonright C = \emptyset \) (Proof: Suppose \( w \in X \cap T \upharpoonright C \). Then for some \( \lambda \in C \) and some \( x \in T \upharpoonright \lambda \) and some \( q \in Q \cup \{ m(x), 1 \} \), \( w = t_x(x, q) \) and for some \( n < \omega \), \( X \upharpoonright \lambda = W_\lambda^i \). Since \( x_n <_T w \) and \( m(w) < q_n \) (see (d)), \( \exists y \in X \}\{ x_n <_T y \text{ & m(y) < q_n } \} \text{ since } \lambda \in C_0 \). Since \( X \upharpoonright \lambda = W_\lambda^i \), we obtain \( \exists y' <_T w \] \( [y' \in W_\lambda^i] \) (see (e)), which means that the antichain \( X \) contains comparable elements, i.e. \( y' \text{ and } w \); a contradiction, q.e.d.).

We shall give disjoint open sets \( U \) and \( V \) such that \( X \subseteq U \) and \( T \upharpoonright C \subseteq V \).

(1) Definition of \( U \). Let \( \langle \lambda_0 : \nu < \omega \rangle \) be the monotone enumeration of \( C \cup \{ 0 \} \).

For each \( \nu < \omega \), let \( \langle x_\nu^n : n < \omega \rangle \) be a one-to-one enumeration of \( X \cap (T \upharpoonright \lambda_{\nu+1}) \upharpoonright \lambda_\nu \).

Pick \( y_\nu^n \in T \upharpoonright \lambda_{\nu+1} \upharpoonright \lambda_\nu \) by induction on \( n \) so that the following hold:

\[ \begin{align*}
  (g) & : y_\nu^n <_T x_\nu^n ; \\
  (h) & : y_\nu^n \in (x_\nu^n) \cap \bigcup_{i < n} (y_i^n, x_i^n) = \emptyset .
\end{align*} \]

(This is possible since \( X \cap T \upharpoonright C = \emptyset \) and \( X \) is an antichain). Then \( \{ (y_\nu^n, x_\nu^n) : \nu < \omega \text{ & } n < \omega \} \) is a family of pairwise disjoint open sets and \( x_\nu^n \in (y_\nu^n, x_\nu^n) \).

Let \( z_\nu^n \) be the \( <_T \text{-least element of} \):

\[ \{ z : x_\nu^n < z < (m(y_\nu^n) + m(x_\nu^n))/2 < m(z) \} . \]

Notice that \( \text{ht}(z_\nu^n) \in \Omega \) (note that whenever \( \text{ht}(x) \in \Omega \), \( m(x) = \sup \{ m(y) : y <_T x \} \) by the construction). Hence \( [z_\nu^n, x_\nu^n] \) is an open nbd of \( x_\nu^n \). Now put:

\[ U = \bigcup_{\nu < \omega} \bigcup_{n < \omega} [z_\nu^n, x_\nu^n] \subseteq X \).

(II) Definition of \( V \). Let \( w \in T \upharpoonright C \). We first define its open nbd \( (w', w) \).

Put \( \lambda = \text{ht}(w) \). There is an \( m < \omega \) such that \( X \upharpoonright \lambda = W_m^i \). By the definition of \( T_\lambda \), there are \( x \in T \upharpoonright \lambda \) and \( q \in Q \cap \{ m(x), 1 \} \) such that \( w = t_x(x, q) \). Let \( x \) and \( q \) be one of such pairs. We define an element \( w' <_T w \) as follows (for the notation, refer to the construction of \( t_x(x, q) \)):

Case 1: \( x_\nu^n \in W_m^i \). Put \( w' = x_\nu^n \).

Case 2: \( x_\nu^n \in W_m^i \) & \( (x_m, w] \cap U = \emptyset \). Put \( w' = x_m \).

Case 3: \( x_\nu^n \in W_m^i \) & \( (x_m, w] \cap U \neq \emptyset \). Pick an interval \( [z_\nu^n, x_\nu^n] \) such that \( (x_m, w] \cap [z_\nu^n, x_\nu^n] \neq \emptyset \) and put \( w' = \) the \( <_T \text{-least element of} \( (x_m, w] \cap [z_\nu^n, x_\nu^n] \).

Claim. \( (w', w] \cap U = \emptyset \).

Proof. Suppose that \( w' \) has been defined in Case 1. Then \( w' = x_\nu^n \in W_m^i \)

\( = X \upharpoonright \lambda \subseteq X \). Note that \( X \) is an antichain and \( U \subseteq \{ y : y <_T x \text{ for some } x \in X \} \).

So \( \{ z : w' <_T z \} \cap U = \emptyset \) which implies the assertion. If \( w' \) has been defined in Case 2, it is trivial. Suppose \( w' \) has been defined in Case 3. Then \( w' \in (x_m, w] \cap [z_\nu^n, x_\nu^n] \). Suppose \( (w', w] \cap U = \emptyset \).

Then we would be able to take an interval...
\[ [z_k^n, x_k^n] \text{ such that } (w', w) \cap [z_k^n, x_k^n] \neq \emptyset. \] Clearly \( x_m <_T x_m^n \) and \( x_k^n \in T \upharpoonright \lambda = W_m^n. \) By (f), \( q_m = m(x_m), q_m = m(x_k^n) \) and \( m(w) < (m(x_m) + q_m)/2. \) On the other hand, \( (m(y_k^n) + m(x_k^n))/2 < m(x_k^n). \) Let \( v \in (w', w) \cap [z_k^n, x_k^n]. \) Then \( m(y_k^n) + q_m)/2 \leq m(x_k^n) \leq m(w) < (m(x_m) + q_m)/2. \) It follows that \( y_k^n <_T x_m, \) since \( y_k^n <_T x_k^n. \) Hence \( x_m \in (y_k^n, x_k^n) \cap (y_v^n, x_v^n) \); this is absurd since \( x_k^n, x_v^n \) are clearly different and the elements of \( [(y_v^n, x_v^n) \cup \langle w', w \rangle \cap [z_k^n, x_k^n]. \) This is a case of disjoint set, q.e.d.

Now put:

\[ V = \bigcup \{(w', w) : w \in T \upharpoonright C\}. \]

Obviously by the claim, \( V \cap U = \emptyset \) and \( T \upharpoonright C \subset V, \) q.e.d.

**Theorem 7 (\diamond).** There is an \( R \)-embeddable \( \omega_1 \)-tree which has no club antichain and is not almost-Souslin nor special Aronszajn.

**Proof.** (If we assume \( \diamond^*, \) such a tree can be easily obtained; for instance, a direct union of \( T_2 \in \mathcal{RE} \cap \mathcal{AST} \) (see Theorem 3 or 6) and \( T_1 \in \mathcal{SAT} \cap \mathcal{NCA} \) (see Theorem 4)). Let \( \langle Z_\alpha : \alpha < \omega_1 \rangle \) be a \( \diamond^* \)-sequence. We define an \( \omega_1 \)-subtree \( T \) of \( \mathcal{R} \) by induction on the levels. As we define \( T_\lambda \) for \( \lambda \in \Omega, \) we shall also define a (singleton or void) subset \( A_\lambda \) of \( T_\lambda \) so that \( A = \bigcup \{ A_\lambda : \lambda \in \Omega \} \) is a stationary antichain of \( T. \) The construction is carried out to ensure that \( T \) satisfies the following condition:

1. If \( \alpha < \beta < \omega_1 \) and \( x \in T_\alpha \) and \( q \in Q \cap (m(x), 2), \) there is a \( y \in T_\beta \) such that \( x <_T y, (x, y) \cap A = \emptyset \) and \( m(y) < q. \)

We set:

\[ T_0 = \{ 0 \}; \]

\[ T_{\alpha+1} = (x \cup \{ q, \alpha+1 \}) : x \in T_\alpha \text{ and } q \in Q \cap (m(x), 2) \].

Let \( \lambda \in \Omega. \) We shall define \( T_\lambda. \) Fix a sequence \( \langle \lambda_\alpha : n < \omega \rangle \) such that \( \sup_n \lambda_\alpha = \lambda. \) For each \( x \in T \upharpoonright \lambda \) and each \( q \in Q \cap (m(x), 2), \) we pick \( x_n \in T \upharpoonright \lambda \) inductively for \( n < \omega \) so that the following hold:

a) If \( Z_\lambda \) is an antichain of \( T \upharpoonright \lambda \) and \( (\exists y \in T \upharpoonright \lambda)[x <_T y \in Z_\lambda] \) \( (x, y) \cap A = \emptyset \) \( m(y) < q], \) then \( x <_T x_n \in Z_\lambda \) \( (x, x_n) \cap A = \emptyset \) \( m(x_n) < q \); otherwise, \( x_0 = x \); put \( q^* = (m(x_0) + q)/2; \)

b) \( x_n <_T x_{n+1} \) \( \lambda_n < \text{ht}(x_{n+1}) \) \( (x_n, x_{n+1}) \cap A = \emptyset \) \( m(x_{n+1}) < q^*. \)

And we put:

\[ l_\lambda(x, q) = \cup_{n \in \omega} x_n \cup \{ \sup_n \omega m(x_n), \lambda \}. \]

Case 1: If \( Z_\lambda \) is not an embedding: \( T \upharpoonright \lambda \to Q \cap [0, 1], \) then we set:
On a classification of aronszajn trees

\[ T_\lambda = \{ t_\lambda(x, q) : x \in T \upharpoonright \lambda & q \in Q \cap (m(x), 2) \}. \]

Case 2: If \( Z_\lambda \) is an embedding: \( T \upharpoonright \lambda \to Q \cap [0, 1) \), we need more preliminaries to define \( T_\lambda \). Let \( \langle q_n : n < \omega \rangle \) be an enumeration of all elements of \( Q \cap [0, 1) \).

We pick \( y_n, y_n^* \in T \upharpoonright \lambda \) and \( r_n \in R \) inductively for \( n < \omega \) as follows:

(d) \( y_0 = 0, r_0 = 1 \);

(e) if \( \langle y \in T \upharpoonright \lambda \rangle y_n < r y \& Z_\lambda(y) = q_n \& m(y) < r_n \rangle \), then \( y_n < r y_n^* \& Z_\lambda(y_n^*) = q_n \& m(y_n^*) < r_n \rangle \);

otherwise, \( y_n^* = y_n \).

(f) \( y_n^* < T y_{n+1} \& \lambda_n \leq \text{ht}(y_{n+1}) \& m(y_{n+1}) < r_n \rangle \);

(g) \( r_{n+1} = (m(y_{n+1}) + r_n) / 2 \).

And we put:

\[ u_\lambda = \bigcup_{n < \omega} y_n \cup \{ \langle \sup_{n < \omega} m(y_n), \lambda \rangle \}. \]

Now we set:

\[ T_\lambda = \{ u_\lambda \} \cup \{ t_\lambda(x, q) : x \in T \upharpoonright \lambda & q \in Q \cap (m(x), 2) \}. \]

Note that \( (x, t_\lambda(x, q)) \cap A \upharpoonright \lambda = \emptyset \). For \( \lambda \in \mathcal{O} \) we put:

\[ A_\lambda = \begin{cases} 0 & \text{if } Z_\lambda \text{ is an antichain of } T \upharpoonright \lambda, \\ \{ t_\lambda(0_r, 1) \} & \text{otherwise.} \end{cases} \]

Clearly \( T \upharpoonright (\lambda + 1) \) satisfies (1). (Although \( t_\lambda(0_r, 1) \) is put in \( A \), we can easily find \( x \) such that \( (0_r, t_\lambda(x, q)) \) \( \cap A = \emptyset \).)

We shall show that \( T = \bigcup_{\alpha < \omega_1} T_\alpha \) is as required. Clearly \( T \in RE \).

Claim 1. \( T \in \text{AST}^c \)

**Proof.** Put \( E = \{ \lambda \in \mathcal{O} : Z_\lambda \) is not an antichain of \( T \upharpoonright \lambda \} \). Clearly \( E \in \mathcal{G} \).

Since \( (0_r, t_\lambda(0_r, 1)) \cap A \upharpoonright \lambda = \emptyset \) for all \( \lambda \in E \), \( A = \bigcap_{\lambda \in E} A_\lambda \) is an antichain of \( T \).

\[ \{ \text{ht}(x) : x \in A \} = E. \]

Thus \( A \) is a stationary antichain.

Claim 2. \( T \in \text{NCA} \).

**Proof.** Suppose that there were a club antichain \( X \). Put:

\[ C = \{ \lambda \in \mathcal{O} : (\forall x \in T \upharpoonright \lambda)(\forall q \in Q \cap (m(x), 2))[(\exists y \in T)[x < r y \in X \\
& (x, y) \cap A = \emptyset \& m(y) < q] \rightarrow (\exists y \in T \upharpoonright \lambda)[x < r y \in X \& (x, y) \cap A = \emptyset \& m(y) < q]] \}. \]

Clearly \( C \in \mathcal{G} \). Since \( \{ \alpha : X \upharpoonright \alpha = Z_\alpha \} \) \( \in \mathcal{G} \), we can pick a \( \lambda \in C \cap \{ \text{ht}(x) : x \in X \} \) such that \( X \upharpoonright \lambda = Z_\lambda \). Then \( X \cap T_\lambda \neq \emptyset \) and \( Z_\lambda \) is an antichain of \( T \upharpoonright \lambda \). Since \( Z_\lambda \) is not an embedding: \( T \upharpoonright \lambda \to Q \), \( T_\lambda \) is of the form \( \{ t_\lambda(x, q) : x \in T \upharpoonright \lambda \} \)
Pick $t_2(x, q) \in X \cap T_2$. Since $(x, t_2(x, q)] \cap A = \emptyset$ (note that $t_2(0, 1) \in A$ if $Z_2$ is not an antichain of $T \upharpoonright \lambda$) and $x < t_2(x, q) \in X$ and $m(t_2(x, q)) \leq q^* < q$ and $\lambda \in C$, it holds that $(\exists y \in X \upharpoonright \lambda)[x < t_2(x, q) & (x, y) \cap A = \emptyset & m(y) < q]$. Since $X \upharpoonright \lambda = Z_2$, by (a), $x < t_2(x, q) \in Z_2 \subseteq X$. This means that an antichain $X$ contains comparable elements $x, t_2(x, q)$; a contradiction.

Claim 3. $T \equiv SAT$.

Proof. Suppose that there were a $Q$-embedding $e : T \to Q \cap [0, 1)$. Put:

$$C = \{ \lambda \in \Omega : (\forall q, r \in Q)(\forall x \in T \upharpoonright \lambda)(\exists y \in T)[x < t_2(x, q) & m(y) < q & e(y) = r \iff (\exists y \in T)[x < t_2(x, q) & m(y) < q & e(y) = r]] \}. $$

Clearly $C \subseteq \Omega$. Since $\{ \lambda \in \Omega : e \upharpoonright (T \upharpoonright \lambda) = Z_2 \} \subseteq \emptyset$, we can pick $\lambda \in C$ such that $e \upharpoonright (T \upharpoonright \lambda) = Z_2$. Since $Z_2$ is an embedding: $T \upharpoonright \lambda \to Q$, $u_2 \in T_2$. Pick $n$ so that $q_n = e(u_2) \in Q \cap [0, 1)$. Let $\langle y_\omega : i < \omega \rangle$ be the sequence that was used to define $u_2$. Then $y_\omega < t_2 u_2 & e(u_2) = q_n & m(u_2) \leq r_{n+1} < r_n$. Since $\lambda \in C$, it implies that $\langle y_\omega \in T \upharpoonright \lambda \rangle[y_\omega < t_2 y & m(y) < r_n & e(y) = q_n]$. Since $e \upharpoonright (T \upharpoonright \lambda) = Z_2$, by (e), $Z_2(y_\omega) = q_n$. Thus $e(y_\omega) = q_n = e(u_2)$ and $y_\omega < t_2 u_2$; This is absurd since $e$ is a $Q$-embedding.

Theorem 7 is thus proved.

Theorem 8 (\textcircled{\textcircled{C}}). \textit{There is an $R$-embeddable $\omega_1$-tree with club antichains which is not special Aronszajn.}

Proof. A slight modification of the proof of Theorem 5. The construction of $T$, an $\omega_1$-subtree of $T$, is carried out to ensure that $T$ satisfies the following condition:

(1) if $\alpha < \beta < \omega_1$ and $x \in T_\alpha$ and $q \in Q \cap (m(x), 2)$, there is a $y \in T_\beta$ such that $x < t_2 y & m(y) = q$.

Let $\langle Z_\alpha : \alpha < \omega_1 \rangle$ be a $\Diamond_{\omega_1}$-sequence.

We set:

$$T_0 = \{ 0 \};$$

$$T_{\alpha+1} = \{ x \cup \{ \langle q, \alpha + 1 \rangle : x \in T_\alpha \text{ and } q \in Q \cap (m(x), 2) \} \} .$$

Let $\lambda \in \Omega$. With each $x \in T \upharpoonright \lambda$ and each $q \in Q \cap (m(x), 2)$, we associate a sequence $\langle x_\alpha : n < \omega \rangle$ such that:

$$x_0 = x \text{ and } x_n < T x_{n+1} \in T \upharpoonright \lambda \text{ and } \lim_{n < \omega} \text{ht}(x_n) = \lambda \text{ and } \lim_{n < \omega} m(x_n) = q.$$ 

And we put:

$$t_2(x, q) = \bigcup_{n < \omega} x_n \cup \{ \langle q, \lambda \rangle \} .$$
On a classification of aronszajn trees

Case 1. If $Z_\lambda$ is not an embedding: $T \uparrow \lambda \rightarrow Q$, then we set:

$$T_\lambda = \{ t_\lambda(x, q): x \in T \uparrow \lambda \& q \in (m(x), 2) \}.$$ 

Case 2. If $Z_\lambda$ is an embedding: $T \uparrow \lambda \rightarrow Q$, we need preliminaries to define $T_\lambda$. Let $<q_i: i < \omega>$ be an enumeration of all elements of $Q \cap (0, 1)$. Let $\langle \lambda_i: i < \omega \rangle$ be a sequence of ordinals such that $\sup_{i < \omega} \lambda_i = \lambda$. Pick $y_n, y_n^\ast \in T \uparrow \lambda$ and $r_n \in R$ inductively for $n < \omega$ so that the following hold:

(a) $y_0 = 0 \& r_0 = 1$;
(b) if $(\exists y \in T \uparrow \lambda)[y_n <_T y \& Z_\lambda(y) = q_n \& m(y) < r_n]$, then $y_n <_T y_n^\ast \& Z_\lambda(y_n^\ast) = q_n \& (m(y_n^\ast) < r_n)$; otherwise, $y_n^\ast = y_n$;
(c) $y_n^\ast <_T y_{n+1} \& \lambda_n \subseteq \text{ht}(y_{n+1}) \& m(y_{n+1}) < r_n$;
(d) $r_{n+1} = (m(y_{n+1}) + r_n)/2$.

Put $u_\lambda = \bigcup_{n < \omega} y_n \cup \{ \sup_{n < \omega} m(y_n), \lambda \}$. We set:

$$T_\lambda = \{ u_\lambda \} \cup \{ t_\lambda(x, q): x \in T \uparrow \lambda \& q \in Q \cap (m(x), 2) \}.$$ 

The tree $T = \bigcup \{ T_\alpha: \alpha < \omega_1 \}$ is as required. Clearly $T \in RE$. It is easily checked that the set $\{ x \in T: m(x) = 1 \}$ is a club antichain. $T \in SAT$ can be proved by the same argument as in the proof of $T \in SAT$ in the proof of Theorem 7, q.e.d.

**Theorem 9 ($\Diamond^*$).** There is an almost-Souslin, Aronszajn tree which is not $R$-embeddable and has not property $\gamma$.

**Proof.** By $\Diamond^*$, we can obtain $T_\theta \in RE \cap AST \setminus ST$ from Theorem 3. Let $T_1 \in ST$. We may assume $T_\theta \cap T_1 = \emptyset$. Put $T = T_\theta \cup T_1$ and define an ordering $<_T$ on $T$ by $<_T y \iff [x, y \in T_\theta \& x < y$ in $T_\theta] \lor [x, y \in T_1 \& x < y$ in $T_1]$. It is easily checked that the tree $T$ is as required, q.e.d.

**Theorem 10 ($\Diamond$).** There is an Aronszajn tree which contains no club antichain and is neither almost-Souslin nor $R$-embeddable.

**Proof.** Let $T_\theta \in NCA \cap RE \setminus AST$ (see Theorem 4 or 7). Let $T_1 \in ST$. Define $T$ from $T_\theta$ and $T_1$ in the same way as in the proof of Theorem 9. The tree $T$ is obviously as required.

**Theorem 11 ($\Diamond$).** There is an Aronszajn tree which contains a club antichain and is not $R$-embeddable.

**Proof.** Let $T_\theta \in RE \setminus NCA$ (see Theorem 5) and $T_1 \in ST$. Then the tree $T$
obtained from $T_0$ and $T_1$ in the same way as in the proof of Theorem 9 is as required.

§ 4. Remarks.

An Aronszajn tree $T$ is called non-Souslin if every uncountable subset of $T$ contains an uncountable antichain (see [1]). This property follows from $R$-embeddability. But J.E. Baumgartner has announced that the converse is not true, in Theorem 2 in [1] in the following form:

If $V=L[A]$ for some $A \subseteq \omega_1$, then there are non-Souslin trees which are not $R$-embeddable.

This property is also interesting. Unfortunately we have not obtained positive results about it.

For $x \in T$, let $T_x$ mean the $\omega_1$-tree $\{y \in T : x \leq_T y\}$ with $<_T$. In connection with Theorem 9, 10 and 11, we could show the following, although they might be less interesting:

1. $(\exists T \in AT)(\forall x \in T)[T_x \in AST \setminus (RE \cup \gamma ST)];$
2. $(\exists T \in AT)(\forall x \in T)[T_x \in NCA \setminus (AST \cup RE)];$
3. $(\exists T \in AT)(\forall x \in T)[T_x \not\in NCA \cup RE].$

References


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