ON MINIMAL SUBMANIFOLDS IN PRODUCT MANIFOLDS WITH A CERTAIN RIEMANNIAN METRIC

By
Masatoshi KOKUBU

Abstract. We generalize Ejiri's theorem about minimal submanifolds in warped product manifolds and see that there exist minimal immersions of the plane and the catenoid into other Riemannian manifolds.

§1. Introduction

Let \((B, g_B)\) and \((F, g_F)\) be Riemannian manifolds, and \(f\) a positive smooth function on \(B\). The warped product manifold of \((B, g_B)\) and \((F, g_F)\) by the warped function \(f\) is defined to be a product manifold \(B \times F\) provided with a Riemannian metric \(g_B + f^2 g_F\), and is denoted by \(B \times_f F\). N. Ejiri proved the following theorem:

THEOREM A ([E]). Let \((B, g_B), (F, g_F)\) and \(f\) be as above. Let \(M\) be an \(m\)-dimensional submanifold in \(B\) and \(N\) an \(n\)-dimensional submanifold in \(F\). Then the product submanifold \(M \times N\) in \(B \times_f F\) is minimal if and only if both \(M \hookrightarrow (B, f^{2m} g_B)\) and \(N \hookrightarrow (F, g_F)\) are minimal submanifolds.

For example, the catenoid, which is a minimal surface of revolution in \(\mathbb{R}^3\), can be considered as a product submanifold in a warped product manifold of the flat upper half-plane \(\{(y > 0) \subset \mathbb{R}^2, dx^2 + dy^2\}\) and the circle \((S^1, d\theta^2)\) of radius 1, whose warped function is \(f(x, y) = y\). So Theorem A implies that the generating curve of the catenoid is a geodesic in the upper half-plane provided with a Riemannian metric \(y^2(dx^2 + dy^2)\). And it gives a reason why the catenoid is generated by the catenary which is a plane curve formed by a flexible inextensible cable of uniform density hanging from two supports.

In this paper, we deal with product immersions whose ambient manifold possesses a Riemannian structure belonging to $\mathcal{M}$. $\mathcal{M}$ is a set consisting of a certain kind of Riemannian structures of the ambient manifold, which contains warped product structures, and is defined in the following section. We define an equivalent relation in $\mathcal{M}$ and show that the minimality for such immersions is invariant in equivalent classes in $\mathcal{M}$. Especially when the metric in $\mathcal{M}$ is of more distinctive form, we give a necessary and sufficient condition for the minimality. The second result is a generalization of Theorem A. We also give some applications of the results.

The author would like to express his gratitude to Professor Koichi Ogiue and Professor Takao Sasai for their suggestions and encouragement.

§2. Statement and Proof of Main Theorem

Throughout this paper, manifolds are assumed to be smooth and connected. At first we prepare some notations.

Let $(N, g_1), \ldots, (N, g_i)$ be Riemannian manifolds and $\dim N_a = n_a$. We put $N := N_1 \times \cdots \times N_i$, and denote by $\mathcal{M}'$ the set of all Riemannian metrics on $N$. A subset $\mathcal{M}$ of $\mathcal{M}'$ is defined to be

$$\mathcal{M} = \{ g \in \mathcal{M}'; g = f^2_1 g_1 + \cdots + f^2_i g_i \},$$

where $f_1, \ldots, f_i$ are positive smooth functions on $N$ and $g_1, \ldots, g_i$ are considered as tensor fields on $N$.

$\mathcal{M}$ is bijectively corresponded to $C^+(N) \times \cdots \times C^+(N)$ (the set of $l$-tuples of positive smooth functions on $N$). Hence we often denote an element $g = \sum f^2_a g_a$ by $(f_1, \ldots, f_i)$.

Let $d_1, \ldots, d_i$ be positive integers. We say that elements $g = (f_1, \ldots, f_i)$ and $\tilde{g} = (\tilde{f}_1, \ldots, \tilde{f}_i)$ in $\mathcal{M}$ are $(d_1, \ldots, d_i)$-equivalent if $f_1^{d_1} \cdots f_i^{d_i} = \tilde{f}_1^{d_1} \cdots \tilde{f}_i^{d_i}$ holds, and denote it by $g \sim_{(d_1, \ldots, d_i)} \tilde{g}$. The relation $\sim_{(d_1, \ldots, d_i)}$ is an equivalent relation in $\mathcal{M}$. We denote by $\mathcal{M} / \sim_{(d_1, \ldots, d_i)}$ the quotient set.

Let $\varphi_a : M_a \to N_a$ be an immersion of a $d_a$-dimensional manifold $M_a$ into $N_a = (1, \ldots, l)$. We denote by $\Phi$ the product immersion of $M = M_1 \times \cdots \times M_i$ into $N = N_1 \times \cdots \times N_i$. It is an easy observation that if $g \sim_{(d_1, \ldots, d_i)} \tilde{g}$ then the volume elements of $(M, \Phi^* g)$ and $(M, \Phi^* \tilde{g})$ are coincide.

That the minimality is equivalent to the stationariness of a variational problem about the volume (cf.[L]) gives an implication of the following.

**Theorem 2.1.** Assume that $g \sim_{(d_1, \ldots, d_i)} \tilde{g}$. Then $\Phi : M \to (N, g)$ is minimal if
and only if $\Phi: M \to (N, g)$ is minimal, i.e., the minimality for $\Phi$ depends only on elements in $\mathcal{M}_{(d_1, \ldots, d_l)}$.

Remark 2.2. Theorem 2.1 implies that it does not depend on representative elements of an equivalent class whether $\Phi$ is stationary with respect to the first variation of volume or not. However, such assertion does not hold for the second variation, that is, the stability property is not preserved in the equivalent class. We can see an example for this in §3.

Theorem 2.3. Let $F_\alpha$ be a positive smooth function on $N_{\alpha}(\alpha = 1, \ldots, l)$. Assume that $g \sim_{(d_1, \ldots, d_l)} (F_1, \ldots, F_l)$ where $F_\alpha$ is identified with a function on $N$. Then $\Phi: M \to (N, g)$ is minimal if and only if each $\varphi_\alpha: M_\alpha \to (N_{\alpha}, F_\alpha^2 g_\alpha)$ is minimal.

Remark 2.4. The assumption of Theorem 2.3 means the separation of variables of the function $f_1^{d_1} \cdots f_l^{d_l}$.

Proof of Theorem 2.1 and 2.3. We shall prove Theorem 2.1 and 2.3 by the moving frame method.

Let the convention on the ranges of indices be the following:

$$1 \leq \alpha, \beta \leq l, \quad 1 \leq i_\alpha, j_\alpha \leq n_\alpha.$$

Let $e_{(\alpha)1}, \ldots, e_{(\alpha)d_\alpha}, \ldots, e_{(\alpha)n_\alpha}$ be a local orthonormal frame field of $(N_{\alpha}, g_\alpha)$ adapted to $\varphi_\alpha$, i.e., the restrictions of $e_{(\alpha)1}, \ldots, e_{(\alpha)d_\alpha}$ to $M_\alpha$ are tangent to $M_\alpha$, and $\theta_{(\alpha)1}, \ldots, \theta_{(\alpha)n_\alpha}$ be the dual coframe field. We denote by $(\theta_{(\alpha)j_\alpha})$ the connection form of $(N_{\alpha}, g_\alpha)$ with respect to $\theta_{(\alpha)1}, \ldots, \theta_{(\alpha)n_\alpha}$, i.e., $(n_\alpha \times n_\alpha)$ matrix-valued 1-form uniquely determined by the structure equations

$$d\theta_{(\alpha)j_\alpha} = -\sum_{j_\alpha} \theta_{(\alpha)i_\alpha} \wedge \theta_{(\alpha)j_\alpha},$$

$$\theta_{(\alpha)i_\alpha} + \theta_{(\alpha)j_\alpha} = 0.$$
Therefore if we put
\[ \Theta^{\mu}_{ja} = \frac{1}{f_a} \{ (e_{(\alpha)ja} f_a) \theta^{(\alpha)ja} - (e_{(\alpha)ja} f_a) \theta_{(\alpha)ja} + f_a \theta^{(\alpha)ja} \}, \]
\[ \Theta^{\mu}_{ja} = \frac{1}{f_{\beta}} \{ (e_{(\beta)ja} f_a) \theta^{(\alpha)ja} - \frac{1}{f_a} (e_{(\alpha)ja} f_{\beta}) \theta_{(\beta)ja} \}, \]
then the following equations hold:
\[ d(f_a \theta^{(\alpha)ja}) = -\sum \sum \Theta^{\mu}_{ja} \wedge f_{\beta} \theta_{(\beta)ja} \]
\[ \Theta^{\mu}_{ja} - \Theta^{\mu}_{ja} = 0. \]
So \( (\Theta^{\mu}_{ja}) \) is the connection form of \((N, g)\).

From now on we shall use the same notations of tensor fields on an ambient space and the restrictions of them to a submanifold, and use the following convention on ranges of indices:
\[ 1 \leq \alpha, \beta \leq l, \]
\[ 1 \leq i_a, j_a \leq d_a, \quad 1 \leq i_{a+1}, j_{a+1} \leq n_a. \]
The mean curvature normal is the trace of the second fundamental form \( h \) divided by the dimension of the submanifold. The minimality is equivalent to that \( trh \) is identically zero.

The second fundamental form \( h \) of \( \Phi \) can be written locally as
\[ h = \sum \sum \frac{1}{f_a} e_{(\alpha)ja} \otimes \sum \sum f_{\beta} \theta_{(\beta)ja} \Theta^{\mu}_{ja}, \]
by definition. Hence,
\[ trh = \sum \sum \frac{1}{f_a} e_{(\alpha)ja} \otimes \sum \sum \frac{1}{f_{\beta}} \Theta^{\mu}_{ja} (e_{(\beta)ja}). \]
On the other hand, it is computed that
\[ \sum \Theta^{\mu}_{ja} (e_{(\beta)ja}) = \begin{cases} -d_\alpha \frac{(e_{(\alpha)ja} f_a)}{f_a} + \sum \theta_{(\alpha)ja} (e_{(\alpha)ja}) & \text{if } \beta = \alpha, \\ -d_\beta \frac{(e_{(\alpha)ja} f_{\beta})}{f_a}, & \text{if } \beta \neq \alpha. \end{cases} \]
Therefore
trh = \sum_{\alpha} \sum_{i_\alpha} \frac{1}{f_\alpha} e^{(\alpha)}_{i_\alpha} \otimes \left( \frac{1}{f_\alpha} \left( -d_\alpha \frac{(e^{(\alpha)}_{i_\alpha} f_\alpha)}{f_\alpha} + \sum_{j_\alpha} \theta^{i_\alpha}_{(\alpha) j_\alpha} (e^{(\alpha)}_{(\alpha) j_\alpha}) \right) \right)

+ \sum_{\beta \neq \alpha} \frac{1}{f_\beta} \left( -d_\beta \frac{(e^{(\alpha)}_{i_\alpha} f_\beta)}{f_\beta} \right) \right) \right)

\sum_{\alpha} \sum_{i_\alpha} \frac{1}{f_\alpha} e^{(\alpha)}_{i_\alpha} \otimes \frac{1}{f_\alpha} \left( -\sum_{\beta} \frac{1}{f_\beta} \frac{(e^{(\alpha)}_{i_\alpha} f_\beta)}{f_\beta} + \sum_{j_\alpha} \theta^{i_\alpha}_{(\alpha) j_\alpha} (e^{(\alpha)}_{(\alpha) j_\alpha}) \right) \right)

\sum_{\alpha} \sum_{i_\alpha} \frac{1}{f_\alpha} e^{(\alpha)}_{i_\alpha} \otimes \frac{1}{f_\alpha} \left[ -e^{(\alpha)}_{i_\alpha} \left( \log f^{d_\alpha} \prod_{i=1}^{d_\alpha} f_i^{d_i} \right) + \sum_{j_\alpha} \theta^{i_\alpha}_{(\alpha) j_\alpha} (e^{(\alpha)}_{(\alpha) j_\alpha}) \right].

So Theorem 2.1 is proved.

Next we assume that \( f^{d_\alpha} \prod_{i=1}^{d_\alpha} f_i^{d_i} = F^{d_\alpha} \prod_{i=1}^{d_\alpha} F_i^{d_i} \) holds for some positive functions \( F_i : M_i \to \mathbb{R}^+ \). Then

\[ \text{trh} = \sum_{\alpha} \sum_{i_\alpha} \frac{1}{f_\alpha} e^{(\alpha)}_{i_\alpha} \otimes \frac{1}{f_\alpha} \left[ -e^{(\alpha)}_{i_\alpha} \left( \log f^{d_\alpha} \right) + \sum_{j_\alpha} \theta^{i_\alpha}_{(\alpha) j_\alpha} (e^{(\alpha)}_{(\alpha) j_\alpha}) \right]. \]

Thus \( \Phi \) is minimal if and only if

\[ -e^{(\alpha)}_{i_\alpha} \left( \log f^{d_\alpha} \right) + \sum_{j_\alpha} \theta^{i_\alpha}_{(\alpha) j_\alpha} (e^{(\alpha)}_{(\alpha) j_\alpha}) = 0 \]

holds for all \( i_\alpha = d_\alpha + 1, \ldots, n_\alpha \) and for all \( \alpha = 1, \ldots, l \).

Finally, we have only to prove that the above equation for \( i_\alpha = d_\alpha + 1, \ldots, n_\alpha \) is the minimality condition for the immersion \( \varphi_\alpha : M_\alpha \to (N_\alpha, F^{d_\alpha}_{\alpha} g_\alpha) \). However, the proof is of similar computations as above, hence we omit it.

§3. Applications

Theorem 2.1 implies that if there is a product minimal submanifold \( M \) in a product manifold \( N \) provided with a Riemannian metric \( g \in \mathcal{M} \) then it is also a minimal submanifold in \( M \) with any metric which is \( (d_1, \ldots, d_\ell) \)-equivalent to \( g \). So we can obtain examples of minimal submanifolds from known examples.

Example 3.1. A totally geodesic plane in \( \mathbb{R}^3 \) is a cylindrical minimal surface, which is also interpreted as a product minimal surface in \( (\mathbb{R}^2 \times \mathbb{R}, (dx^2 + dy^2) + dz^2) \). Therefore it is also a minimal surface in \( (U, g_f = (dx^2 + dy^2) + f^2 dz^2) \) where \( U \) is a open set in \( \mathbb{R}^2 \times \mathbb{R} \) and \( f \) is an arbitrary positive smooth function on \( U \). In particular, if \( U \) and \( f \) are the following (1)–(3) then \( (U, g_f) \) is a Riemannian homogeneous space in each case.

(1) Let \( U = \{ y > 0 \} \subset \mathbb{R}^2 \times \mathbb{R} \) and \( f(x, y, z) = y \). Then \( g_f = (dx^2 + dy^2) + y^2 dz^2 \). So \( (U, g_f) \) is a warped product manifold of the Poincaré upper half-plane.
and \( R \). Moreover it is isometric to the following Lie group \( G_1 \) provided with a left invariant metric:

\[
G_1 = \left\{ \begin{bmatrix} y & 0 & 0 & x \\ 0 & y & 0 & 0 \\ 0 & 0 & 1/y & z \\ 0 & 0 & 0 & 1 \end{bmatrix} ; (x, z) \in R^2, y \in R^+ \right\},
\]

which is a semi-direct product of \((R^2,+)\) and \((R^+,\times)\).

In fact, for an arbitrary \( a_i = (x_i, y_i, z_i) \in G_1 \), let \( L_{a_i} \) denote the left translation by \( a_i \), then

\[
L_{a_i} = \begin{bmatrix} y & 0 & 0 & x \\ 0 & y & 0 & 0 \\ 0 & 0 & 1/y & z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_i & 0 & 0 & y_i x + x_i \\ 0 & y_i & 0 & 0 \\ 0 & 0 & 1/y_i & z_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\]

and

\[
L_{a_i}^* g_f = \frac{(d(y_i x + x))^2 + (d(y_i y))^2}{y_i^2 y^2} + \frac{(y_i y)^2 (d(z / y_i + z_i))^2}{y_i^2 y^2}
\]

\[
= \frac{y_i^2 dx^2 + y_i^2 dy^2}{y_i^2 y^2} + \frac{1}{y_i^2 y^2} dz^2
\]

The obtained minimal surface is totally geodesic if it is defined by the equation \( x = \text{constant} \), otherwise it is not totally geodesic.

(2) Let \( U = \{ z > 0 \} \subset R^2 \times R \) and \( f(x, y, z) = 1/z \). In the similar way to (1), we can prove that \((U, g_f)\) is isometric to the following Lie group \( G_2 \) provided with a left invariant metric:

\[
G_2 = \left\{ \begin{bmatrix} z & 0 & 0 & 0 \\ 0 & 1/z & 0 & x \\ 0 & 0 & 1/z & y \\ 0 & 0 & 0 & 1 \end{bmatrix} ; (x, y) \in R^2, z \in R^+ \right\},
\]

which is a semi-direct product of \( R^2 \) and \( R^+ \). Moreover it is easily checked that \((U, g_f)\) has constant sectional curvature \(-1\), that is, \((U, g_f)\) is isometric to the hyperbolic space, and that the obtained minimal surface is totally geodesic.

(3) Let \( U = R^2 \times R - \{ x = y = 0 \} \subset R^2 \times R \) and \( f(x, y, z) = (x^2 + y^2)^{1/2} \). Making use of the polar coordinate \( x = r \cos \theta, y = r \sin \theta, \) \((U, g_f)\) can be written as

\[
((R^+ \times S^1) \times R, \frac{1}{r^2} dr^2 + d\theta^2 + r^2 dz^2),
\]
and is isometric to the following Lie group $G_3$ provided with a left invariant metric, that can be proved similarly to (1):

$$G_3 = \left\{ \begin{bmatrix} r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/r \end{bmatrix} : r \in \mathbb{R}^*, \theta \in \mathbb{R}, \theta \in S^1 \right\}.$$  

which is a semi-direct product of $\mathbb{R} \times S^1$ and $\mathbb{R}^*$.

The obtained minimal surface is totally geodesic if it is defined by the equation $\theta = \text{constant}$, otherwise it is not totally geodesic.

**Example 3.2.** A totally geodesic plane minus one point in $\mathbb{R}^3$ can be considered as a minimal cone over a great circle in $S^2$, i.e., a product minimal surface in $\mathbb{R}^+ \times S^2$, where $t$ is the canonical coordinate for $\mathbb{R}^+$. Therefore it is also a minimal surface in $(\mathbb{R}^+ \times S^2, g_f = dt^2 + f^2 \gamma g_{S^2})$ where $f$ is an arbitrary positive smooth function on $\mathbb{R}^+ \times S^2$ and $g_{S^2}$ is the Riemannian metric of $S^2$ of constant Gaussian curvature $+1$.

In particular, we consider the case of $f = 1/t$. Then $(\mathbb{R}^+ \times S^2, g_f)$ is isometric to the Riemannian product manifold $\mathbb{R}^+ \times S^2$. In this case, it may be considered to be trivial that the surface is minimal in $(\mathbb{R}^+ \times S^2, g_f)$, more precisely it is totally geodesic. However this is an easy example for Remark 2.2. In fact, the surface is a stable minimal surface in $\mathbb{R}^+ \times S^2$ but is an unstable minimal surface in $(\mathbb{R}^+ \times S^2, g_f)$.

As another special case, we take the function $f$ to be $\frac{1}{2} \cos \rho(t)$, where $\rho(t)$ is defined by $\sin \rho(t) = t^2 / 2$ in an appropriate interval. Then $g_f$ is a metric of constant sectional curvature $+1$ defined on some open set in $\mathbb{R}^+ \times S^2$. It is remarked that the minimal surface obtained in this case is totally geodesic.

**Example 3.3.** Let $H^2 = \{(x,y) \in \mathbb{R}^2; y > 0\}$ be the upper half-plane. As mentioned in §1, the catenoid is a product minimal surface in $(H^2 \times S^1, (dx^2 + dy^2) + y^2 d\theta^2)$, hence is also a minimal surface in $(U, g_f = (dx^2 + dy^2) / f^2 + f^2 y^2 d\theta^2)$ where $U$ is an open set in $H^2 \times S^1$ and $f$ is an arbitrary positive smooth function on $U$.

In particular, we consider the function $f(x,y,\theta) = y$ and $U = H^2 \times S^1$. Then $(U, g_f) = (H^2, \text{the Poincaré metric}) \times_y S^1$.

Moreover, in the similar way to Example 3.1, it can be shown that this Riemannian manifold is isometric to a locally homogeneous space $2\pi \mathbb{Z} \setminus G$.
Masatoshi Kokubu

defined as follows:

\[ G \text{ is a Lie group of semi-direct product of } \mathbb{R}^2 \text{ and } \mathbb{R}^+ \text{, which is realized as a subgroup of } GL(4; \mathbb{R}) \text{ as follows:} \]

\[
G = \begin{bmatrix}
  y & 0 & 0 & x \\
  0 & y & 0 & 0 \\
  0 & 0 & 1/y^2 & \theta \\
  0 & 0 & 0 & 1 \\
\end{bmatrix}; x \in \mathbb{R}, \theta \in \mathbb{R}, y \in \mathbb{R}^+ .
\]

The metric \((dx^2 + dy^2)/y^2 + y^4d\theta^2\) on \(G\) is left invariant. \(2\pi \mathbb{Z}\) is a discrete subgroup of \(G\) defined by

\[
2\pi \mathbb{Z} = \begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 2\pi n \\
  0 & 0 & 0 & 1 \\
\end{bmatrix}; n \in \mathbb{Z} .
\]

The obtained minimal surface is not totally geodesic. To see this, we may assume that the surface is defined by \(y = c \cosh(x/c)\) for some non-zero constant \(c\), and have only to show that the curve defined by the equation \(x = 0\) is a geodesic on this surface but is not a geodesic in \(2\pi \mathbb{Z} \setminus G\).

As an another application we give the following.

\textbf{Example 3.4.} In [L-F], L. Lyusternik and A. I. Fet proved that there exists a closed geodesic in any compact Riemannian manifold. So by this theorem together with Theorem 2.1, we immediately have the following:

\textbf{Corollary.} In any product manifold of \(k\) numbers of compact manifolds provided with a Riemannian metric which is \((1,\ldots,1)\)-equivalent to any product metric, there exists a \(k\)-dimensional minimal torus.

\textbf{References}


Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1-1, Hachioji
Tokyo, 192-03
Japan