A NOTE ON SOME STRONG WHITNEY-REVERSIBLE PROPERTIES

By

Akira Koyama

1. All spaces considered in this paper are assumed to be metric. A continuum means a compact connected space and a map means a continuous function. The letter $X$ will always denote a continuum. Let $C(X)$ denote the hyperspace of all non-empty subcontinua of $X$ with the Hausdorff metric (see [7]). Whitney [10] proved that for every continuum $X$ there exists a map $\mu : C(X) \to [0, +\infty)$ satisfying

1. if $A, B \subseteq C(X)$ and $A \subseteq B$, then $\mu(A) \leq \mu(B)$ and
2. $\mu(\{x\}) = 0$ for every $x \in X$.

We shall call any map from $C(X)$ to $[0, +\infty)$ satisfying the above conditions (1) and (2) a Whitney map for $C(X)$.

Nadler [7] introduced the concept of a strong Whitney-reversible property. Let $P$ be a topological property. We say that $P$ is a strong Whitney-reversible property provided whenever $X$ is a continuum such that $\mu^{-1}(t)$ has the property $P$ for some Whitney map $\mu$ for $C(X)$ and every $0 < t \leq \mu(X)$, then so does $X$. Moreover he has shown that some topological properties are strong Whitney-reversible properties. For example hereditary indecomposability and local connectedness are such properties.

We refer readers to see [1] and [7] for the shape theory and the hyperspace theory respectively if necessary.

2. We shall show that some topological properties are strong Whitney-reversible properties.

**Theorem 1.** Let $\mu$ be a Whitney map for $C(X)$. If there is a sequence $\{t_n\}$ in $(0, \mu(X)]$ such that $t_n \to 0$ as $n \to +\infty$ and $\mu^{-1}(t_n)$ is an FAR for each $n = 1, 2, 3, \ldots$, then $X$ is also an FAR.

Hence the property of being an FAR is a strong Whitney-reversible property.

**Proof.** Let $M$ be an arbitrary ANR and $f : X \to M$ be an arbitrary map. Since $M$ is an ANR and we can identify $X$ with $\mu^{-1}(0) = \{x \mid x \in X\}$, there are

Received May 1, 1980. Revised July 7, 1980
an open neighborhood $U$ of $X$ and a map $\tilde{f}: U \to M$ such that $\tilde{f}|X=f$. Then there is an integer $n \geq 1$ such that $\mu^{-1}([0, t_n]) \subset U$. Since $\mu^{-1}(t_n)$ is an FAR, $\tilde{f}|\mu^{-1}(t_n)=0$, where 0 is a constant map. Hence there exists a map $g: \mu^{-1}([t_n, \mu(X)]) \to M$ such that $g|\mu^{-1}(t_n)=\tilde{f}|\mu^{-1}(t_n)$. Now we can define a map $h: C(X) \to M$ as the following formula:

$$h|\mu^{-1}([0, t_n])=\tilde{f}|\mu^{-1}([0, t_n]) \quad \text{and} \quad h|\mu^{-1}([t_n, \mu(X)])=g.$$ 

Since $C(X)$ is an FAR (see [3]), $h \cong 0$. Hence $f=h|X \cong 0$. Therefore $X$ is an FAR.

**Remark 1.** By the example of Petrus [8] the converse of Theorem 1 is false.

**Remark 2.** By the proof of Theorem 1 the property of being acyclic is a strong Whitney-reversible property. But it is not Whitney property (see [5]) by the same example of Petrus [8].

**Theorem 2.** Let $\mu$ be a Whitney map for $C(X)$. Let $\mathcal{B}$ be a class of compact connected polyhedra. If there is a sequence $\{t_n\} n \geq 1$ in $(0, \mu(X))$ such that $t_n \to 0$ as $n \to +\infty$ and $\mu^{-1}(t_n)$ is an hereditarily indecomposable $\mathcal{B}$-like continuum (see [6]) for each $n=1, 2, 3, \ldots$, then $X$ is also an hereditarily indecomposable $\mathcal{B}$-like continuum.

Hence the property of being an hereditarily indecomposable $\mathcal{B}$-like continuum is a strong Whitney-reversible property.

**Proof.** By [7] $X$ is hereditarily indecomposable. Hence it is sufficient to show that $X$ is $\mathcal{B}$-like. Without loss of generality we may assume that the sequence $\{t_n\} n \geq 1$ is decreasing. Now for each $n=1, 2, 3, \ldots$ we define a function $\eta_n: X \to \mu^{-1}(t_n)$ such that $x \in \eta_n(x) \in \mu^{-1}(t_n)$ for every $x \in X$. Since $X$ is hereditarily indecomposable, for each $n=1, 2, 3, \ldots$, $\eta$ is well-defined and continuous (see [2]). Similarly for each $n=1, 2, 3, \ldots$ we can define a map $p_n: \mu^{-1}(t_{n+1}) \to \mu^{-1}(t_n)$ such that $A \subset p_n(A)$ for each $A \in \mu^{-1}(t_{n+1})$. Then $\{\mu^{-1}(t_n), p_n\}$ is an inverse sequence of $\mathcal{B}$-like continua and onto bonding maps. Moreover we hold that $p_n \eta_{n+1}=\eta_n$ for each $n=1, 2, 3, \ldots$. Then it is clear that $X$ is homeomorphic to the inverse limit $\lim_{\leftarrow} \{\mu^{-1}(t_n), p_n\}$. Therefore $X$ is $\mathcal{B}$-like.

In particular the converse of the result of Krasinkiewicz (4.2. [4]) is hold.

**Corollary 1.** Let $\mu$ be a Whitney map for $C(X)$. If there exists a sequence
A note on some strong whitney-reversible properties

\{t_n\} n \geq 1 in (0, \mu(X)] such that \( t_n \rightarrow 0 \) as \( n \rightarrow +\infty \) and \( \mu^{-1}(t_n) \) is an hereditarily indecomposable tree-like continuum for each \( n = 1, 2, 3, \ldots \), then \( X \) is also an hereditarily indecomposable tree-like continuum.

The next lemma is useful for our results.

**Lemma** (Krasinkiewicz and Nadler [5]). Let \( \mu \) be a Whitney map for \( C(X) \). If \( X \) contains an \( n \)-odd \( (n \geq 3) \), there exists \( t_0 > 0 \) such that \( \mu^{-1}(t_0) \) contains an \( (n-1) \)-disk.

**Theorem 3.** Let \( \mu \) be a Whitney map for \( C(X) \). If \( \dim \mu^{-1}(t) \leq n < +\infty \) for every \( t \in (0, \mu(X)] \) and one of the following conditions is satisfied, then \( \dim X \leq n \):

1. \( \dim X < +\infty \),
2. \( \mu^{-1}(t) \) is locally connected for every \( t \in (0, \mu(X)] \),
3. \( \mu^{-1}(t) \) is hereditarily indecomposable for every \( t \in (0, \mu(X)] \).

**Proof.** First we shall show the case (1). The following inequality is clearly hold.

\[
\dim C(X) \leq 1 + \max \{ \dim \mu^{-1}(t) \mid t \in [0, \mu(X)] \} < +\infty.
\]

Then by the result of Rogers [9] \( \dim C(X) \leq \dim \mu^{-1}(t) \) for some \( t \in (0, \mu(X)] \). Hence \( \dim X \leq n \).

Next we shall the case (2). Then \( X \) is locally connected by [7]. If \( \dim X \geq 2 \), for every \( m \geq 3 \) \( X \) contains an \( (m+1) \)-odd. But by Lemma this fact contradicts the assumption. Hence \( \dim X = 1 \).

In the case (3) by the same way of the proof of Theorem 2 we can show that \( \dim X \leq n \).

**Corollary 2.** Let \( \mu \) be a Whitney map for \( C(X) \). If \( \mu^{-1}(t) \) is locally connected and \( \dim \mu^{-1}(t) \leq n < +\infty \) for every \( t \in (0, \mu(X)] \), then \( X \) is a finite graph. In particular if \( \dim \mu^{-1}(t) = 1 \) for every \( t \in (0, \mu(X)] \), \( X \) is an arc or a circle.

**Proof.** By the proof of Theorem 3 \( X \) is one-dimensional and locally connected. If \( X \) has infinitely many ramification points or a point with an infinite order, for every \( m > 1 \) \( X \) contains \( (m+1) \)-odd. Then by Lemma \( \dim \mu^{-1}(t) = n \) for some \( t \in (0, \mu(X)] \). This contradicts our assumption. Hence \( X \) has at most finitely many ramification points and the order of each point of \( X \) is finite. Therefore \( X \) is a finite graph.

The following corollary is an easy consequence of Theorem 1 and Corollary 2.
COROLLARY 3. Let $\mu$ be a Whitney map for $C(X)$. If $\mu^{-1}(t)$ is locally connected, $\dim \mu^{-1}(t) \leq n < +\infty$ and an FAR for every $t \in (0, \mu(X)]$, $X$ is a tree. In particular if $\dim \mu^{-1}(t) = 1$ for every $t \in (0, \mu(X)]$, $X$ is an arc.

REMARK 3. Corollary 1 also can be proved by Theorem 1, Theorem 3 and the fact that hereditary indecomposability is a strong Whitney-reversible property.

REMARK 4. The author does not know whether the conditions of Theorem 3 are essential. But it seems not to be essential.

Related to Theorem 1 the following problem is open.

PROBLEM. Is the property of being an FANR or a movable continuum a strong Whitney-reversible property?

References


Department of Mathematics
Osaka Kyoiku University
Tennoji, Osaka, Japan