A RELATION BETWEEN $k$-th $UV^{k+1}$ GROUPS AND $k$-th STRONG SHAPE GROUPS

By

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1. Introduction

Compacta $X$ and $Y$ are $UV^n$-equivalent provided that there exist sequences \( \{E_i\}_{i \in \mathbb{N}} \) and \( \{F_i\}_{i \in \mathbb{N}} \) of compacta and sequences \( \{f_i\}_{i \in \mathbb{N}} \) and \( \{g_i\}_{i \in \mathbb{N}} \) of $UV^n$-maps \( f_i : E_i \to F_{i-1} \) and \( g_i : E_i \to F_i \), where \( F_0 = X \) and \( F_m = Y \). Replacing $UV^n$-maps with $CE$-maps, we have the definition of $CE$-equivalence.

![Diagram](image)

It is well known that finite-dimensional $CE$-equivalent compacta are shape equivalent (see [D-S]). The first example that shows the gap between shape equivalence and $CE$-equivalence was found by Ferry [Fe1]. In [Fe3], it was shown that $UV^n$-equivalent $n$-dimensional compacta are shape equivalent. Next Daverman and Venema [D-V] constructed an $n$-dimensional $LC^{n-2}$-continuum which is shape equivalent but not $UV^{n-1}$-equivalent to $S^1$. Mrozik [Mr1] obtained a method to have continua which are shape equivalent but not $UV^1$-equivalent to each other. Moreover Mrozik [Mr2] improved the method and had a strategy to construct a $LC^n$-continuum $Y$ from any $LC^{n+1}$-continuum $X$ with $\pi_t(X)$ infinite such that they are shape equivalent but not $UV^{n+1}$-equivalent.

As a criterion of $UV^n$-equivalence he introduced the notions of $UV^n$-component $\pi^{(n)}_k(X)$ [Mr1], $k$-th $UV^n$-homotopy group $\pi^{(n)}_{k+1}(X)$ and $k$-th $CE$-homotopy group $\pi^{CE}_k(X)$ [Mr2]. Venema [Ve] investigated the groups and showed that $\pi^{(k+1)}_k(X) = \pi^{(k+2)}_k(X) = \cdots = \pi^{CE}_k(X)$ for every continuum $X$ and that $\pi^{(n)}_n(Y) = 0$ for every $UV^n$-continuum $Y$.

In this paper we consider a relation between $\pi^{(k+1)}_k(X)$ and the $k$-th strong shape group $\pi^*_k(X)$ [Q]. We define a natural homomorphism $s_k : \pi^{(k+1)}_k(X) \to \pi^*_k(X)$ and show that, if pro-$\pi_k(X)$ is profinite, $s_k$ is an isomorphism. As its
consequence we have that if \( \text{pro-} \pi_{X}(X) \) is profinite, and \( \pi_{n}^{(k)}(X) = \{X\} \) and \( \pi_{*}^{(k+1)}(X) = 0 \) for \( k = 1, \ldots, n \), then a continuum \( X \) is \( UV^n \).

2. Definitions and lemmas.

By the Hilbert cube \( Q \), we mean the countable product of closed unit intervals \( I = [0, 1] \). By \( S^k \) and \( D^k \), we denote the \( k \)-sphere and the \( k \)-ball, respectively. For each \( k \in \mathbb{N} \), a compactum is a \( UV^k \)-compactum provided that for every embedding \( i : X \to M \) of \( X \) into an ANR \( M \) and every neighborhood \( U \) of \( i(X) \) in \( M \), there is a neighborhood \( V \) of \( i(X) \) in \( M \) such that \( U \supseteq V \) and the homomorphism \( \pi_j(V) \to \pi_j(U) \) induced by the inclusion is trivial for \( j \geq k \). For each compacta \( X \) and \( Y \), a surjective map \( f : X \to Y \) is \( UV^k \)-compact provided that each point preimage \( f^{-1}(y) \) is a \( UV^k \)-compactum. For a subspace \( Z \) of \( X \) and \( x \in X \), by \( d(x, Z) \) we denote the number \( \inf \{d(x, z) | z \in Z\} \), and set \( N_{\varepsilon}(Z) = \{x \in X | d(x, Z) < \varepsilon\} \).

If \( X \) and \( Y \) are compact metric spaces and \( j : Y \to W \) is an embedding into a compact AR \( W \), then an approaching map \( \xi : X \to Y \) is a pair \( (f, j) \), where \( f \) is a map \( f : X \times [0, \infty) \to W \) such that for each neighborhood \( U \) of \( j(Y) \), there is an \( m \in \mathbb{N} \) such that \( f(X \times [m, \infty)) \subseteq U \). Two approaching maps \( f, g : X \to Y \) \((f = (f, j), \ g = (g, j)) \) are homotopic through approaching maps, if there is an approaching map \( H : X \times I \to Y \) \((H = (H, j)) \) such that \( H|X \times \{0\} = f \) and \( H|X \times \{1\} = g \) [Fe2].

Let \( h : X \to Y \) be a map and let \( i : X \to Q \) and \( j : Y \to Q \) be embeddings. Define an embedding \( l : X \to Q \times Q \) by \( l(x) = (j \circ h(x), i(x)) \) and the projection \( \text{proj} : Q \times Q \to Q \) by \( \text{proj}(a, b) = a \). We assume that \( X \subseteq Q \times Q \) by the above embedding \( l \), and \( \text{proj} | X = h \). We take the metric on \( Q \times Q \) to be the supremum of the metrics on two factors.

**Lemma 1.** Let \( h : X \to Y \) be a \( UV^k \)-map as above. If \( P \) is a finite \( k \)-dimensional polyhedron, \( S \) is a subpolyhedron of \( P \), \( \xi = (f, j) : P \to Y \) is an approaching map, and \( g = (g, l) : S \to X \) is an approaching map with \( \text{proj} \circ g = f | S \times [0, \infty) \), then there is an extension \( g^* : P \times [0, \infty) \to Q \times Q \) of \( g \) such that \( (g^*, l) \) is an approaching map and that \( \xi \) and \( (\text{proj} \circ g^*, f) \) are homotopic through approaching maps.

**Proof.** By Corollary 1.2 of [Fe3], we get a sequence \( \{\delta_n\}_{n \geq 0} \) of positive numbers satisfying:

1. \( \delta_n < \min \{\delta_{n-1}, 1/2^n\} \) for \( n \geq 1 \), \( \delta_0 < 1 \) and
2. for any finite \((k+1)\)-dimensional polyhedron \( K \), subpolyhedron \( L \) of \( K \), map \( \alpha : K \to N_{\delta_n}(Y) \) and map \( \alpha_0 : L \to N_{\delta_n}(X) \) with \( \text{proj} \circ \alpha_0 = \alpha | L \), there exists an extension \( \alpha^* : K \to N_{\delta_{n-1}}(X) \) of \( \alpha_0 \) such that \( \text{proj} \circ \alpha^* = \alpha \).
Since $\bar{f}$ is an approaching map, there is a monotone increasing sequence $\{i_n\}_{n \geq 1}$ with $f(P \times [i_n, \infty)) \subseteq \mathbb{N} \delta_n (X)$ for $n \geq 1$. For each $n \in \mathbb{N}$ set $f_n = f|_{P \times [i_n, i_{n+1}]}$. By (2) we get an extension $g_n : P \times [i_n, i_{n+1}] \to \mathbb{N} \delta_n (X)$ of $g|_{P \times [i_n, i_{n+1}]}$ with proj$g_n = f_n$. For each $n \in \mathbb{N}$, define $H_n : P \times I \to \mathbb{N} \delta_n (Y)$ by $H_n(x, t) = f_n(x, i_{n+1})$ for each $x \in P$ and $t \in I$, and $H_{n,0} : P \times \{0, 1\} \to \mathbb{N} \delta_n (X)$ by $H_{n,0}(x, 0) = g_n(x, i_{n+1})$, $H_{n,0}(x, 1) = g_{n+1}(x, i_{n+1})$ for each $x \in P$. And by (2) there exists an extension $H^*_n : P \times I \to \mathbb{N} \delta_n (X)$ of $H_{n,0}$ with proj$H^*_n = H_n$. Define $g^*_n : P \times [i_n, i_{n+1}] \to Q \times Q$ as $g^*_n(x, (1-t)i_n + ti_{n+1}) = \begin{cases} g_n(x, (1-2t)i_n + 2ti_{n+1}) & \text{if } t \in [0, 1/2] \\ H^*_n(x, 2t-1) & \text{if } t \in [1/2, 1] \end{cases}$

Then $g^* = \bigcup_{n \in \mathbb{N}} g^*_n : P \times [i_0, \infty) \to Q \times Q$ is a desired extension of $g$ and the proof is finished.

For each pointed compactum $(X, x_0)$ and each $k \geq 1$, let $UV^m_k(X, x_0)$ be the class of all triples $\Delta = (C, \alpha, \beta)$, where $C$ is a $UV^m$ compactum and $\alpha : S^{k-1} \to C$, $\beta : C \to X$ are maps with $\beta \circ \alpha(S^{k-1}) = \{x_0\}$. Given two such triples $\Delta = (C, \alpha, \beta)$ and $\Delta' = (C', \alpha', \beta')$, we write $\Delta' \geq \Delta$ if there exists a map $\gamma : C' \to C$ such that commutativity holds in each triangle of the following diagram.

Let $\equiv$ denote the equivalence relation generated by $\geq$ (i.e. $\Delta' \equiv \Delta$ iff there exists a sequence of triples $\Delta_i = \Delta, \Delta_2, \ldots, \Delta_{2r+1} = \Delta'$ in $UV^m_k(X, x_0)$ such that $\Delta_i \geq \Delta_{i+1}, i = 1, \ldots, r$) and let $\pi_k^{(m)}(X, x_0) = UV^m_k(X, x_0) / \equiv$. The equivalence class of $\Delta = (C, \alpha, \beta)$ in $\pi_k^{(m)}(X, x_0)$ will be denoted by $[\Delta] = [C, \alpha, \beta]$.

Let $\kappa : S^{k-1} \to (S^{k-1}, *) \vee (S^{k-1}, *)$ denote the usual comultiplication map on the $H$-cogroup $S^{k-1}$ and $\mu : (X, x_0) \vee (X, x_0) \to X$ the folding map. For $[\Delta_i] = [C_i, \alpha_i, \beta_i] \in \pi_k^{(m)}(X, x_0), i = 1, 2$, define a multiplication by

$$([\Delta_1][\Delta_2] = [(C_1, \alpha_1(*)) \vee (C_2, \alpha_2(*)), (\alpha_1 \vee \alpha_2) \kappa, \mu \circ (\beta_1 \vee \beta_2)].$$

Obviously this is a group multiplication on $\pi_k^{(m)}(X, x_0)$: The neutral element is $\Delta x_0 = [(*), \text{const}, \text{const}]$, where const is the constant map. An inverse for $[\Delta] = [C, \alpha, \beta]$ is given by $[\Delta^{-1}]$, where $\Delta^{-1} = (C, \alpha \vee \beta)$ and $\nu : S^{k-1} \to S^{k-1}$ is the usual homotopy inverse on the $H$-cogroup $S^{k-1}$ (see [Mr2]).

Lemma 2. Let $(X, x_0)$ be a pointed compactum and $k \geq 1$. Then for each $[C, \alpha, \beta] \in \pi_k^{(k+1)}(X, x_0)$, there exists a $[C', \alpha', \beta'] \in \pi_k^{(k+1)}(X, x_0)$ such that
(1) \( \alpha' : S^{k-1} \to C' \) is an embedding.
(2) \( \dim C' \leq k + 2 \) and
(3) \( [C, \alpha, \beta] = [C', \alpha', \beta'] \).

**Proof.** By Theorem 2.1.9 of [Be], there exists a compactum \( C^* \) with \( \dim C^* \leq k + 2 \), and a \( UV^{k+1} \)-map \( f : C^* \to C \). Since \( C \) is \( UV^{k+1} \), \( C^* \) is \( UV^{k+1} \). Let \( i : C^* \to Q \) and \( j : C \to Q \) be embeddings. Define \( l : C^* \to Q \times Q \) by \( l(x) = (j \circ f(x), i(x)) \). For convenience we may assume that \( \text{proj}|C^* = f \) as before. Moreover define a \( UV^{k+1} \)-map \( \varphi : S^{k-1} \times [0, \infty) \to Q \) by \( \varphi(x, t) = a(x) \) for each \( x \in S^{k-1} \) and \( t \in [0, \infty) \). By the proof of Lemma 1 there exists a map \( \varphi^* : S^{k-1} \times [0, \infty) \to Q \times Q \) such that \( (\varphi^*, I) \) is an approaching map and \( \text{proj} \circ \varphi^* = \varphi \). The mapping cylinder \( M(\varphi^*) \) of \( \varphi^* \) is the space obtained from \( (S^{k-1} \times [0, \infty) \times I) \cup (\varphi^*(S^{k-1} \times [0, \infty)) \cup C^*) \) by identifying for each \( y \in \varphi^*(S^{k-1} \times [0, \infty)) \) the set \( (\varphi^*(y) \times \{1\}) \cup \{y\} \) to a single point. Identifying of \( C^* \) and \( S^{k-1} \times [0, \infty) \times [0, 1) \) as subspaces of \( M(\varphi^*) \), we set
\[
M^*(\varphi^*) = C^* \cup \{x, s/(1+s) \in M(\varphi^*) | x \in S^{k-1}, s \in [0, \infty)\}.
\]
Then \( M^*(\varphi^*) \) is \( UV^{k+1} \). Let \( r : M(\varphi) \to \varphi^*(S^{k-1} \times [0, \infty)) \cup C^* \) be the natural retraction of the mapping cylinder and define an embedding \( \alpha' : S^{k-1} \to M^*(\varphi^*) \) by \( \alpha'(x) = [x, 0, 0] \). Since we can obtain a commutative diagram:

\[
\begin{array}{ccc}
S^{k-1} & \longrightarrow & X \\
\alpha' \downarrow & & \downarrow \text{proj} \circ r \\
\alpha \downarrow & & \downarrow \beta \\
C & \longrightarrow & X
\end{array}
\]

we infer \( [M^*(\varphi^*), \alpha', \beta^* \circ \text{proj} \circ r | M^*(\varphi^*)] = [C, \alpha, \beta] \in \pi_k^{k+1}(X, x_0) \).

If \( X \) and \( Y \) are compact metric ANR's, a map \( p : X \to Y \) is said to have the **approximate homotopy lifting property** (AHLP) with respect to a compact space \( Z \) if for every homotopy \( f : Z \times I \to Y \), map \( F_0 : Z \to X \) with \( p \circ F_0 = f \upharpoonright Z \times \{0\} \), and \( \varepsilon > 0 \) there is a map \( F : Z \times I \to X \) such that \( F_0 = F \upharpoonright Z \times \{0\} \) and \( d(p \circ F(z, t), f(z, t)) < \varepsilon \) for each \( (z, t) \in Z \times I \). We will call \( p \) an **AF*-map** if \( p \) has the AHLP for all \( n \)-dimensional compacta.

For a finite or infinite inverse sequence \( \{(X_i, f_i)\} \) of compacta, \( \text{CMap}^\ast((X, f_i)) \) is defined by S. Ferry, (Definition 5.2, [Fe2]). We remark that the inverse limit \( \varprojlim (X_i, f_i) \) is regarded a subspace of \( \text{CMap}^\ast((X, f_i)) \) and that if the spaces \( X_i \)'s are ANR's, then \( \text{CMap}^\ast((X_i, f_i)) \) is an AR.

Next we shall define a homomorphism \( t_k : \pi_k(X, x_0) \to \pi_k^{k+1}(X, x_0) \). For
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each $\beta \in \pi_k(X, x_0)$, where $\beta : D^k \to X$ is a map with $\beta(S^{k-1}) = \{x_0\}$, define $t_k(\beta) = [D^k, \text{incl}, \beta]$. Here, incl : $S^{k-1} \to D^k$ is the inclusion map \cite{Mr2}.

**Lemma 3.** If $X = \lim(K_i, f_i)$, where each $K_i$ is a finite polyhedron and each $f_i$ is an $AF^i$-map, then the homomorphism $t_k : \pi_k(X, x_0) \to \pi_k^{(k+1)}(X, x_0)$ is isomorphic for each $k \geq 1$.

**Proof.** a) Injectivity. Let $\beta$ be a map $\beta : D^k \to X$ such that $t_k(\beta) = [D^k, \text{incl}, \beta] \in \pi_k^{(k+1)}(X, x_0)$. By the proof of Theorem 2.7 in \cite{Mr2}, there exist $\beta^* : D_+^{k+1} \to X$ isomorphic compactum $C$ and maps satisfying the following commutative diagram:

Define $\gamma : S^k \to C$ by $\gamma|_{\text{upper hemisphere}} = \gamma_+$, $\gamma|_{\text{lower hemisphere}} = \gamma_-$. Let $i : C \to Q$ be an embedding. Since $C$ is $UV^{k+1}$, we get a map $\gamma^* : D^{k+1} \times [0, \infty) \to Q$ such that $(\gamma^*, i)$ is an approaching map and $\gamma^*(x, t) = \gamma(x)$ for each $x \in S^k$, $t \in [0, \infty)$. There is an extension $\beta^* : Q \to \text{CMap}^*(\langle K_i, f_i \rangle)$ of $\beta^*$. By Corollary 5.5 of \cite{Fe2}, there exists a map $g^* : D^{k+1} \times [0, \infty) \to \text{CMap}^*(\langle K_i, f_i \rangle)$ such that $g^*(x, \infty) = \beta^* \gamma(x)$ for each $x \in S^k$, and that $g^*(S^k \times \{\infty\}) \subset X$. Since $[g^*(S^k \times \{\infty\}]) = [\beta^* \gamma] = [\beta] \in \pi_k(X, x_0)$, $[\beta] = 0$.

b) Surjectivity. Let $[C, \alpha, \beta] \in \pi_k^{(k+1)}(X, x_0)$. By Lemma 2 we may assume that $\dim C \leq k+2$ and $\alpha$ is an embedding. Since $C$ is $UV^k$, we get a map $\varphi : D^k \times [0, \infty) \to Q$ such that $\varphi(x, t) = \alpha(x)$ for each $x \in S^{k-1}$ and $t \in [0, \infty)$, and that $(\varphi, i)$ is an approaching map, where $i : C \to Q$ is an embedding. The mapping cylinder $M(\varphi)$ of $\varphi$ is the space obtained from $(S^{k-1} \times [0, \infty) \times I) \cup \{\varphi(D^k \times [0, \infty) \cup C)\}$ by identifying for each $y \in \varphi(D^k \times [0, \infty))$ the set $(\varphi^{k-1}(y) \times \{1\}) \cup \{y\}$ to a single point. Identifying of $C$ and $D^k \times [0, \infty) \times [0, 1)$ as subspaces of $M(\varphi)$, we set

$$M^*(\varphi) = C \cup [x, s, s/(1+s)] \in M(\varphi) | x \in D^k, s \in [0, \infty)) \sqcup M^*(\varphi)$$

(see Lemma 2).

Let $j : M^*(\varphi) \to Q$ be an embedding. We will construct a map $\phi : M^*(\varphi) \times [0, \infty) \to Q$ with $(\phi, j)$ an approaching map satisfying the following condition:
For a while, we assume that there exists a map \( \phi \) as above. Let \( \beta^*: Q \to \text{CMap}^*(K_i, f_i) \) be an extension of \( \beta \) satisfying \( \beta^*([x, s, s/(1+s)]) = \beta(x) \) for each \( x \in S^k - 1 \) and \( s \in [0, \infty) \) and apply Corollary 5.5 of \cite{Fe2} to \( \beta^* \phi \), then there exists a map \( \phi^*: M^*(\varphi) \to X \) with \( \phi^*|M^*(\varphi) = \beta^*|M^*(\varphi) \). Identifying \( D^k \) with \( \{[x, 0, 0] \in M(\varphi) | x \in D^k \} \), and from the following commutative diagram:

and the fact that \( M^*(\varphi) \) and \( C \) are shape equivalent, we have \( t_k([\phi|D^k]) = [C, \alpha, \beta] \). Therefore it is sufficient to construct a map \( \phi \) with the condition (2).

Since \( C \) and \( M^*(\varphi) \) are shape equivalent, \( M^*(\varphi) \) is \( UV^{k+1} \). There exists a sequence \( \{U_n\}_{n \geq -2} \) of neighborhoods of \( M^*(\varphi) \) in \( Q \) such that

1. \( U_n \supset U_{n+1} \) for each \( n \geq -2 \), and
2. for each \( n \geq -2 \), \( l \leq k+1 \) and map \( \alpha: S^l \to U_{n+1} \), there exists an extension \( \alpha^*: D^{k+1} \to U_n \) of \( \alpha \).

Since \( M(\varphi) \supset M^*(\varphi) \), there exists a monotone sequence \( \{s_m\}_{m \geq 0} \) of positive numbers such that \( D^k \times \{s_m\} = \{[x, s_m, s_m/(1+s_m)] \in M(\varphi) | x \in D^k \} \subset U_{m+1} \) for each \( m \geq 0 \). By (2), there exists a map \( \alpha_m: D^k \times U_{m+1} \) with \( \alpha_m(x) = [x, 0, 0] \in M^*(\varphi) \) for each \( x \in S^k - 1 \). Identifying \( D^k \times [0, s_m] \) with \( \{[x, s, s/(1+s)] \in M(\varphi) | x \in D^k \} \subset U_{m+1} \), by (2) we have a map \( \phi_m^*: D^k \times [0, s_m] \to U_{m+1} \) such that

\[
\phi_m^*(x, t) = \alpha_m(x) \quad \text{for each } x \in D^k, \quad \text{and} \quad \phi_m^*(x, t) = [x, t/(1+t)] \quad \text{for each } (x, t) \in S^{k-1} \times [0, s_m] \cup D^k \times \{s_m\}.
\]

Since \( \phi_m^*(D^k \times [0]) \cup \phi_{m+1}^*(D^k \times [0]) \subset U_{m+1} \), by (2) there exists a map \( \phi_{m+1}^*: D^{k+1} \to U_{m+1} \) with \( \phi_{m+1}^* \) the upper hemisphere \( = \phi_m^*|D^k \times [0] \) and \( \phi_{m+1}^* \) the lower hemisphere \( = \phi_{m+1}^*|D^k \times [0] \). Applying (2) to \( D^k \times [s_m, s_{m+1}] \subset U_{m+1} \), and three maps \( \phi_m^* \), \( \phi_{m+1}^* \) and \( \phi_{m+1}^* \), then we get a map \( \phi_{m, m+1}^*: D^k \times [0, s_m] \times [m, m+1] \to D_{m+1} \) satisfying that

\[
\phi_{m, m+1}^*(x, t, m) = \phi_m^*(x, t) \quad \text{if } (x, t) \in D^k \times [0, s_m],
\]
\[
\phi_{m, m+1}^*(x, t, m) = [x, t/(1+t)] \quad \text{if } (x, t) \in D^k \times [s_m, s_{m+1}] \text{ and}
\]
\[
\phi_{m, m+1}^*(x, t, m+1) = \phi_{m+1}^*(x, t) \quad \text{if } (x, t) \in D^k \times [0, s_m].
\]

For each \( m \geq 0 \) define \( p_m: \{D^k \times [s_m, \infty) \cup C \} \times [m, m+1] \to M^*(\varphi) \) by \( p_m(x, t, s) = [x, t, t/(1+t)] \) for each \( (x, t, s) \in D^k \times [s_m, \infty) \times [m, m+1] \), and
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\[ p_m(y, s) = y \text{ for each } (y, s) \in C \times [m, m+1]. \]

We set

\[ \phi_{m, m+1} = \phi^*_{m, m+1} \cup p_m : M**(\varphi) \times [m, m+1] \to U_{m-2}, \]

and

\[ \phi = \bigcup_{m \in \mathbb{N}} \phi_{m, m+1} : M**(\varphi) \times [0, \infty) \to U_{-2}. \]

Clearly by the construction as above, the map \( \phi \) satisfies the condition (\#).

### 3. Main results

The \( k \)-th homotopy pro-group, the \( k \)-th shape group and the strong shape group of a space \( X \) are denoted \( \pi_k(X), \pi^s_k(X) \) and \( \pi^{sk}_k(X) \), respectively. We will construct a homomorphism \( s_\alpha : \pi_k(X, x_0) \to \pi^s_k(X, x_0) \). Let \( [C, \alpha, \beta] \in \pi_k^{k+1}(X, x_0) \) and let \( i : C \to Q \) be an embedding. Since \( C \) is \( UV^{k+1} \), there exists a map \( \phi_C : D^k \times [0, \infty) \to Q \) such that \( \phi_C(x, t) = \alpha(x) \) for each \( x \in S^{k-1} \) and \( t \in [0, \infty) \), and that \( (\phi_C, i) \) is an approaching map. Suppose that \( X = \lim (K_i, f_i) \), where \( K_i \)'s are finite polyhedra, then there exists a map \( \beta^* : Q \to \text{Map}^{*}(\{K_i, f_i\}) \) which is an extension of \( \beta \). Define \( s_\alpha : \pi_k^{k+1}(X, x_0) \to \pi^s_k(X, x_0) \) by \( s_\alpha([C, \alpha, \beta]) = [\beta^* \phi_C] \). Since \( C \) is \( UV^{k+1} \), the definition as above is independent of a choice of \( \phi_C \). By the proof of Theorem 2.7 in [Mr2], if \( [C, \alpha, \beta] = [C', \alpha', \beta'] \), there exists the following commutative diagram:

Here \( \gamma_+ \) and \( \gamma_- \) are embeddings and \( [C'', \alpha'', \beta''] \in \pi_k^{k+1}(X, x_0) \). By the commutative diagram as above,

\[ [\beta^* \phi_C] = [\beta^* r^* \phi_{M(\alpha)}] = [\beta^* \phi_{\alpha''}] = [\beta^* r^* \phi_{M(\alpha')} ] = [\beta^* \phi_{\alpha''} ] \in \pi^s_k(X, x_0). \]

\( s_\alpha \) turns out to be well-defined. Clearly \( s_\alpha \) is a homomorphism.

An inverse sequence \( \{G_i, h_i\} \) of groups and homomorphisms is profinite if for each \( i \) there is a \( j > i \) such that \( \text{im} h_{i+1} \cdots h_j(G_j) \subseteq G_i \) is finite. A continuum \( X \) has pro-\( \pi_k(X) \) profinite if whenever \( X \) is written as an inverse limit \( X = \lim (K_i, \alpha_i) \) of finite CW complexes, the system \( \{\pi_k(K_i), \alpha_i\} \) is profinite.

**Main Theorem.** If \( (X, x_0) \) is a pointed continuum with pro-\( \pi_1(X) \) profinite,
then $\pi_k^{(k+1)}(X, x_0)$ and $\pi_k(X, x_0)$ are isomorphic for each $k \geq 1$.

PROOF. We will show that $s_k$ is an isomorphism.

First we may consider a special case that $f'_i$ is an $AF^i$-map for each $i \geq 1$. Then we will construct a homomorphism $u_k : \pi_k(X, x_0) \to \pi_k^{(k+1)}(X, x_0)$. Let $\varphi : S^k \times [0, \infty) \to \text{CMap}^*((K_i, f'_i))$ such that $\varphi([s_0] \times [0, \infty)) = \{x_0\}$, where $s_0$ is the basepoint of $S^k$, and such that $(\varphi, f)$ is an approaching map, where $j : X \to \text{CMap}^*((K_i, f'_i))$ is the inclusion. By Corollary 5.5 of [Fe2], there exists a map $\varphi' : S^k \times [0, \infty) \to \text{CMap}^*((K_i, f'_i))$ for each $i \geq 1$. Then we will construct a homomorphism $u_k : \pi_k(X, x_0) \to \pi_k^{(k+1)}(X, x_0)$. Let $\varphi : S^k \times [0, \infty) \to C_{\text{Map}^*((K_i, f'_i))}$ such that $\varphi([s_0] \times [0, \infty)) = \{x_0\}$, where $s_0$ is the basepoint of $S^k$, and such that $(\varphi, j)$ is an approximating map, where $j : X \to C_{\text{Map}^*((K_i, f'_i))}$ is the inclusion. By Corollary 5.5 of [Fe2] and [Mr2], $u_k$ is well-defined. It is clear that $s_k \circ u_k = \text{id}$. Since $t_k : \pi_k(X, x_0) \to \pi_k^{(k+1)}(X, x_0)$ is an isomorphism by Lemma 3, for each $[C, \alpha, \beta] \in \pi_k^{(k+1)}(X, x_0)$ there exists a map $\gamma : D^k \to X$ such that $\gamma(S^k-1) = \{x_0\}$ and $[C, \alpha, \beta] = [D^k, \text{incl}, \gamma]$, where $\text{incl} : S^k-1 \to D^k$ is the inclusion. Because of Corollary 5.5 of [Fe2] and [Mr2], $u_k \circ s_k = \text{id}$. That is, $s_k$ is an isomorphism.

Next we consider the general case. Since pro-$\pi_i(X)$ is profinite, by Theorem 3' and Lemma 3.2 of [Fe2], there exists a continuum $X'$ such that $X'$ and $X$ are shape equivalent and $X' = \lim (K'_i, f'_i)$, where $K'_i$'s are finite polyhedra and $f'_i$'s are $AF^i$-maps. Moreover, by Theorem 2 of [Fe3], $X'$ and $X$ are $UV^n$-equivalent for each $n \geq 0$. There exist a compactum $X''$ and $UV^{k+1}$-maps $\xi_1 : X'' \to X$, $\xi_2 : X'' \to X'$. Let $x''_0 \in X''$ with $\xi_1(x''_0) = x_0$. 

\[
\begin{array}{cccccc}
\pi_k^{(k+1)}(X, x_0) & \xrightarrow{\xi_1^*} & \pi_k^{(k+1)}(X'', x''_0) & \xrightarrow{\xi_2^*} & \pi_k^{(k+1)}(X', \xi_2^*(x''_0)) \\
\downarrow \quad s_k & \downarrow s_k' & \downarrow s_k' & \downarrow s_k' & \downarrow s_k' \\
\pi_k(X, x_0) & \xrightarrow{\xi_1^*} & \pi_k(X'', x''_0) & \xrightarrow{\xi_2^*} & \pi_k(X', \xi_2^*(x''_0)) .
\end{array}
\]

By Theorem 1.6 of [Mr2], $\xi_1$ and $\xi_2$ are isomorphisms, and by Lemma 1, $\xi_1^*$ and $\xi_2^*$ are isomorphisms. Since this diagram is commutative and $s_k'$ is an isomorphism, so is $s_k$.

A space $X$ will be called $UV^n$-connected provided that for any two points $x, x' \in X$ there exist a $UV^n$-compactum $C$ and a map $\gamma : C \to X$ with $x, x' \in \gamma(X)$. By a $UV^n$-component of $X$ we mean a maximal $UV^n$-connected subspace of $X$. Denote $\pi_0^{(n)}(X)$ the set of all $UV^n$-components of $X$.

LEMMA 4. Let $X$ be a continuum. If $\pi_0^{(1)}(X) = \{X\}$, then $\pi_0(X, x_0) = 0$ for
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PROOF. Let $x_0 \in X$ be an arbitrary point. Since $\pi_0(X) = 0$, for each $x_i \in X$ there exist a $UV^1$-compactum $C$ and a map $\gamma : C \to X$ with $x_0, x_i \in \gamma(C)$. Let $M$ and $M'$ be AR's, $i : C \to M$ and $j : X \to M'$ be embeddings and $\gamma^* : M \to M'$ be an extension of $\gamma$. Taking points $y_0, y_1 \in C$ with $\gamma^*(y_0) = x_0, \gamma^*(y_1) = x_1$, since $C$ is $UV^1$, there exists a map $\phi : I \times [0, \infty) \to M$ such that $(\phi, i)$ is an approaching map and $\phi(t, \delta) = y_2$ for each $t \in [0, \infty)$ and $\delta \in [0, 1]$. Since $(\gamma^* \phi, j)$ is an approaching map, $\pi_0(X, x_0) = 0$.

COROLLARY. Let $X$ be a continuum with $pro-\pi_k(X)$ profinite. If $\pi_0(X) = X$ and $\pi_k^{(k+1)}(X, x_0) = 0$ for each $x_0 \in X$ and $k = 1, 2, \ldots, n$, then $X$ is $UV^n$.

PROOF. It follows from Main theorem and Lemma 4 that $\pi_k(X, x_0) = 0$ for each $x_0 \in X$ and $k = 0, 1, \ldots, n$. By [Wa] $\lim\sup (pro-\pi_{k+1}(X, x_0)) = 0 = \pi_k(X, x_0)$. Moreover by Theorem 11 and Lemma 2 of Theorem 12 in §6.2 [M-S], $pro-\pi_k(X, x_0) = 0$ for each $x_0 \in X$ and $k = 1, 2, \ldots, n$. Since $X$ is connected, $X$ is $UV^n$.

4. Remarks and problems.

Mrozik [Mr2] and Venema [Ve] gave fundamental properties of $k$-th $UV^n$-groups for an arbitrary continuum $X$: $\pi_k^{(k)}(X) = \pi_k^{(k+1)}(X, x_0) = \ldots = \pi_k^{(k+1)}(X) = 0$ and $\pi_k^{(k+1)}(X) = \pi_k^{(k+2)}(X) = \ldots = \pi_k^{(n)}(X)$. Thus the groups have some meaning only in the cases $n = k$ and $k+1$. Moreover Venema showed that, for every $UV^n$-compactum $X$, $\pi_n^{(n)}(X) = 0$. Considering Corollary and Venema's result, we have a natural problem:

PROBLEM 1. Is a continuum $X$ with $\pi_k^{(k)}(X) = 0$ for $k = 1, \ldots, n$, a $UV^n$-compactum?

On the other hand, we clearly have a natural homomorphism $h_{k, k+1}: \pi_k^{(k+1)}(X, x_0) \to \pi_k^{(k)}(X, x_0)$ as follows: for each $[C, \alpha, \beta] \in \pi_k^{(k+1)}(X, x_0)$ where $C$ is $UV^{k+1}$, and $\alpha : S^{k+1} \to C$ and $\beta : C \to X$, define

$$h_{k, k+1}([C, \alpha, \beta]) = [C, \alpha, \beta].$$

However, we do not have any information about $h_{k, k+1}$. It is obvious that if $h_{k, k+1}$ is a monomorphism, Problem 1 has the affirmative answer. Therefore we pose the following problem:

PROBLEM 2. When is the homomorphism $h_{k, k+1}$ a monomorphism? In particular, consider the case that $pro-\pi_k(X)$ is profinite.
References


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