OPTIMUM PROPERTIES OF THE WILCOXON SIGNED RANK TEST UNDER A LEHMANN ALTERNATIVE

By

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1. Introduction.

Let \( X_1, \ldots, X_n \) be a random sample from an absolutely continuous distribution function \( F(x) \). The problem is to test the null hypothesis \( H: F(x) = G(x) \) where \( G'(x) = g(x) \) is assumed to be symmetric about zero. When \( G(x) \) is a logistic distribution function, Hájek and Šidák [1] reviewed that the Wilcoxon signed rank test is locally most powerful among all rank tests against the location alternative \( A: F(x) = G(x - \theta) \) for \( \theta > 0 \) and showed that the test is asymptotically optimum under the contiguous sequence of alternatives \( A_n: F(x) = G(x - d/\sqrt{n}) \) for some \( d > 0 \).

In this paper, we consider the alternative of the contaminated distribution

\[
K: F(x) = (1 - \theta)G(x) + \theta(G(x))^2 \quad \text{for } 0 < \theta < 1.
\]

The alternative \( K \) was introduced by Lehmann [2] for a two-sample problem. In order to get an asymptotic optimum property, we consider the sequence of alternatives

\[
K_n: F(x) = (1 - d/\sqrt{n})G(x) + (d/\sqrt{n})G(x)^2 \quad \text{for } d > 0,
\]

which is included in \( K \) and approaches the null hypothesis \( H \) as \( n \to \infty \). In the following Section, we shall show that the Wilcoxon signed rank test is locally most powerful among all rank tests under \( K \) and is asymptotically most powerful under \( K_n \). Further in Section 3, we shall compare the Wilcoxon signed rank test with the one-sample \( t \)-test by the asymptotic relative efficiency under the contiguous sequence of alternatives of general contaminated distributions

\[
K_n': F(x) = (1 - d/\sqrt{n})G(x) + (d/\sqrt{n})H(G(x)) \quad \text{for } d > 0.
\]

2. Optimum properties.

Taking the absolute values of observations, let \( R_i \) be the rank of \( |X_i| \) among the observations \( \{|X_i|; i = 1, \ldots, n| \) and define \( \text{sign} X = 1 \) for \( X > 0 \), \( 0 \) for \( X = 0 \) and \( -1 \) for \( X < 0 \).

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\(-1\) otherwise. Note that \(\Pr\{\text{sign } X_i = 0\} = 0\) since we consider only absolutely continuous distribution. Then we can describe the Wilcoxon signed rank statistic as the following.

\[
T = \sum_{i=1}^{n}(\text{sign } X_i)R_i. 
\]

At first, we investigate the property of "locally most powerful".

**Theorem 1.** The Wilcoxon signed rank test based on \(T\) defined by (2.1) is locally most powerful for \(H\) versus \(K\) defined by (1.1) among all rank tests.

**Proof.** Putting \(\text{sign } X = (\text{sign } X_1, \ldots, \text{sign } X_n)\) and \(R = (R_1, \ldots, R_n)\), we get for any vector \(v = (v_1, \ldots, v_n)\) such that \(v_i = 1\) or \(-1\) and any permutation \(r = (r_1, \ldots, r_n)\) of \((1, \ldots, n)\), under \(H\), \(\Pr\{\text{sign } X = v\} = 1/2^n\) and \(\Pr\{R = r\} = 1/n!\). Here since the likelihood function of \((X_1, \ldots, X_n)\) under \(K\) is given by

\[
\eta(x) = \prod_{i=1}^{n}((1-\theta)g(x_i) + 2\theta G(x_i)g(x_i))
\]

the joint probability of sign vector \(\text{sign } X\) and rank vector \(R\) is expressed by

\[
\beta(\theta) = \Pr\{\text{sign } X = v, R = r\}
= \sum_{\text{sign } X = v, R = r} \eta(x) dx
= 1/(2^n \cdot n!) + \sum_{i=1}^{n} \prod_{k=1}^{n-1} g(x_k) \prod_{k=1}^{n} ((1-\theta)g(x_k) + 2\theta G(x_k)g(x_k))
\times [(1-\theta)g(x_i) + 2\theta G(x_i)g(x_i)] - \theta g(x_i) dx ,
\]

It follows that

\[
\beta'(0) = \sum_{i=1}^{n} \prod_{k=1}^{n-1} \{ -1 + 2G(x_k) \} \prod_{k=1}^{n} g(x_k) dx .
\]

Let \(|X|^{(i)}\) be the \(i\)-th order statistic among the absolute values \(|X_i|; \, i = 1, \ldots, n\). Since \(|X|^{(i)}\), \(\text{sign } X\) and \(R\) are mutually independent under \(H\) from II 1.3 theorem of Hájek and Šidák [1], we can get

\[
\beta'(0) = 1/(2^n \cdot n!) \cdot \sum_{i=1}^{n} E\{ -1 + 2G(v_i)|X|^{(i)} \}
= 1/(2^n \cdot n!) \cdot \sum_{i=1}^{n} E[v_i G(|X|^{(i)}) - 1]
= 1/(2^n \cdot n!) \cdot \sum_{i=1}^{n} v_i n/(n+1) ,
\]

which implies the result.
Next we shall show the asymptotic optimum property. Corresponding to (2.2), the joint density of \((X_i, \cdots, X_n)\) under \(K_n\) defined by (1.2) is given by

\[
q_d(X) = \prod_{i=1}^{n} \left[ (1 - d/\sqrt{n}) + 2dG(X_i)/\sqrt{n} \right] g(X_i)
\]

**Theorem 2.** The asymptotic power of the Wilcoxon signed rank test is equal to that of the most powerful test for \(H\) versus \(K_n\) when \(d\) and \(G(u)\) are known, having critical region \(\{x; \log \{q_d(x)/g(x)\} \geq t_{na}\}\).

**Proof.** Taylor's series expansion of the logarithm of the likelihood ratio yields

\[
L_d = \log \left( \frac{q_d(X)/g(X)}{\prod_{i=1}^{n} \{ (1 - d/\sqrt{n}) + 2dG(X_i)/\sqrt{n} \} \} \right) \\
= \left( d/\sqrt{n} \right) \sum_{i=1}^{n} [2G(X_i) - 1] - d^2/(2n) \sum_{i=1}^{n} (2G(X_i) - 1)^2 \\
\quad + d^3/(3n\sqrt{n}) \sum_{i=1}^{n} [ (2G(X_i) - 1)^3/(1 + \delta_i(d/\sqrt{n})(2G(X_i) - 1))]^2,
\]

where \(\delta_i\) satisfies \(0 < \delta_i < 1\). Under the null hypothesis \(H\), the first term of the last expression of (2.3), namely \(d/\sqrt{n} \sum_{i=1}^{n} [2G(X_i) - 1]\), has asymptotically a normal distribution with mean 0 and variance \(d^2/3\) by the central limit theorem, the second term converges to \(-d^2/6\) in probability by the law of large numbers and the third term tends to zero in probability.

Thus we get

\[
L_d \xrightarrow{\text{law}} N(\mu, \sigma^2),
\]

where \(\xrightarrow{\text{law}}\) denotes convergence in law and

\[
\mu = -d^2/6 \quad \text{and} \quad \sigma^2 = -2d^2/3.
\]

From VI 1.2 corollary of Hájek and Šidák [1], the family of densities \(\{q_d(x)\}\) is contiguous to \(\{g(x)\}\). So from LeCam’s third lemma stated in VI 1.4 of Hájek and Šidák [1], under \(\{q_d(x)\}\), \(L_d \xrightarrow{\text{law}} N(-\mu, \sigma^2)\), where \(\mu\) and \(\sigma^2\) are defined by (2.5).

Therefore the asymptotic power of the test of level \(\alpha\) with critical region \(L_d > t_{na}\) under \(\{q_d(x)\}\) is

\[
1 - \Phi(z_\alpha - d/\sqrt{3}),
\]

where \(t_{na} = -d^2/6 + z_\alpha d/\sqrt{3} + o(1)\), \(\Phi(\cdot)\) is a distribution function of the standard normal and \(z_\alpha\) is the upper 100\(\alpha\) percentage point of the standard normal distri-
bution. On the other hand, let us put $S = \sum_{i=1}^{n} \frac{(\text{sign } X_i)[2G(|X_i|) - 1]}{\sqrt{n}}$, then $T/(n+1)\sqrt{n} - S$ converges to zero in probability under $H$ from V 1.7 theorem of Hájek and Šidák [1]. Hence $(L_{u}, T/((n+1)\sqrt{n}))$ and $(L_{u}, S)$ have asymptotically the same normal distribution. Also it follows under $H$ that $(L_{u}, S)$ has asymptotically a bivariate normal distribution with mean $(\mu, 0)$ and singular covariance matrix $(\sigma_{11}, \sigma_{22})$, where $\mu$ and $\sigma$ are defined by (2.5), $\sigma_{12} = \sqrt{3}/3$ and $\sigma_{22} = 1/3$. Hence, from LeCam’s third lemma, under $H$ from $q_{l}(x)$, we get that $S$ has asymptotically the normal distribution with mean $\sigma_{11}$ and variance $\sigma_{22}$. Thus the asymptotic power of the test based on $T$ for $H$ versus $K_{n}$ at level $\alpha$ is given by the expression (2.6). This completes the proof.

3. Comparison with the $t$-test under a contiguous sequence of alternatives of general contaminated distributions.

We extend $K_{n}$ defined by (1.2) to the contiguous sequence of alternatives of general contaminated distributions $K'_{n}$ defined by (1.3) and compare the Wilcoxon signed rank test with the $t$-test based on

$$U = \sqrt{n}^{-1} \sum_{i=1}^{n} X_i / \sqrt{n} \sum_{i=1}^{n} (X_i - \bar{X})^2.$$  

Then we get

**Theorem 3.** Suppose that the derivative of $H(u)$ exists and the derivative $h(u) = H'(u)$ is bounded. Then the asymptotic relative efficiency of the Wilcoxon signed rank test with respect to the $t$-test based on $U$ under $K'_{n}$ defined by (1.3) is given by

$$\text{ARE}(T, U) = 3\sigma^{2} \left[ \int_{0}^{1} (2u-1)h(u)du \right]^{2} / \left[ \int_{-\infty}^{\infty} th(G(t))g(t)dt \right]^{2}$$

where $\sigma^{2} = \int_{-\infty}^{\infty} t^{2}dG(t)$.

**Proof.** From the straight similar way to the proof of Theorem 2, $\sqrt{3} \cdot T/(n+1)\sqrt{n}$ has asymptotically a normal distribution with mean $\sqrt{3} \int_{0}^{1} (2u-1)h(u)du$ and variance 1. Further the similar argument as in the proof of Theorem 2 shows that $U$ defined by (3.1) has asymptotically a normal distribution with mean $\int_{-\infty}^{\infty} \frac{th(G(t))dt}{\sigma}$ and variance 1 under $K'_{n}$. The ratio of squares of the two asymptotic means gives the result.

This asymptotic relative efficiency (ARE) equals the ARE of the two-sample
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Wilcoxon test with respect to the two-sample $t$-test under a contiguous sequence of alternatives of contaminated distributions which is given by corollary 2 of Shiraishi [3]. So we find that this ARE is 1 for any bounded function $h(u)$ if $G(x)$ is the distribution function from the uniform random variable on a finite interval. In Table 1 of Shiraishi [3], we showed the values of this ARE for $H(u) = u^k$, $1 - (1 - u)^k$ with $k = 1, 1.1, 1.3, 1.6, 2, 3, 5, 10$ and $G(x) =$ uniform, normal, logistic, double exponential distributions. As the numerical results, ARE's are always nearly equal to 1 irrespective of the form of $H(u)$, $G(x)$ and $k$ chosen.

4. Conclusion.

About the exact power, Theorem 1 gives an admissibility of the Wilcoxon signed rank test for the alternative of contaminated distribution $F(x) = (1 - \theta)G(x) + \theta H(G(x))$, which includes $K$ defined by (1.1), as far as we intend to seek a test having higher exact power among all rank tests. Though we found that there does not exist asymptotically a most powerful rank test under a contiguous sequence of alternatives of contaminated distributions for the two-sample problem from corollary 1 of Shiraishi [3], Theorem 2 shows that the Wilcoxon signed rank test is asymptotically most powerful for $K_n$ defined by (1.2) which is included by $K'_n$. Further we find that the numerical values of ARE of the Wilcoxon signed rank test with respect to the $t$-test stated by Theorem 3 give no loss of the relative efficiency even against the alternative hypothesis of contaminated distributions discussed in Section 3.

References


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