ON COMPLETE HYPERSURFACES WITH HARMONIC CURVATURE IN A RIEMANNIAN MANIFOLD OF CONSTANT CURVATURE

Dedicated to Professor Morio Obata on his 60th birthday

By

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0. Introduction.

This paper is concerned with hypersurfaces with harmonic curvature in a Riemannian manifold of constant curvature. The classification of curvature-like tensor fields on a Riemannian manifold has been studied by K. Nomizu [10], in which the Codazzi equation for the curvature-like tensor played an important role. The subject is also treated by S. Y. Cheng and S. T. Yau [3] from the different point of view. A Riemannian curvature tensor is said to be harmonic if the Ricci tensor \( S \) satisfies the Codazzi equation \( \delta S = 0 \), namely, in local coordinates

\[
R_{ijk} = R_{ikj},
\]

where \( R_{ijk} \) denotes the covariant derivative of the Ricci tensor \( R_{ij} \). Although the concept is closely related to a parallel Ricci tensor, it was shown by A. Derdziński [5] and A. Gray [6] that it is essentially weaker than the latter one. In the Yang-Mills theory the harmonic curvature is also weighty, and some studies for these topics are made. In particular, J. P. Bourguignon conjectured that on a 4-dimensional compact Riemannian manifold with harmonic curvature the Ricci tensor must be parallel. This is negatively answered by A. Derdziński [4], who gave an example of a 4-dimensional compact Riemannian manifold with harmonic curvature and non-parallel Ricci tensor. Certain kinds of Riemannian manifolds with harmonic curvature are investigated by J. P. Bourguignon [1], A. Derdziński [5], T. Kashiwada [7], S. Tachibana [13] and so on. In particular, A. Derdziński [5] gave also other examples of higher dimensional Riemannian manifolds.

On the other hand, hypersurfaces with parallel Ricci tensor in a Riemannian manifold of constant curvature are studied by H. B. Lawson Jr. [8] and I. Mogi

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and one of the present authors [9], and hypersurfaces with harmonic curvature in a Riemannian manifold of constant curvature are recently investigated by E. Ōmachi [11] and one of the present authors [15], who determined the situation of the principal curvatures, provided that the mean curvature is constant. Especially, one of the present authors [15] treated also them without the assumption that the mean curvature is constant.

In this paper a class of hypersurfaces with harmonic curvature in a Riemannian manifold of constant curvature will be considered. The purpose is to classify completely hypersurfaces with harmonic curvature in the case where a multiplicity of each principal curvature is greater than one, and to show that there exist infinitely many hypersurfaces with harmonic curvature and non-parallel Ricci tensor.

1. Preliminaries.

In order to fix the notation, the theory of hypersurfaces in a Riemannian manifold of constant curvature is prepared for. Let \( \overline{M} = M^{n+1}(c) \) be an \((n+1)\)-dimensional Riemannian manifold of constant curvature \( c \) and let \( M \) be an \( n \)-dimensional connected Riemannian manifold. By \( \phi \) the isometric immersion of \( M \) into \( \overline{M} \) is denoted. When the argument is local, \( M \) need not be distinguished from \( \phi(M) \) and therefore, to simplify the discussion a point \( x \) in \( M \) may be identified with the point \( \phi(x) \) and a tangent vector \( X \) at \( x \) may be also identified with the tangent vector \( d\phi(X) \) at \( \phi(x) \) via the differential \( d\phi \) of \( \phi \).

To begin with, we choose an orthonormal local frame field \( \{e_1, \ldots, e_n, e_{n+1}\} \) in \( \overline{M} \) in such a way that, restricted to \( M \), the vectors \( e_1, \ldots, e_n \) are tangent to \( M \) and hence the other \( e_{n+1} \) is normal to \( M \). With respect to this field of frames on \( \overline{M} \), let \( \{\overline{e}_1, \ldots, \overline{e}_n, \overline{e}_{n+1}\} \) be the dual field. Here and in the sequel, the following convention on the range of indices are adopted, unless otherwise stated:

\[
A, B, \ldots = 1, \ldots, n, n+1, \\
i, j, \ldots = 1, \ldots, n.
\]

Then, associated with the frame field \( \{e_1, \ldots, e_n, e_{n+1}\} \) there exist differential 1-forms \( \overline{\omega}_{AB} \) on \( \overline{M} \), which are called connection forms on \( \overline{M} \), so that they satisfy the following structure equations on \( \overline{M} \):

\[
\begin{align}
\label{eq:1.1}
d\overline{\omega}_A + \sum B \overline{\omega}_{AB} \wedge \overline{\omega}_B &= 0, \\
\overline{\omega}_{AB} + \overline{\omega}_{BA} &= 0.
\end{align}
\]

\[
\begin{align}
\label{eq:1.2}
d\overline{\omega}_{AB} + \sum C \overline{\omega}_{AC} \wedge \overline{\omega}_{CB} &= \varepsilon \overline{\omega}_A \wedge \overline{\omega}_B.
\end{align}
\]
By restricting these forms $\overline{\omega}_A$ and $\overline{\omega}_{AB}$ to $M$, they are denoted by $\omega_A$ and $\omega_{AB}$ without bar, respectively. Then

\begin{equation}
\omega_{n+1} = 0.
\end{equation}

The metric on $M$ induced from the Riemannian metric $\tilde{g}$ on the ambient space $\tilde{M}$ under the immersion $\phi$ is given by $g = 2\sum\omega_i \omega_i$. Then $\{e_1, \ldots, e_n\}$ is an orthonormal local field with respect to the induced metric and $\{\omega_1, \ldots, \omega_n\}$ is the dual field, which consists of real valued, linearly independent 1-forms on $M$. They are called canonical forms on the hypersurface $M$. It follows from (1.3) and the Cartan lemma that

\begin{equation}
\omega_{n+1, i} = \sum_i h_{ij} \omega_j, \quad h_{ij} = h_{ji}.
\end{equation}

The quadratic form $\sum_i j h_{ij} \omega_i \otimes \omega_j$ is called a second fundamental form of $M$. We call also a form $\sigma$ defined by

\begin{equation}
\sigma(X, Y) = \sum_i j h_{ij} \omega_i(X) \omega_j(Y) e_{n+1}
\end{equation}

for any vector fields $X$ and $Y$ a second fundamental form on $M$. A linear transformation $A$ on the tangent bundle $TM$ is defined by $g(AX, Y) = g(\sigma(X, Y), e_{n+1})$. Then $A$ is called a shape operator of $M$. By the structure equations (1.1), (1.2) and (1.3), the following structure equations on the hypersurface $M$ are given:

\begin{align}
\begin{aligned}
&\sum_i j h_{ij} \omega_i \otimes \omega_j = 0, \quad \omega_{ij} + \omega_{ji} = 0, \\
&\sum_i j h_{ij} \omega_i \otimes \omega_j = Q_{ij}, \quad Q_{ij} = -\frac{1}{2} \sum R_{ijkl} \omega_k \otimes \omega_l,
\end{aligned}
\end{align}

where $\omega_{ij}$ (resp. $Q_{ij}$ and $R_{ijkl}$) denotes a connection form (resp. a curvature form and a curvature tensor) on $M$. From (1.2) and (1.6) the Gauss equation

\begin{equation}
R_{ijkl} = c(\delta_{ij} \delta_{lk} - \delta_{ik} \delta_{lj}) + h_{il} h_{jk} - h_{ik} h_{jl}
\end{equation}

is obtained, and the Ricci tensor $R_{ij}$ and the scalar curvature $R$ can be expressed as follows:

\begin{align}
\begin{aligned}
R_{ij} &= (n-1)c \delta_{ij} + h_{ij} - \sum_k h_{ik} h_{kj}, \\
R &= n(n-1)c + h^2 - \sum_i \sum_j h_{ij} h_{ij},
\end{aligned}
\end{align}

where $h$ is a function defined by $h = \sum_i h_{ii}$, namely, for the mean curvature $H$ it satisfies $h = nH$.

Now, the covariant derivative $h_{ijk}$ and $R_{ijk}$ of $h_{ij}$ and $R_{ij}$ are respectively defined by

\begin{align}
\begin{aligned}
\sum_k h_{ijk} \omega_k &= d h_{ij} - \sum_k h_{ik} \omega_k - \sum_k h_{ik} \omega_{kj}, \\
\sum_k R_{ijk} \omega_k &= d R_{ij} - \sum_k R_{ik} \omega_k - \sum_k R_{ik} \omega_{kj}.
\end{aligned}
\end{align}
Differentiating (1.4) exteriorly, we have the Codazzi equation on the hypersurface \( M \)
\[
(1.11) \quad h_{i;k} - h_{k;i} = 0,
\]
since the ambient space \( \mathcal{M} \) is of constant curvature, and by differentiating (1.8) exteriorly the covariant derivative \( R_{i;jk} \) satisfies
\[
\sum_k R_{i;jk} \omega_k = \sum_k (h_{k;i} h_{j;k} + h_{k;j} h_{i;k} - \sum_l h_{l;i} h_{l;j;k}) \omega_k,
\]
where \( dh = \sum_k h_k \omega_k \), and hence
\[
(1.12) \quad \sum_{j,k} R_{i;jk} \omega_k \wedge \omega_j = \sum_k (h_{k;i} h_{j;k} - \sum_l h_{l;i} h_{l;j;k}) \omega_k \wedge \omega_j.
\]
A Riemannian curvature tensor is said to be harmonic if the Ricci tensor satisfies the Codazzi equation (0.1), namely, \( R_{i;jk} \) is symmetric with respect to all indices \( i, j \) and \( k \). It follows from (1.12) that it is necessary and sufficient for \( M \) to be of harmonic curvature that it satisfies
\[
(1.13) \quad h_{k;i} h_{j;k} - h_{j;i} h_{k;j} - \sum_l h_{l;i} h_{l;j;k} + \sum_l h_{l;i} h_{l;k;j} = 0
\]
for any indices.

2. The gradient of the mean curvature.

Let \( M \) be an \( n \)-dimensional hypersurface with harmonic curvature in \( M^{n+1}(c) \) and let \( H \) be the mean curvature on \( M \). In this section, assume that the gradient of \( H \) is an eigenvector associated with an eigenvalue \( 0 \) of the shape operator \( A \). In other words, we shall assume that it satisfies
\[
(2.1) \quad A \text{ grad } H = 0, \text{ namely } \sum_j h_{j;i} h_{j;j} = 0
\]
holds true. In this assumption the case where \( \text{ grad } H = 0 \) is included, that is, the situation that the mean curvature \( H \) has critical points is admitted. For simplification, a tensor \( h_{ij}^m \) and a function \( h_m \) on \( M \) for any integer \( m \) are introduced as follows;
\[
(2.2) \quad h_{ij}^m = \sum_{i_1,...,i_m} h_{i_1;i_2} h_{i_2;i_3} \cdots h_{i_m;i_j},
\]
\[
\text{ and } h_m = \sum_i h_{i;i}^m,
\]
where \( h_{i}=h=nH \). By taking account of the second Bianchi identity, it is easily seen that the scalar curvature is constant, and therefore the function \( h^2 - h_z \) is constant. This implies
\[
(2.3) \quad dh_z = 2hdh.
\]
First of all, the generalization of (2.3) is requested. Namely, the relation
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(2.4) \[ dh_m = mh_{m-1}dh \]
is true for any integer \( m \geq 2 \). In fact, the relation (2.4) is proved by induction on \( m \). At first, (2.3) shows that the case where \( m = 2 \) in (2.4) holds. By the property of derivations for the exterior differential, it is easily seen that the following equation

\[ dh_m = \sum_{i,j} mh_{ij} h_{ij}^{-1} dh_{ij}. \]

The definition (1.10) of the covariant derivative \( h_{ijk} \) and the above equation imply

\[ dh_m = m \sum_{i,j,k} (h_{ij} \omega_k + h_{kj} \omega_i + h_{ik} \omega_j) h_{ij}^{-1} \]

\[ = m \sum_{i,j,k} h_{ijk} h_{ij}^{-1} \omega_k + 2m \sum_{i,j} h_{ij}^{-1} \omega_{ij}, \]

Thus

\[ (2.5) \]

\[ dh_m = m \sum_{i,j,k} h_{ijk} h_{ij}^{-1} \omega_k, \]

because \( h_{ij}^{-1} \) is symmetric with respect to \( i \) and \( j \) and the connection form \( \omega_{ij} \) is skew-symmetric with respect to \( i \) and \( j \). This yields

\[ dh_m = m \sum_{i,j,k} h_{ijk} h_{ij}^{-1} \omega_k \]

\[ = m \sum_{i,j,k} (\sum h_{ijk} h_{ijk} + h_{kj} h_{ij} - h_{ij} h_{ik}) h_{ij}^{-1} \omega_k \]

\[ = m (\sum_{i,j,k} h_{ijk} h_{ij}^{-1} \omega_k + h_{m-1} dh - \sum h_{k} h_{k}^{-1} dh), \]

where we have used (1.10) and (1.13). By the assumption (2.1) the last term in the right hand side vanishes identically. It follows from the case where \( m = 1 \) in (2.5) and the supposition of the induction that we get

\[ \sum_{i,j,k} h_{ijk} h_{ij}^{-1} \omega_k = \frac{1}{m-1} dh_{m-1} = dh_{m-2}, \]

which yields

\[ \sum_{i,j,k} h_{ijk} h_{ij}^{-1} = dh_{m-2}(e_j), \]

This means that the first term in the right hand side of the above equation vanishes also identically, which completes the proof.

A function \( H_m \) for any integer \( m \geq 2 \) is next defined by

\[ (2.6) \]

\[ H_m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} h_{m-k} h_k, \quad h_0 = 1. \]

By making use of (2.4) it follows from the straightforward calculation that

\[ dH_m = \sum_{k=0}^{m-1} (-1)^k \binom{m}{k} (m-k)h_{m-k-1} h_k dh + \sum_{k=1}^{m} (-1)^k \binom{m}{k} h_{m-k} h_{k-1} dh \]

\[ = \sum_{k=0}^{m} (-1)^k \binom{m}{k} (m-k) \binom{m}{k+1} h_{m-k-1} h_k dh, \]
which shows that $H_m$ is constant on $M$. Thus we have

**Lemma 2.1.** Let $M$ be a hypersurface with harmonic curvature in $M^{n+1}(c)$. If the shape operator $A$ of $M$ satisfies $A^g H = 0$, then $H_m$ is constant on $M$ for any integer $m (\geq 2)$.

By rewriting (2.6), the relation

$$h_m = h^m + \sum_{k=0}^{m} (-1)^k \binom{m}{k} h^{m-k}$$

is true for any integer $m \geq 2$. In fact, the equation is also verified by induction on $m$. At first, the case where $m=2$ in (2.6) is considered. Then it shows that (2.7) holds for $m=2$. Next, suppose that (2.7) holds for integers less than $m$. Since the constant $H_m$ is expressed as

$$H_m = \binom{m}{0} h^m - \binom{m}{1} h^{m-1} h + \sum_{k=2}^{m} (-1)^k \binom{m}{k} h^{m-k} h_k + (-1)^m h_m,$$

the supposition of the induction is applied to the third term in the right hand side, so it is reduced to

$$H_m = (-1)^m h_m + \left\{ \binom{m}{0} - \binom{m}{1} \right\} h^m + \sum_{k=2}^{m} (-1)^k \binom{m}{k} \left\{ h^k + \sum_{l=0}^{k-1} (-1)^l \binom{k}{l} H_l h^{k-l} \right\}$$

$$= (-1)^m h_m + \sum_{k=0}^{m} (-1)^k \binom{m}{k} h^m$$

$$+ \sum_{l=1}^{m-1} (-1)^l \left\{ \sum_{k=l}^{m-1} (-1)^k \binom{m}{k} \binom{k}{l} \right\} H_l h^{m-l}.$$

On the other hand, the binomial theorem $(1-x)^m = \sum_{k=0}^{m} (-1)^k \binom{m}{k} x^k$ and the derivative of $l$-order for variable $x$ yield the following relation for the binomial coefficients:

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{k}{l} = 0.$$

Accordingly we have

$$H_m = (-1)^m h_m + \left\{ \sum_{k=0}^{m} (-1)^k \binom{m}{k} - (-1)^m \right\} h_m$$

$$+ \sum_{l=1}^{m-1} (-1)^l \left\{ \sum_{k=l}^{m-1} (-1)^k \binom{m}{k} \binom{k}{l} \right\} H_l h^{m-l}$$

$$= (-1)^m h_m - (-1)^m h_m - (-1)^m \sum_{l=1}^{m-1} (-1)^l \binom{m}{l} H_l h^{m-l},$$
which implies that (2.7) holds for any integer $m \geq 2$.

3. No simple roots.

This section is devoted to the study the case where the hypersurfaces with harmonic curvature in $M^{n+1}(c)$ has principal curvatures all of whose multiplicities are greater than one. The second fundamental form may be diagonalized so that $\sum_{i,j} h_{ij} \omega_i \otimes \omega_j = \sum \lambda_i \omega_i \otimes \omega_i$. A principal curvature $\lambda_i$ is called a simple root at $x$ if the multiplicity at $x$ is equal to 1.

First of all, we prove

**Lemma 3.1.** Let $M$ be a hypersurface of $M^{n+1}(c)$ with harmonic curvature. If the shape operator has no simple roots on $M$, then $A \text{grad} H = 0$.

**Proof.** Since we have $h_{ij} = \lambda_i \delta_{ij}$ at a point $x$ on $M$, then equation (1.13) says that

$$
\lambda_i h_{k} \delta_{ij} - \lambda_k h_{i} \delta_{kj} + (\lambda_k - \lambda_j) h_{ij} = 0
$$

at $x$, where we have used

$$
\sum_i h_{i} h_{ij} = \lambda_j h_j.
$$

Because of the assumption that the second fundamental form $h_{ij}$ has no simple roots, for any fixed index $j$ there is an index $k$ different from $j$ such that $\lambda_j = \lambda_k$, and therefore (3.1) reduces to

$$
\lambda_j (h_k \delta_{ij} - h_i \delta_{kj}) = 0
$$

at the point $x$, which implies that if $x$ is not a zero point of the principal curvature $\lambda_j$, then we have $h_j = 0$ at $x$. From these data, we conclude, using (3.2), that $\sum_i h_i h_{ij} = 0$. This completes the proof of the lemma.

In the next place, using Lemma 3.1 we are going to prove that the mean curvature $H$ of $M$ is constant.

By taking account of (2.7), it is easily seen that

$$
h_{n+1} - hh_n = \sum_{k=2}^{n} (-1)^k \binom{n+1}{k} H_k h^{n+1-k} + (-1)^n H_{n+1}
$$

which is a polynomial of degree $n-1$ with respect to $h$ with constant coefficient, because of Lemma 2.1. Since $\lambda_1, \ldots, \lambda_n$ are the principal curvatures of the second fundamental form $h_{ij}$, $h_m$ can be written as
(3.4) \[ h_0 = 1, \quad h_1 = h = \sum_{i=1}^{n} \lambda_i, \quad h_m = \sum_{i=1}^{m} \lambda_i^m, \quad m = 2, 3, \ldots. \]

Now, let \( f_i(\lambda), \ldots, f_n(\lambda) \) be elementary symmetric functions of \( \lambda = (\lambda_1, \ldots, \lambda_n) \), namely,

\[
\begin{align*}
  f_1 &= f_1(\lambda) = (-1)^{1} \sum_i \lambda_i,
  f_2 &= f_2(\lambda) = (-1)^{2} \sum_{i<j} \lambda_i \lambda_j,
  & \vdots \\
  f_n &= f_n(\lambda) = (-1)^{n} \lambda_n.
\end{align*}
\]

Then it is well known that \( f_1, \ldots, f_n \) and \( h_1, \ldots, h_n, h_{n+1} \) are related by the Newton formulas (cf. [14]) as follows:

\[
\begin{align*}
  h_1 + f_1 &= 0, \\
  h_2 + f_1 h_1 + 2f_2 &= 0, \\
  & \vdots \\
  h_n + f_1 h_{n-1} + \cdots + f_{n-1} h_1 + nf_n &= 0, \\
  h_{n+1} + f_1 h_n + \cdots + f_{n-1} h_2 + f_n h_1 &= 0.
\end{align*}
\]

When these formulas are regarded as the linear homogeneous simultaneous equations with respect to \( (1, f_1, \ldots, f_n) \), we see, using the principle of elimination, that the determinant of coefficients vanishes identically. If we take account of (3.3) and the Laplace expansion to this determinant, we can get

\[
((n+1)/2)H_n h^{n-1} - ((n-1)(n+1)/3)H_n h^{n-2} + \cdots = 0.
\]

Therefore, it follows from (3.7) that \( h_1 \) is the root of the algebraic equation with constant coefficients unless all \( H_m \) vanishes. According to Lemma 2.1, we have

**Lemma 3.2.** Let \( M \) be a hypersurface with harmonic curvature in \( M^{n+1}(c) \). If the shape operator \( A \) of \( M \) satisfies \( A \text{grad} H = 0 \), then the mean curvature of \( M \) is constant, provided that there exists a nonzero \( H_m \) defined by (2.6).

On the other hand, if all \( H_m \)'s are zero and the shape operator of \( M \) has no simple roots, then it is easily derived, by using (2.7), (3.5) and (3.6), that \( h \) is also constant.

Combining this fact, Lemma 3.1 and Lemma 3.2, we have

**Proposition 3.3.** Let \( M \) be a hypersurface with harmonic curvature in \( M^{n+1}(c) \). If the shape operator of \( M \) has no simple roots, then the mean curvature of \( M \) is constant.
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Under the property of Proposition 3.3, (2.4) means that each principal curvature of $M$ is constant and hence, by means of Umehara's theorem [15] the number of distinct principal curvatures is at most two, say $\lambda$ and $\mu$, such that $c+\lambda\mu=0$, which is applied to the situation where the ambient space is a sphere, a Euclidean one or a hyperbolic one. So, in the case, the above result for the number of distinct principal curvatures is simply proved from a different point of view. In fact, $M$ is an isoparametric hypersurface in the sense of E. Cartan and the basic identity for principal curvatures shows that the above is true, provided that $c\leq 0$ [2]. If $c>0$, then it is evident in [11]. Moreover, the second fundamental form of $M$ is parallel.

By the way, we shall here give a model of hypersurfaces with parallel Ricci tensor in a hyperbolic space $H^{n+1}(c)$ (cf. Lawson [8]). $H^{n+1}(c)$ is covered by a coordinate system $\{x_1, \cdots, x_{n+1}\}$ such that the Riemannian metric $ds^2$ of $H^{n+1}(c)$ is given by

$$ds^2=\sum_{a=1}^{n+1}dx_a^2-(\sum_{a=1}^{n+1}x_a dx_a)^2/(r^2+\sum_{a=1}^{n+1}x_a^2),$$

where $r^2=-1/c$. The space $H^{n+1}(c)$ is a complete and simply connected Riemannian manifold of constant negative curvature $c$. A family of hypersurfaces $M(s)$ in $H^{n+1}(c)$ is defined by

$$M(s)=\{x\in H^{n+1}(c): \sum_{a=1}^{n+1}x_a^2=s^2-r^2\}$$

for $s>r$. Then a hypersurface $M(s)$ for a fixed $s$ is a space of constant curvature $c_1=1/(s^2-r^2)$ in $H^{n+1}(c)$, which is totally umbilic. As another family of hypersurfaces $M(t)$, the following subject is defined:

$$M(t)=\{x\in H^{n+1}(c): x_1=t\geq 0\}.$$ 

The hypersurface $M(t)$ for an arbitrary fixed $t$ is totally umbilic and hence it is a hyperbolic space of constant curvature $c_1=-1/(r^2+t^2)$. A flat hypersurface $F^n$ is constructed as follows:

$$F^n=\{x\in H^{n+1}(c): \sum_{a=1}^{n+1}x_a^2=2r x_{n+1}\}.$$ 

Then $F^n$ is covered by one coordinate system $\{x_1, \cdots, x_n\}$ such that the Riemannian metric induced from the Riemannian metric in $H^{n+1}(c)$ is given by $ds^2=\sum_{a=1}^n dx_a^2$. Accordingly, $F^n$ is flat. Lastly, a family of product hypersurfaces $S^{k}(c_1)\times H^{n-k}(c_2)$ in $H^{n+1}(c)$ is considered. They are defined by

$$S^{k}(c_1)\times H^{n-k}(c_2)=\{x\in H^{n+1}(c): \sum_{a=1}^{n+k}x_a^2=1/c_1\},$$

where $c_1$ is a positive constant and $1/c_1+1/c_2=1/c$, and $1\leq k\leq n-1$. Any hypersurface of the family is the product manifold of a sphere of constant curvature.
c_1 and a hyperbolic space of constant curvature c_2 and consequently it has exactly two distinct principal curvatures (c_1-c)^{1/2} and (c_2-c)^{1/2} of multiplicity k and n-k, respectively.

Combining Proposition 3.3 together with Umehara's theorem, we can see the following

**Theorem 3.4.** Let M be an n (⊇3)-dimensional complete and simply connected Riemannian manifold with harmonic curvature and let \( \phi \) be an isometric immersion of M into an (n+1)-dimensional complete and simply connected Riemannian manifold of constant curvature c. If the multiplicity of each principal curvature is greater than one, then \( \phi(M) \) is isometric to one of the following spaces:

1. **The case where** \( c>0 \). The great sphere, the small sphere and \( S^k(c_1) \times S^{n-k}(c_2) \), where \( 2 \leq k \leq n-2 \) and \( 1/c_1+1/c_2=1/c \). In particular, \( \phi \) is an imbedding.
2. **The case where** \( c=0 \). The sphere, the Euclidean space and \( S^k(c_1) \times R^{n-k} \).
3. **The case where** \( c<0 \). The sphere, the hyperbolic space, the flat space \( F^{n} \) and \( S^k(c_1) \times H^{n-k}(c_2) \), where \( 2 \leq k \leq n-2 \) and \( 1/c_1+1/c_2=1/c \). In particular, \( \phi \) is an imbedding.

4. **Hypersurfaces with harmonic curvature and non-parallel Ricci tensor.**

This section is devoted to the investigation of examples of hypersurfaces with harmonic curvature and non-parallel Ricci tensor in \( M^{n+1}(c) \). By taking account of Theorem 3.4, it is seen that at least one principal curvatures ought to be of multiplicity 1.

Let M be a hypersurface immersed in \( M^{n+1}(c) \), and assume that the principal curvatures \( \lambda_i \) on M satisfy

\[
\begin{cases}
\lambda_i = \cdots = \lambda_{n-1} = \lambda \neq 0 ,
\end{cases}
\]

such that \( \lambda \neq \mu \). Without loss of generality, we may suppose that \( \lambda > 0 \). As is well known, the distribution of the space of eigenvectors corresponding to the eigenvalue \( \lambda \) is completely integrable, because the multiplicity of each principal curvature is constant. Now, since \( \lambda \) and \( \mu \) are smooth functions on M, we have, using the covariant derivative \( h_{ijk} \),

\[
d\lambda = d\lambda_a = h_{aa} \omega_a + \sum_{b \neq a} h_{a}^{ab} \omega_b + h_{aa} \omega_a ,
\]

where indices \( a, b, \cdots \) run over the range \( \{ 1, \cdots , n-1 \} \). Because of \( \omega_{n+1} a = \lambda_a \omega_a \), we have
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$$d \omega_{n+1} = d \lambda_a \wedge \omega_a + \lambda_a d \omega_a$$

$$= d \lambda \wedge \omega_a + \lambda (\sum d \omega_{ab} \wedge \omega_b - \omega_{an} \wedge \omega_n).$$

On the other hand, the structure equation (1.2) yields

$$d \omega_{n+1} = - \sum \omega_{n+1 a} \wedge \omega_a$$

$$= - \lambda \sum \omega_b \wedge \omega_{ba} - \mu \omega_n \wedge \omega_{na}.$$

Combining with above two equations, we have

$$(4.3) \quad \sum \lambda_b \omega_b \wedge \omega_a + \{ (\mu - \lambda) \omega_{an} - \lambda \omega_a \} \wedge \omega_n = 0$$

for a fixed index $a$, where $\lambda = \sum \lambda_b \omega_b + \lambda \omega_n$. This implies

$$(4.4) \quad \begin{cases} 
\lambda_{,a} = 0, \\
(\mu - \lambda) \omega_{an} - \lambda \omega_a = \sigma_a \omega_n 
\end{cases}$$

for any index $a$, where $\sigma_a$ is a function on $M$. From (4.2) and the first equation of (4.4) it follows that we have

$$h_{aa} \omega_a + \sum \omega b h_{aba} + h_{aan} \omega_n = \lambda \omega_n,$$

and hence

$$h_{aa} = 0, \quad h_{aa} = 0 \quad (b \neq a), \quad h_{aan} = \lambda n.$$

Similarly, for the other $\mu$ we have

$$d \mu = \sum h_{nnb} \omega_b + h_{nnn} \omega_n.$$

Because of $\omega_{n+1 n} = \mu \omega_n$, by the same argument as that of $\lambda$ we have

$$d \omega_{n+1} = - \lambda \sum \omega_b \wedge \omega_b = d \lambda \wedge \omega_n - \mu \sum \omega_{nb} \wedge \omega_b,$$

and hence

$$d \mu \wedge \omega_n + (\lambda - \mu) \sum \omega_{nb} \wedge \omega_b = 0.$$

This together with (4.4) implies

$$(4.5) \quad \mu_{,a} = \sigma_a \quad \text{for any index} \quad a.$$

On the other hand, for distinct indices $a$ and $b$, we have

$$(4.6) \quad h_{ab} = 0.$$

In the case where $M$ is with harmonic curvature, principal curvatures $\lambda_j$ satisfy (3.1) and because of $h = (n-1) \lambda + \mu$ and $dh = \sum h_k \omega_k$, we see

$$h_k = (n-1) \lambda_k + \mu_k,$$

for any index $k$. Considering the case where $j = a$ and $k = n$ in (3.1), one gets

$$\lambda h_{a} \delta_{a1} - \mu h_{a} \delta_{an} - (\lambda - \mu) h_{an} = 0.$$
for any indices $a$ and $i$. This means that it follows from the above equation and (4.5) that

\[
\begin{cases}
(n-2)\lambda + \mu \lambda_{,n} + \lambda \mu_{,n} = 0, \\
\mu h_a + (\lambda - \mu)h_{ann} = 0.
\end{cases}
\]

(4.7)

Consequently, making use of the above relations, we have $h_a = \mu_a = h_{ann}$ and $\lambda h_{ann} = 0$, namely

\[
h_a = 0, \quad \sigma_a = 0.
\]

Thus, by (4.4) we have

\[
h_{ann} = \mu_{,n},
\]

(4.8)

\[
\omega_{aa} = \frac{\lambda_{,n}}{\lambda - \mu} \omega_a.
\]

Accordingly, in order for $M$ to be with harmonic curvature, principal curvatures $\lambda$ and $\mu$ must satisfy (4.7) and (4.8). Moreover we have $d\omega_n = 0$, which shows that we may put

\[
\omega_n = dv.
\]

Thus we have

\[
(4.9)
\]

\[
(4.10)
\]

where the prime denotes the derivative with respect to $v$. This means that the integral submanifold $M^{n-1}(v)$ corresponding to $\lambda$ and $v$ is umbilic in $M$ and hence in $M^{n+1}(c)$.

By the simple calculation the following properties for the Ricci tensor are obtained:

\[
R_{abc} = 0, \quad R_{ann} = 0, \quad R_{abn} = [2(n-2)\lambda + \mu] \lambda_{,n} + \lambda \mu_{,n} \delta_{ab},
\]

\[
R_{nnn} = (n-1)(\lambda \mu_{,n} + \mu \lambda_{,n}).
\]

Therefore, in order for $M$ to be with parallel Ricci tensor, it is necessary and sufficient that $\lambda$ and $\mu$ are both constant.

**Example.** $M = S^{n-1}(c_1) \times S^{1}(c_2) \subset R^n \times R^2$ such that $1/c_1 + 1/c_2 = 1$. The principal curvatures $\lambda_j$ are given by

\[
\lambda_1 = \cdots = \lambda_{n-1} = \lambda = \pm (c_1 - 1)^{1/2},
\]

\[
\lambda_n = \mu = \mp (c_2 - 1)^{1/2}.
\]

Actually $M$ is with harmonic curvature and parallel Ricci tensor. In particular, when $c_1 = n/(n-2)$ and $c_2 = n/2$, $\lambda$ and $\mu$ are given by $\lambda = \pm (2/(n-2))^{1/2}$ and
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\( \mu = \mp (n-2)/2 \) and moreover they satisfy \((n-2)\lambda + \mu = 0\). In the latter case, the scalar curvature \( R \) is equal to \( n(n-1) \).

Now, substituting (4.10) into the structure equation

\[ d\omega_{a} + \sum \omega_{b} \wedge \omega_{a} = (c + \lambda \mu) \omega_{a} \wedge \omega_{a}, \]

we have

\[ d\left( \frac{\lambda'}{\lambda - \mu} \omega_{a} \right) = - \frac{\lambda'}{\lambda - \mu} \sum \omega_{b} \wedge \omega_{a} + (c + \lambda \mu) \omega_{a} \wedge \omega_{a}. \]

Since the left hand side is reduced to

\[ \left( \frac{\lambda'}{\lambda - \mu} \right) \omega_{a} \wedge \omega_{a} + \frac{\lambda'}{\lambda - \mu} \left( - \sum \omega_{b} \wedge \omega_{a} - \omega_{a} \wedge \omega_{a} \right), \]

the following equation is obtained:

\[ \left( \frac{\lambda'}{\lambda - \mu} \right)' - \left( \frac{\lambda'}{\lambda - \mu} \right)^{2} - (c + \lambda \mu) = 0, \]

and hence we have

(4.11) \( \lambda'(\lambda - \mu) - \lambda'(-\mu' - \lambda') - \lambda'^{2} - (c + \lambda \mu)(\lambda - \mu)^{2} = 0. \)

Furthermore, under the condition (4.7) we have

(4.12) \( (n-2)\lambda + \mu \lambda' + \lambda \mu' = 0. \)

Thus the distinct principal curvatures \( \lambda \) and \( \mu \) satisfy a system of ordinary differential equations (4.11) and (4.12) of order 2. (4.12) is however equivalent to

\[ (n-2)\lambda^{2} + 2\lambda \mu = c_{1}, \]

where \( c_{1} \) is the integral constant. Then the scalar curvature \( R \) is given by \( R = n(n-1)c + (n-1)c_{1} \), and by taking account of (4.12), the ordinary differential equation of order 2 for \( \lambda \) is given by

(4.13) \[ 4\lambda(n \lambda^{2} - c_{1}) \lambda'' - 4(n + 2) \lambda^{2} + c_{1} \lambda' \]

\[ - (n \lambda^{2} - c_{1})^{2} \{ 2c + c_{1} - (n - 2) \lambda^{2} \} = 0, \]

where \( n \lambda^{2} - c_{1} \neq 0 \). Putting \( \omega = \lambda^{2/n} \), (4.13) can be replaced by

\[ \frac{d^{2} \omega}{dv^{2}} + \frac{(n+1)c_{1} \omega^{n-1}}{n - c_{1} \omega^{n}} \omega^{2} + \frac{\omega}{2n} (n - c_{1} \omega^{n}) \left( 2c + c_{1} - \frac{n - 2}{\omega^{n}} \right) = 0. \]

Integrating the differential equation of degree 2, we obtain

\[ \left( \frac{d \omega}{dv} \right)^{2} = (n - c_{1} \omega^{n})^{2(n+1)/n} \left( c_{2} - \frac{1}{n} \int \omega(n - c_{1} \omega^{n})^{-\frac{1}{n}} \left( 2c + c_{1} - \frac{n - 2}{\omega^{n}} \right) d\omega \right), \]
where $c_2$ is the integral constant. In the case where $c_1=0$, this is reduced to
\[
\left( \frac{d\omega}{dv} \right)^2 + \frac{1}{\omega^{n-2}} + c\omega^2 = c_2,
\]
which is the differential equation similar to that treated by T. Otsuki [12].
Thus there exist infinitely many hypersurfaces with harmonic curvature in $M^{n+1}(c)$ corresponding to the constants $c_1$ and $c_2$, and the hypersurfaces have non-parallel Ricci tensor and the scalar curvatures are equal to $n(n-1)c + (n-1)c_1$.

By the same method as that of Otsuki's theory, we have the following construction theorem concerning for hypersurfaces with harmonic curvature.

**Theorem 4.1.** Let $M$ be an $n$-dimensional hypersurface with scalar curvature $n(n-1)c$ and the harmonic curvature immersed in $M^{n+1}(c)$. If it has exactly two distinct principal curvatures, one's multiplicity of which is equal to 1, and the other has no zero points, then the following assertions are true:

1. $M$ is a locus of moving $(n-1)$-dimensional submanifold $M^{n-1}(v)$ along which the principal curvature $\lambda$ of multiplicity $n-1$ is constant and which is umbilic in $M$ and of constant curvature $(d/dv(\log(n\lambda^2-c_1)^{1/n}))^2 + \lambda^2 + c$, where $v$ is the arc length of an orthogonal trajectory of the family $M^{n-1}(v)$, and $\lambda = \lambda(v)$ satisfies the ordinary differential equation (4.13) of order 2.

2. If $\bar{M} = S^{n+1}(c) \subset R^{n+2}$, then $M^{n-1}(v)$ is contained in an $(n-1)$-dimensional sphere $S^{n-1}(v) = E^n(v)/S^{n+1}$ of the intersection of $S^{n+1}$ and an $n$-dimensional linear subspace $E^n(v)$ in $R^{n+2}$ which is parallel to a fixed $E^n$. The center $q$ moves on a plane curve in a plane $R^2$ through the origin of $R^{n+2}$ and orthogonal to $E^n$.

**Corollary.** There exist infinitely many hypersurfaces with harmonic curvature and non-parallel Ricci tensor in $M^{n+1}(c)$, which is not congruent to each other in it.

In the next place, the condition under which the plane curve figured with the center $q$ is controlled will be required. Since the matter discussed in [12, section 4] can be completely applied to this case, the necessary subjects for the explanation of the statement of the theorem are only quoted from [12], and the precise argument is omitted. The sphere $S^{n+1}$ is regarded as $S^{n+1} \subset R^{n+2}$ = $R^n \times R^2$, and $\{e_1, \ldots, e_3\}$ denotes the orthonormal frame in $R^n$ at the origin. Let $C$ be a plane curve in $R^2$ with a given surporting function $h(\theta)$, then the generic point $q(\theta)$ of $C$ is given by
\[
q(\theta) = e^{i(\theta-n/2)}(h(\theta) + ih'(\theta)).
\]
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by considering $\mathbb{R}^2$ as the complex plane. The Frenet formula of $C$ at $q(\theta)$ is given by $\vec{e}_{n+1} = e^{i\phi}$ and $\vec{e}_{n+2} = e^{i(\phi + \pi/2)}$. Suppose that the curve $C$ is contained in the unit circle. Then a positive function $\rho$ can be defined by $\rho^2 = 1 - \|q\|^2$, and a hypersurface $M$ is defined in $S^{n+1}(1)$ by

$$(4.15) \quad \rho = q + \rho \vec{e}_n.$$ 

A unit vector $e_n$ is defined by

$$e_n = (\rho'\vec{e}_n + (h + h^*)\vec{e}_{n+1})/((\rho')^2 + (h + h^*)^2)^{1/2}.$$ 

If the hypersurface $M$ in $S^{n+1}(1)$ is with harmonic curvature and $R = n(n-1)$, then the function $h$ satisfies the following ordinary differential equation

$$(4.16) \quad nh(1-h^*)\frac{d^2h}{d\theta^2} + 2(h'\frac{dh}{d\theta}^2 + (1-h^*)(nh^2 - 2) = 0.$$ 

Conversely, if a function $h(\theta)$ satisfying (4.16) gives a plane curve by the equation (4.14) in $\mathbb{R}^2$ contained in the unit circle, then a hypersurface $M$ with harmonic curvature and $R = n(n-1)$ is obtained by (4.15). The hypersurfaces depend completely on properties of $h(\theta)$.

**Theorem 4.2.** Any complete hypersurface $M$ with harmonic curvature and $R = n(n-1)$ in $S^{n+1}(1)$ of the type of Theorem 4.1 is given by the following method.

1. $C$ is a plane curve in $\mathbb{R}^2$ given by

$$q(\theta) = e^{i(\phi - \pi/2)}(h(\theta) + ih'(\theta))$$

where $h(\theta)$ is a solution of the differential equation (4.16) with $0 < h(0) \leq (2/n)^{1/2}$ and $h'(0) = 0$.

2. $M \ni p = (1 - h(\theta)^2 - h'(\theta)^2)^{1/2} \vec{e}_n + q(\theta)$, where $\vec{e}_n \in \mathbb{R}^n$, $\|\vec{e}_n\| = 1$ and $S^{n+1} \subset \mathbb{R}^n \times \mathbb{R}^2$.

There exist countable number of compact hypersurfaces of this type, and the special case $S^{n-1}(n/(n-2)) \times S^1(n/2)$ corresponds to $h(0) = (2/n)^{1/2}$ and $h'(0) = 0$.

**Bibliography**


