ON CL-ISOCOMPACTNESS AND WEAK 
BOREL COMPLETENESS

By

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Introduction.

A space $X$ is said to be isocompact $[1]$ if every countably compact closed subset of $X$ is compact. In this paper we introduce a new class of spaces called $CL$-isocompact spaces. We call a space $X$ $CL$-isocompact if the closure of each countably compact subset of $X$ is compact. $CL$-isocompact spaces are isocompact. The class of $CL$-isocompact spaces behaves well with respect to topological operations. For example the class is productive and closed hereditary. After showing various properties of $CL$-isocompact spaces, we investigate the relationship between $CL$-isocompact spaces, weakly $\theta$-refinable spaces $[6]$ and weakly Borel complete spaces $[3]$. We show that every weakly $\theta$-refinable space of non-measurable cardinal is weakly Borel complete and every weakly Borel complete space is $CL$-isocompact.

All spaces are assumed to be completely regular. But this is not always needed.

§ 1. Fundamental properties.

Definition 1.1. A space $X$ is said to be $CL$-isocompact if the closure of each countably compact subset of $X$ is compact.

Obviously $CL$-isocompact spaces are isocompact.

Proposition 1.2. The following facts hold.

(a) Let $f$ be a perfect map from $X$ onto $Y$. Then, $X$ is $CL$-isocompact iff $Y$ is $CL$-isocompact.

(b) Let $X$ be $CL$-isocompact, and $Y$ be an $F_\sigma$-subset of $X$. Then, $Y$ is $CL$-isocompact.

(c) If $X=\prod_\alpha X_\alpha$, with $X_\alpha$ $CL$-isocompact for $\alpha \in A$, then $X$ is $CL$-isocompact.

(d) If $X=\bigoplus_\alpha X_\alpha$, with $X_\alpha$ $CL$-isocompact for $\alpha \in A$, then $X$ is $CL$-isocompact.
(e) If each $X_\alpha$ is a CL-isocompact subset of $X$, then $\bigcap_\alpha X_\alpha$ is CL-isocompact.

(f) The following (1), (2) and (3) are equivalent.

1. $X$ is hereditarily CL-isocompact.
2. $X$ is hereditarily isocompact.
3. For each $x \in X$, $X \setminus \{x\}$ is CL-isocompact.

Proof. (a) Compactness and countably compactness are preserved by perfect maps. From this fact, it is easy to show (a). (b) We set $Y = \bigcup_{i=1}^\infty Y_i$, each $Y_i$ is closed in $X$. Let $E$ be any countably compact subset of $Y$. Since each $Y_i$ is CL-isocompact, $\overline{\text{Cl}(E \cap Y_i)}$ is compact. $\bigcup_i \overline{\text{Cl}(E \cap Y_i)}$ contains $E$ as a dense subset. Since $\bigcup_i \overline{\text{Cl}(E \cap Y_i)}$ is pseudocompact, it is compact. We get $\overline{\text{Cl}(E \cap Y_i)} = \bigcap_i \overline{\text{Cl}(E \cap Y_i)}$. (c) Let $E$ be any countably compact subset of $X$. Since each $\text{Pr}_aE$ is countably compact, $\overline{\text{Cl}(\text{Pr}_aE)}$ is compact. Here $\text{Pr}_a$ is the projection of $X$ onto $X_a$. The closure of $E$ in $X$ is contained in the compact space $\prod_a \overline{\text{Cl}(\text{Pr}_aE)}$. $\overline{\text{Cl}(E \cap Y_i)}$ is compact. (d) is trivial. (e) $\bigcap_i X_\alpha$ can be naturally embedded as a closed subspace into $\prod_a \overline{\text{Cl}(E \cap Y_i)}$. By (b) and (c), $\bigcap_i X_\alpha$ is CL-isocompact. (f) The equivalence of (1) and (2) is obvious. We assume (3). Let $Y$ be any subspace of $X$. Since $Y = \bigcap \{X \setminus \{x\} \mid x \in X \setminus Y\}$, $Y$ is CL-isocompact by (e).

Bacon proved in [1] that the product of an isocompact space and a hereditarily isocompact space is isocompact. The following result generalizes it.

Proposition 1.3. Let $X$ be CL-isocompact, and $Y$ be isocompact. Then $X \times Y$ is isocompact.

Proof. Let $E$ be any countably compact closed subset of $X \times Y$. Since $\text{Pr}_X E$ is countably compact, $\overline{\text{Cl}(\text{Pr}_X E)}$ is compact. Therefore $\text{Pr}_Y E$ is closed countably compact in $Y$. So, $\text{Pr}_Y E$ must be compact. $E$ is contained in the compact space $\overline{\text{Cl}(\text{Pr}_X E) \times \text{Pr}_Y E}$. The proof is complete.

Proposition 1.4. The following (a) and (b) hold.

(a) For each space $X$, there exists a CL-isocompact space $pX$ with the following properties.

1. $X \subseteq pX \subseteq \beta X$. Here $\beta X$ is the Stone-Čech compactification of $X$.
2. If $f$ is a map from $X$ onto a CL-isocompact space $Y$, then $f$ has a continuous extension $f^p$ that maps $pX$ onto $Y$.

(b) If $X$ has a dense countably compact subspace, then $pX = \beta X$. Conversely,
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if \( pX = \beta X \), then \( X \) is pseudocompact.

PROOF. (a) is obtained from Proposition 1.2. (b), (c) and Theorem 2.1. in [7]. (b) is trivial. Note that \( pX \subseteq \nu X \), \( \nu X \) is the Hewitt’s realcompactification.

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§ 2. Weak Borel completeness.

A space \( X \) is said to be weakly Borel complete [3] if each Borel ultrafilter \( \mathcal{B} \) on \( X \) with c.i.p. (countable intersection property) has the property that \( \cap \{ Z | Z \in \mathcal{B} \cap \mathcal{Z}(X) \} = \cap \{ F | F \in \mathcal{B}, \ F \text{ is closed in } X \} \) is non-void. Here \( \mathcal{Z}(X) \) is the set of zero sets of \( X \).

THEOREM 2.1. Weakly Borel complete spaces are CL-isocompact.

PROOF. Weak Borel completeness is closed hereditary [3]. So, we show that a weakly Borel complete space which has a dense countably compact subset is compact. Let \( X \) be weakly Borel complete, and \( Y \) be a dense countably compact subset of \( X \).

Suppose that \( X \) is not compact. Since \( X \) is pseudocompact, \( X \) is not realcompact. We take a free zero ultrafilter \( \mathcal{Z} \) on \( X \) with c.i.p.. Each element of \( \mathcal{Z} \) must intersect with \( Y \). Put \( \mathcal{A} = \{ \mathcal{K} | \mathcal{K} \text{ is a closed family such that (1) } \mathcal{Z} \subseteq \mathcal{K} \text{. (2) If } H \in \mathcal{K}, \text{ then } H \cap Y \neq \emptyset \text{. (3) } \mathcal{K} \text{ is closed under the finite intersections.} \} \). Let \( \mathcal{K} \) be a maximal element of \( \mathcal{A} \). It is easily showed that \( \mathcal{K} \) is closed under the countable intersections, and \( X \in \mathcal{K} \) by the maximality.

Put \( \mathcal{B} = \{ B \in Bo(X) | B \supset H \cap Y \text{ for some } H \in \mathcal{K} \} \). Here \( Bo(X) \) is the set of Borel sets of \( X \). We take a Borel ultrafilter \( \mathcal{B} \) on \( X \) containing \( \mathcal{B} \). Put \( \mathcal{E} = \{ B \in Bo(X) | \text{If } B \supset H \cap Y \text{ for any } H \in \mathcal{K}, \text{ then } B \cap H \cap Y = \emptyset \text{ for some } H \in \mathcal{K} \} \).

Now, \( \mathcal{E} \) satisfies the following conditions.

(a) If \( F \) is closed in \( X \), then \( F \in \mathcal{E} \).

(b) If \( B \in \mathcal{E} \), then \( X - B \in \mathcal{E} \).

(c) If \( \mathcal{E} \supset \{ B_i \}_{i=1}^{n} \), then \( \bigcap_{i=1}^{n} B_i \in \mathcal{E} \).

Firstly we show (a). Let \( F \) be a closed subset of \( X \), and suppose that \( F \supset H \cap Y \) for any \( H \in \mathcal{K} \). Obviously \( F \in \mathcal{K} \). Put \( \mathcal{L} = \mathcal{K} \cup \{ F \cap H | H \in \mathcal{K} \} \). \( \mathcal{L} \) satisfies (1), (3) of \( \mathcal{A} \), and \( \mathcal{K} \neq \mathcal{L} \), because \( F \in \mathcal{L} \). By the maximality of \( \mathcal{K} \), there exists \( H \in \mathcal{K} \) such that \( F \cap H \cap Y = \emptyset \). This shows that \( F \in \mathcal{E} \). The proof of (b) and (c) is a routine matter. We omit the proof.

Since \( Bo(X) \) is the smallest \( \sigma \)-field containing the set of closed subsets of \( X \), we get \( \mathcal{E} = Bo(X) \).

Suppose that \( B \in \mathcal{B} \), and \( B \cap H \cap Y = \emptyset \) for some \( H \in \mathcal{K} \). Then \( X - B \in \mathcal{W} \subseteq \mathcal{B} \).
It is a contradiction that $\mathcal{B}$ is a filter. Therefore, for each $B \in \mathcal{B}$, $B \cap H \cap Y \neq \emptyset$ for any $H \in \mathcal{H}$. It follows from $\mathcal{E} = \text{Bo}(X)$ that for each $B \in \mathcal{B}$ there exists some $H(B) \in \mathcal{H}$ such that $B \supset H(B) \cap Y$. This fact gives that $\mathcal{B}$ has c.i.p. Since $Z \subset \mathcal{B}$, we obtain that $\bigcap \{Z | Z \in \mathcal{B} \cap \mathcal{E}(X)\} = \emptyset$. This is a contradiction that $X$ is weakly Borel complete.

**Corollary 2.2.** If $X$ has a countably compact dense subset, then $wX = \beta X$. Here $wX$ is the weak Borel completion of $X$.

**Proof.** Apply Proposition 1.4. (b) and Theorem 2.1. ■

**Corollary 2.3.** If $X$ is a perfect image of a weakly Borel complete space, then $X$ is CL-isocompact.

**Proof.** Apply Proposition 1.2. (a) and Theorem 2.1. ■

It is not known whether perfect images of weakly Borel complete spaces are weakly Borel complete.

**Theorem 2.4.** If $X$ is a weakly $\theta$-refinable space of non-measurable cardinal, then $X$ is weakly Borel complete.

**Proof.** Hardy proved in [2] that a weakly $\theta$-refinable space of non-measurable cardinal is $a$-realcompact. The procedure of the proof is valid for this theorem.

Let $\mathcal{B}$ be a Borel ultrafilter on $X$ with c.i.p.. Let $\mathcal{K} = \{H | H \in \mathcal{B}, H \text{ is closed in } X\}$. Suppose that $\bigcap \mathcal{K} = \emptyset$. Since $\mathcal{U} = \{X - H | H \in \mathcal{K}\}$ is an open cover of $X$, there exists a weak $\theta$-refinement $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ of $\mathcal{U}$. For $n, j$, let $H_{n,j} = \{x \in X | 1 \leq \text{ord}(x, \mathcal{U}_n) \leq j\}$. Then obviously $X = \bigcup_{n,j} H_{n,j}$. By c.i.p. of $\mathcal{B}$, there exist natural numbers $n, j$ such that $H_{n,j} \cap B \neq \emptyset$ for any $B \in \mathcal{B}$. We fix these $n, j$.

By virtue of Zorn’s lemma, we can find a discrete subspace $D \subset H_{n,j}$ such that

(a) $\{\text{St}(x, \mathcal{U}_n) | x \in D\}$ covers $H_{n,j}$,

(b) If $V \in \mathcal{U}_n$, then $|V \cap D| \leq 1$.

Since $|X| < m_i$, $D$ is realcompact. Here $m_i$ is the first measurable cardinal.

For each $F \in \mathcal{K}$, let $F^* = \{x \in D | \text{St}(x, \mathcal{U}_n) \cap F \cap H_{n,j} \neq \emptyset\}$. Then $\mathcal{M} = \{F^* | F \in \mathcal{K}\}$ is a free filter base on $D$. Take a ultrafilter $\mathcal{K}$ on $D$ such that $\mathcal{M} \subset \mathcal{K}$. Since $D$ is realcompact, there exists a countable subcollection $\{K_i\}_{i=1}^{\infty}$
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Let $U_i = \bigcup \{ St(x, \mathcal{V}_n) \mid x \in K_i \}$. If $x \in \bigcap_i U_i$, then for each $i$ there exist $x_i \in K_i$ and $V_i \subseteq \mathcal{V}_n$ with $x, x_i \in V_i$. Since this shows that $\text{ord}(x, \mathcal{V}_n) = \omega$, we have $x \in H_n$. Consequently $H_n \cap \bigcap_i U_i = \emptyset$.

If $X - U_i \in \mathcal{K}$ for some $i$, we can consider $(X - U_i)^*$. But it is easily showed that $K_i \cap (X - U_i) = \emptyset$. Since $K_i$, $(X - U_i)^* \in \mathcal{K}$, this is a contradiction. It must be $X - U_i \in \mathcal{K}$ for every $i$. Therefore $X - U_i \in \mathcal{B}$ for every $i$. Since it must be $U_i \in \mathcal{B}$ for every $i$, we have $\bigcap_i U_i \in \mathcal{B}$. It follows that $H_n \cap \bigcap_i U_i \neq \emptyset$. This is a contradiction.

By the similar procedure of the proof of Theorem 2.4, we can show that each $\theta$-refinable space [6] is weakly Borel complete if the cardinality of each closed discrete subspace is non-measurable.

**Remark 2.5.** Hardy conjectured in [2, Remark 2.8.] that there exists an $a$-realcompact space of non-measurable cardinal which is not weakly $\theta$-refinable. Rudin's Dowker space in [4] is, in fact, such a space. Because Simon proved in [5] that the Rudin's Dowker space is $a$-realcompact, and not weakly Borel complete. This fact answers the third question posed in [9].

**Corollary 2.6.** A quasi-developable space of non-measurable cardinal is Borel complete.

**Proof.** It is known that a quasi-developable space is hereditarily weakly $\theta$-refinable, and that Borel completeness is equivalent to be hereditarily weakly Borel complete [3].

**Addendum**

Theorem 2.4 is extendable to the class of $\theta$-penetrable spaces. Namely each $\theta$-penetrable space of non-measurable cardinal is weakly Borel complete. For $\theta$-penetrable spaces, refer to [8]. For the proof, we use the fact that, for a free closed filter $\mathcal{F}$ on $X$ with c.i.p. which is extendable to a Borel ultrafilter on $X$ with c.i.p., $\{ X-F \mid F \in \mathcal{F} \}$ has a weak $\theta$-refinement if it has a $\theta$-penetration. This fact is proved by the quite similar way of [8, Lemma 2.2].

**References**


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