FOURTH ORDER SEMILINEAR PARABOLIC EQUATIONS

By

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1. Introduction.

The aim of this paper is to give a simple proof of the existence of a smooth solution to the semilinear parabolic equation with fourth order elliptic operator:

\[ u_t = -\varepsilon^4 \Delta^4 u + f(t, x, u, u_x, u_{xx}) = L(t, x, u), \]

where \( x \in \Omega \subset \mathbb{R}^n \), \( \Omega \) is a bounded domain, \( t \in [0, T_{\text{max}}) \), \( T_{\text{max}} \leq +\infty \), and \( \Delta^4 = \Delta \cdot \Delta \). The vector \( u_x \) is a vector of partial derivatives \( (u_{x_1}, \ldots, u_{x_n}) \) and \( u_{xx} \) stands for the Hessian matrix \( [u_{x_i x_j}], i, j = 1, \ldots, n \). We consider (1) together with initial-boundary conditions

\[ u(0, x) = u_0(x), \quad x \in \Omega, \]

\[ \frac{\partial u}{\partial n} = \frac{\partial (\Delta u)}{\partial n} = 0 \quad \text{when} \quad x \in \partial \Omega. \]

Schematically we may write (3) as \( B_1 u = B_2 u = 0 \).

In recent years a rapidly growing interest has been evinced in special problems such as the Cahn-Hilliard or the Kuramoto-Sivashinsky equations covered by our general form (1). Recently weak solutions for these special problems were considered in Temam’s monograph [12]. The methods used here are an extension of those in previous papers [5, 6] devoted to the study of second order equations. General scheme of our proof of local existence (construction of the set \( X \), considerations following (19)) is similar to the classical proof of the Picard theorem for solutions of ordinary differential equations.


We have two tasks in this paper. In Part I we prove local in time classical solvability of (1)-(3). We cannot expect global (that is in an arbitrarily large time interval) solvability of (1)-(3) under the weak assumption of local Lipschitz continuity of the nonlinear term \( f \) only (because of the possible rapid growth

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of $f$ with respect to $u, u_x$ or $u_{xx}$). However, the technique and estimates
developed in reaching our first task allow immediate verification in Part II for
a special problem (Cahn-Hilliard or Kuramoto-Sivashinsky equations) of global
Lipschitz continuity of its specific nonlinearities, which in turn guarantees
global solvability of this problem. Using our technique it is possible (see e.g.
[6]) to find effective estimates of the life time of solutions to various problems
with blowing-up solutions, blowing-up derivatives, etc. The last may be of
special interest for the numerical calculations as an indication of how long the
solution of the approximated problem exists.

3. Assumptions.

Let us assume that $\partial \Omega \in C^{1+\mu}$ with some $\mu \in (0, 1)$, the function $f$ is locally
Lipschitz continuous with respect to its arguments $t, u, u_x, u_{x_ix_j}$ ($i, j = 1, \ldots, n$) and locally Hölder continuous with respect to $x$ (exponent $\mu$) in the set
$[0, T] \times \bar{\Omega} \times R^{1+n+n^2}$. When $n > 3$, for existence of the Hölder solution to (1)-(3) we need additionally to assume that the partial derivatives $f_t, f_u, f_{ux}, f_{uxx}$ fulfill the assumptions just mentioned for $f$ (here and in what follows we use the simplified notation for partial derivatives, e.g. $f_t$ denotes $\partial f/\partial t$). By “Hölder solution” of (1)-(3) we mean the classical solution of the problem being Hölder continuous together with all the derivatives appearing in (1). The initial function $u_0 \in C^{1+\mu}(\bar{\Omega})$ fulfills the compatibility conditions required for a smooth solution:

$$\frac{\partial u_0}{\partial n} = \frac{\partial (\Delta u_0)}{\partial n} = 0 \quad \text{for } x \in \partial \Omega,$$

moreover, when $n > 3$

$$\frac{\partial L(0, x, u_0)}{\partial n} = \frac{\partial (\Delta L(0, x, u_0))}{\partial n} = 0 \quad \text{for } x \in \partial \Omega,$$

4. Basic estimates and inequalities.

It is well known that a system $(\Delta i, \{B_i, B_i^d\}, \Omega)$ defines a “regular elliptic
boundary value problem” in the sense of [7], p. 76 also [11], pp. 165, 221, 273. Moreover, our considerations will remain valid for boundary conditions other
than (3); e.g. for the Dirichlet condition:

$$(3') \quad B_i^d u = 0, \quad B_i^d u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial \Omega,$$

The system $(\Delta i, \{B_i, B_i^d\}, \Omega)$ also defines a regular elliptic boundary value
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problem. It is known ([7], p. 75), that for such problems the Calderon-Zygmund estimates are valid i.e.:

\[ \forall \varepsilon > 0 \exists C(\varepsilon, \Omega) \text{ s.t.: } \| v \|_{W^{1,p}(\Omega)} \leq C(\| \Delta v \|_{L^p(\Omega)} + \| v \|_{L^p(\Omega)}), \]

where \( v \) is an arbitrary \( C^4(\overline{\Omega}) \) function satisfying homogeneous boundary conditions; \( B_v = B_v = 0 \) on \( \partial \Omega \). We need a version of such an estimate valid for second order elliptic operators (known also [9] as "the second fundamental inequality for elliptic operators"):

\[ \| v \|_{W^{1,q}(\Omega)} \leq C(\| \Delta v \|_{L^p(\Omega)} + \| v \|_{L^p(\Omega)}), \]

where \( q \geq 1, p > 1, v \in W^{1,p}(\Omega) \) and \( \partial v/\partial n = 0 \) on \( \partial \Omega \). The second terms on the right sides of (4), (5) will be replaced by \( |\nabla v| = |\nabla \Omega|^{-1} \int_\Omega |v(x)|dx \).

Further, we need a version of the interpolation inequality for intermediate derivatives [1], p. 75: For \( \Omega \subset \mathbb{R}^n \) having the uniform cone property, \( \varepsilon_0 > 0 \) fixed, there exists a constant \( K = K(\varepsilon_0, m, \Omega) \) for every \( \nu \in W^{m,2}(\Omega) \), such that

\[ \forall \varepsilon > 0 \exists \nu \in W^{1,\varepsilon}(\Omega) \text{ s.t.: } \| \nabla^{\varepsilon} \nu \|_{L^p(\Omega)} \leq C \| \nabla \nu \|_{L^p(\Omega)} \]

where \( |\nabla^{\varepsilon} \nu| = \left( \sum_{i=1}^m |D^i \nu|^{1/2} \right)^{1/2}, \varepsilon = 2K^2 \varepsilon^2 \) and \( C_\varepsilon = 2K^2 \varepsilon^{-3/2} \). We also claim an estimate ([1], p. 108);

\[ \exists \varepsilon > 0 \forall \nu \in W^{1,p}(\Omega) \| \nu \|_{L^p(\Omega)} \leq C \| \nu \|_{W^{1,p}(\Omega)}, \quad p \leq n, \]

where \( \Omega \subset \mathbb{R}^n \) has the cone property. Finally ([8], p. 37), when \( \partial \Omega \subset C^m \), then

\[ \| \nu \|_{W^{k,p}(\Omega)} \leq C \| \nu \|_{W^{m,q}(\Omega)} \| \nu \|_{L^p(\Omega)}, \]

with \( p \geq q, p \geq r, 0 \leq \theta \leq 1 \) and \( k-n/p \leq \theta (m-n/q) - (1-\theta)n/r \).

Part I. General theory.

5. Local solvability of the problem (1)-(3).

Let us note that, due to Lipschitz continuity of \( f \), uniqueness of the Hölder (and weaker) solution of (1)-(3) is guaranteed. The proof, in which we consider the difference of two solutions, is very similar to that of Lemma 2 and will be omitted.

We define the range of arguments of the nonlinear function \( f \); let \( \Omega \in \mathbb{R}^n \) and set
(9) \[ X := \left\{ (t, x, v, p, q) ; t \in [0, T], x \in \overline{G}, \left( |v|^2 + \sum_i |p_i|^2 + \sum_{ij} |q_{ij}|^2 \right)^{1/2} \leq R \} \]

where \( T \) and \( R \) are fixed positive numbers. The expression bounded in \( (9) \) by \( R \) corresponds, for the composite function \( f(t, x, u, u_x, u_{xx}) \) in \( (1) \), to \( W^{2,\infty}(\Omega) \) norm of \( u \). Let us denote the Lipschitz constants, inside \( X \), with respect to \( t, v, \pi, q_{ij} (i, j = 1, \cdots, n) \) by \( L_1, L_3, L_4, L_5 \) respectively (e.g. \( L_5 \) is suitable for each \( q_{ij}, i, j = 1, \cdots, n \)). Also let \( |f(t, x, 0, 0, 0)| \leq N \) for \( t \in [0, T], x \in \overline{G} \).

We shall start with the formulation of Lemma 1 necessary to present the main result of Part I; Theorem 1. Because the proof of this lemma is very technical, it will be left until the Appendix.

**Lemma 1.** As long as the Hölder solution of \( (1)-(3) \) remains in \( X \), the following estimates hold; when the dimension \( n \leq 3 \), then

\[
\|u(t, \cdot)\|_{W^{2,\infty}(\Omega)}^2 \leq \nu \left( \int_{\Omega} u_t^2 dx + N^2 |\Omega| \right) + C \int_{\Omega} u^2 dx,
\]

also

\[
\|u(t, \cdot)\|_{W^{2,\infty}(\Omega)}^2 \leq \nu \left( \int_{\Omega} u_t^2 dx + N^2 |\Omega| \right) + C \int_{\Omega} u^2 dx
\]

for the space dimension \( n \geq 4 \). Here \( \nu \in (0, \nu_0] \) (\( \nu_0 \) given in \( (55) \)), \( C_\nu \) increases when \( \nu \) decreases and \( |\Omega| \) denotes the Lebesgue measure of \( \Omega \).

We are now ready to formulate:

**Theorem 1.** For two arbitrary positive numbers \( r, R \) and initial function \( u_0 \) satisfying the condition

\[
\nu \left[ \int_{\Omega} |L^2(0, x, u_0) dx + N^2 |\Omega| \right] + C_\nu \int_{\Omega} u_0^2 dx \leq r^2 < R^2
\]

(the constants \( \nu \) and \( C_\nu \) were chosen in Lemma 1) there is a time \( T_0, 0<T_0 \leq T \), such that the Hölder solution of \( (1)-(3) \) corresponding to \( u_0 \) exists at least until the time \( T_0 \).

**Comment.** Condition \( (12) \) defines certain neighbourhood of the zero function in \( W^{2,\infty}(\Omega) \) to which \( u_0 \) should belong. When \( u_0 \) has too large norm we shall transform the problem \( (1)-(3) \) onto equivalent one for the new unknown function \( v := u - u_0 \);

\[
v_t = -\varepsilon^2 \Delta v + f(t, x, v, v_x, v_{xx})
\]
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with \( \tilde{f}(t, x, v, v_x, v_{xx}) := -\varepsilon^2 \Delta^2 u_0 + f(t, x, v_0, v_0_x, v_0_{xx}) \) and homogeneous (zero) initial and boundary conditions corresponding to (2), (3). The estimate (12) for the transformed problem reads

\[
(12') \quad \nu \left( \int_Q \left| -\varepsilon^2 \Delta^2 u_0 + f(0, x, u_0, u_0_x, u_0_{xx}) \right|^2 dx + N^2 |\Omega| \right)^{\frac{1}{2}} \leq r^2 < R^3,
\]

and is evidently fulfilled, provided \( \nu > 0 \) is chosen sufficiently small. All the results obtained for \( u \) and (1)–(3) stay valid for \( v \) and the transformed problem.

The proof of Theorem 1 is divided into several steps. We start with two simple a priori estimates for \( \|u(t, \cdot)\|_{L^2(\Omega)} \) and \( \|u_t(t, \cdot)\|_{L^2(\Omega)} \) valid while \( u \) remains in \( X \).

**Lemma 2 (First a priori estimate).** As long as \( u \) remains in \( X \), we have an estimate

\[
(13) \quad \int_Q u^2(t, x) dx \leq e^{ct} \left[ \int_Q u_0^2(x) dx + \frac{N|\Omega|}{c} (1 - e^{-ct}) \right],
\]

where \( c = c(L_3, L_4, L_5, N, \varepsilon) \) being a constant.

**Proof.** Multiplying (1) by \( u \) and integrating over \( \Omega \), we get:

\[
\frac{1}{2} \frac{d}{dt} \int_Q u^2 dx = -\varepsilon^2 \int_Q \Delta^2 u_0 dx + \int_Q f u dx.
\]

Integrating by parts, noting (3)

\[-\varepsilon^2 \int_Q \Delta^2 u_0 dx = -\varepsilon^2 \int_Q (\Delta u)^2 dx,
\]

from the Lipschitz continuity of \( f \) inside \( X \) and the Cauchy inequality we find:

\[
(14) \quad \int_Q f(t, x, u, u_x, u_{xx}) u dx
\]

\[
= \int_Q f(t, x, u, u_x, u_{xx}) - f(t, x, 0, u_x, u_{xx}) + f(t, x, 0, 0, u_x, u_{xx})
\]

\[
- f(t, x, 0, 0, u_{xx}) + \cdots + f(t, x, 0, 0, 0) u dx
\]

\[
\leq \frac{\gamma}{2} \left[ L_1 \int_Q \sum_i u^2_i dx + L_2 \int_Q \sum_{i,j} u^2_{ij} dx \right]
\]

\[
+ \left[ L_3 + \frac{N}{2} + \frac{L_4}{2\gamma} + \frac{L_5 n^2}{2\gamma} \right] \int_Q u^2 dx + \frac{N}{2} |\Omega|.
\]

Estimating the first term on the right side of (14) through (5) with \( p = r = 2 \), and choosing \( \gamma = \gamma_0 \) sufficiently small that
we finally get
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega u^2 \, dx \leq \left[ L_s + \frac{N}{2} + \frac{L_c t}{2t_0} + \frac{L_3 h^2}{2t_0} + \varepsilon^2 \right] \int_\Omega u^2 \, dx + \frac{N}{2} |\Omega|,
\]
which is equivalent to (13). The proof is completed.

We proceed to the next a priori estimate:

**Lemma 3** (Second a priori estimate). As long as the solution \( u \) remains in \( X \):
\[
\int_\Omega u_2^2(t, x) \, dx \leq \left[ \int_\Omega L^2(0, x, u_0) \, dx + \frac{c_3}{c_1} (1 - e^{-c_1 t}) \right] e^{c_1 t},
\]
where \( c_3 = c_3(L_3, L_4, L_5, \varepsilon) \) and \( c_3 = c_3(L_1, \varepsilon) \) is proportional to \( c_3^1 \).

**Proof.** The difference quotient \( u_h(t, x) = h^{-1}(u(t+h, x) - u(t, x)) \) \((h > 0 \text{ is fixed})\) solves the equation:
\[
(16) \quad u_h = -\varepsilon \Delta^2 u_h + h^{-1} [f |_{t=t+h} - f |_{t=t}].
\]
Multiplying (16) by \( u_h \), integrating over \( \Omega \) and by parts, we find that:
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega u_h^2 \, dx = -\varepsilon \int_\Omega (\Delta u_h)^2 \, dx
\]
\[
+ h^{-1} \int_\Omega \left[ f(t+h, x, u(t+h, x), u_x(t+h, x), u_{xx}(t+h, x))
\right.
\]
\[
- f(t, x, u(t+h, x), u_x(t+h, x), u_{xx}(t+h, x)) + \cdots
\]
\[
- f(t, x, u(t, x), u_x(t, x), u_{xx}(t, x)) \] \[u_h \, dx \]
\[
\leq -\varepsilon \int_\Omega (\Delta u_h)^2 \, dx + \frac{\tau}{2} \left[ L_1 \int_\Omega \sum_{i,t} u_{x_i}^2 \, dx + L_3 \int_\Omega \sum_{i,t} u_{x_i x_t}^2 \, dx + L_3^2 |\Omega| \right]
\]
\[
+ \frac{1}{2\tau} (1 + L_3 + L_4 + L_5) \int_\Omega u_h^2 \, dx,
\]
making use of the Lipschitz conditions and Cauchy inequality and, in particular, an estimate:
\[
\int_\Omega \left[ f(t+h, x, u(t+h, x), u_x(t+h, x), u_{xx}(t+h, x))
\right.
\]
\[
- f(t, x, u(t+h, x), u_x(t+h, x), u_{xx}(t+h, x)) \] \[u_h \, dx \]
\[
\leq \frac{\tau}{2} \int_\Omega L_1^2 \, dx + \frac{1}{2\tau} \int_\Omega u_h^2 \, dx = \frac{\tau}{2} L_1^2 |\Omega| + \frac{1}{2\tau} \int_\Omega u_h^2 \, dx.
\]
Noting that \( u_h \) fulfils the same boundary conditions as \( u \) did, by (5), for \( r=r_\circ \), we find that

\[
\frac{d}{dt}\int_Q u_h(t, x) dx \leq r_\circ (1 + L_3 + L_4 + L_5 + r_\circ \varepsilon^2) \int_Q u_h(t, x) dx + r_\circ L_4 \| \Omega \|,
\]

which leads to an estimate

\[
\int_Q u_h^2(t, x) dx \leq \left[ \int_Q u_h^2(0, x) dx + \frac{c_r}{c_r} (1 - e^{-\varepsilon t}) \right] e^{\varepsilon t},
\]

with \( c_r = r_\circ (1 + L_3 + L_4 + L_5 + r_\circ \varepsilon^2), \) \( c_r = r_\circ L_4 \| \Omega \|. \) Passing in (18) with \( h \) to \( 0^+ \), noting that for the smooth solution we consider \( u_h \) tends to \( u \) when \( h \to 0^+ \) and \( u_h(0, x) \) will be found from (1) with \( t=0 \), we justify (15). The proof is completed.

For the time being we restrict our considerations to space dimension \( n \leq 3 \), higher dimensions will be treated in the Appendix. For \( n \leq 3 \) we will now specify the value \( T_\circ \) mentioned in the formulation of Theorem 1.

In the definition (9) of \( X \) we have introduced the time interval \([0, T]\), for which the Lipschitz constants for \( f \) were chosen. Next, from Lemmas 2, 3 we have increasing with \( t \) estimates (13), (15), which together with (10) in Lemma 1 give:

\[
\| u(t, \cdot) \|_{H^2, \infty(Q)} \leq \nu \left[ \int_Q u_h^2 dx + N^2 \| \Omega \| \right] + C_s \int_Q u^2 dx
\]

\[
\leq \nu \left[ \int_Q L^2(0, x, u_h) dx + \frac{c_r}{c_r} (1 - e^{-\varepsilon t}) \right] e^{\varepsilon t} + N^2 \| \Omega \|
\]

\[
+ C_s e^{\varepsilon t} \left[ \int_Q u_h^2(x) dx + \frac{N \| \Omega \|}{c} (1 - e^{-\varepsilon t}) \right].
\]

The estimate (19) is valid as long as \( u \) remains in \( X \). But the right side of (19) increases with \( t \), starting for \( t=0 \) from a value not exceeding \( r^3 \) (compare (12)). Defining \( T_\circ \) as equal to \( \min \{ T, \tau \} \), where \( \tau \) is the time for which the right side of (19) reaches the value \( R^3 \), we are sure that \( u(t, \cdot) \) remains in \( X \) for \( t \leq T_\circ \) and \( n \leq 3 \). Moreover, the composite function \( f(t, x, u, u_x, u_{xx}) \) is uniformly Lipschitz continuous (constants \( L_1, L_3, L_4, L_5 \)) and bounded in \( Q_{T_\circ} = [0, T_\circ] \times \bar{\Omega} \).

The remaining part of the proof of Theorem 1 for \( n \leq 3 \) is based on estimates of solutions of linear 2b-parabolic equations (here \( b=2 \)) in \( W_q^{m, 2m}(Q_{T_\circ}) \) space (see [10], Chapt. VII, §10). As a consequence of Theorem 10.4 reported there (with \( m=1, b=2, t=4, s=0, l=0; \) hence \( l+t=4 \)), we have:

\[
u \in W_q^{1, q}(Q_{T_\circ}) \qquad \text{with arbitrary } q \in (1, \infty),
\]
which means boundedness of the $W^{1,4}_q$ norm of $u$;

$$
\sum_{j=0}^{n} \sum_{i_1 \neq i_2 \neq \cdots \neq i_j} \|D^{i_1}_{i_2} D^{i_2}_{i_3} \cdots D^{i_j}_{i_1} u \|_{L^q(Q_{R_0})} < +\infty.
$$

In particular $u \in L^q(Q_{R_0})$ and $x_{i \xi j} \in L^q(Q_{R_0})$ for any $q \in (1, \infty)$.

To obtain a priori estimates for the H"older solution of (1)-(3) we shall use the following:

**Lemma 4.** For $n \leq 3$, under our basic assumption that $f$ is locally Lipschitz continuous with respect to $t$, $u$, $x_i$, $x_{i \xi j}$ ($i = 1, \ldots, n$) and Hölder continuous (exponent $\mu$) with respect to $x$ and that $u \in C^{1,\mu}(\overline{\Omega})$ satisfies compatibility conditions

$$
\frac{\partial u}{\partial n} = \frac{\partial (\Delta u)}{\partial n} = 0 \quad \text{for } x \in \partial \Omega,
$$

the solution $u$ will be estimated a priori in the H"older space $C^{1+\frac{1}{2},\mu}(Q_{R_0})$, $\mu = \min \{2/9, \mu\}$.

**Outline of the proof.** As a consequence of (20) with $q=2n+2$ we find that $u, u_t, u_{x_i} \in L^{2n+2}(Q_{R_0})$ which, with the use of the Sobolev theorem, ensures that

$$
u \in C^{1/2,1/2}(Q_{R_0}).
$$

Since as a consequence of (15) $u \in L^\infty(0, T_0; L^5(\Omega))$, then by (19) and (1)

$$
\varepsilon^2 \Delta u = -u_t + f(t, \cdot, \cdot, u, u_x, u_{xx}) \in L^\infty(0, T_0; L^5(\Omega))
$$

and further, by the elliptic regularity [7, 11], $u \in L^\infty(0, T_0; W^{4,5}(\Omega))$. Again by the Sobolev theorem (in dimension $n \leq 3$) $W^{4,5}(\Omega) \subset C^{1/2}(\overline{\Omega})$, hence

$$
u \in L^\infty(0, T_0; C^{1/2}(\overline{\Omega})).
$$

Using Lemma 3.1, Chapt. II of [10] subsequently to $u_{x_i}$ and then to $u_{x_{i \xi j}}$ ($i, j = 1, \ldots, n$), in the presence of (23), (24) we find that $u_{x_i} \in C^{1/4,1/2}(Q_{R_0})$, moreover

$$
u_{x_{i \xi j}} \in C^{1/8,1/2}(Q_{R_0}).
$$

Finally, from the Lipschitz, Hölder continuity of $f$ inside $X$ and (25) the composite function $f(t, x, u, u_x, u_{xx})$ belongs to $C^{1/18,\mu'}(Q_{R_0})$ for $\mu' = \min \{1/2, \mu\}$.

From Theorem 10.1, Chapt. VII of [10] (with $l-s=\rho$, $t+s=4$ and $l+t=4+\beta$):

$$
u \in C^{1+\frac{1}{18},1+\frac{1}{2}}(Q_{R_0}), \quad \rho = \min \left\{ \frac{2}{9}, \mu' \right\},
$$

(here the letter $C$ is used instead of $H$ in [10]), and we have the required esti-
mate of $u$ in the Hölder space. The proof of Lemma 4 is completed.

Until now a number of a priori estimates for the hypothetical solution of (1)-(3) have been given. With these estimates, however, the proper proof of existence of the Hölder solution to (1)-(3) based on the Leray-Schauder Principle ("method of continuity") is standard and will be omitted here (compare e.g. [10, 5]). The proof of Theorem 1 for $n \leq 3$ is thus finished.

Part II. Applications.


It is simple to conclude from the considerations of Part I, that if we are able to assure global in a time interval $[0, T_1]$ Lipschitz continuity of the function $f(t, x, u, u_x, u_{xx})$ (and its derivatives when $n > 3$), then the solution (being as smooth as the data allow) exists at least for $t \in [0, T_1]$. Obviously we cannot expect such global Lipschitz continuity for general $f$ in (1) (perhaps of a very complicated nature), but we may prove it for a number of special problems such as the Cahn-Hilliard equation. Here we will follow the presentation of this equation in [12], p. 147. Let us consider;

(26) \[ u_t = -s^2 \Delta^2 u + \Delta(F(u)), \]

$x \in \Omega \subset \mathbb{R}^n$, $n \leq 3$, together with conditions (2), (3). Here $F$ is a polynomial of the order $2p-1$ (moreover $p=2$ if $n=3$),

(27) \[ F(u) = \sum_{j=1}^{2p-1} a_j u^j, \quad p \in \mathbb{N}, \quad p \geq 2, \]

with positive leading coefficient; $a_{2p-1} > 0$. The prototype was $F(u) = \beta u^3 - \alpha u$ with $\beta, \alpha > 0$.

Since $\Delta(F(u)) = F'(u) \Delta u + F''(u) |\nabla u|^2$ is locally Lipschitz continuous ($F'$, $F''$ are locally bounded), then the assumptions of Part I are satisfied (provided that $u_0, \partial \Omega$ are smooth and (22) is fulfilled) and we have free local in time existence of the Hölder solution to (26), (2), (3). However, if we can justify, using a priori estimates, Lipschitz continuity of

(28) \[ f(t, x, u, u_x, u_{xx}) = \Delta(F(u)) = F'(u) \Delta u + F''(u) |\nabla u|^2 \]

in $[0, T_1]$ ($T_1 > 0$ will be fixed from now on), we will have proved the existence of the global Hölder solution to the Cahn-Hilliard equation. We need to estimate a priori $\|u(t, \cdot)\|_{L^\infty(\Omega)}$ and $\|\Delta u(t, \cdot)\|_{L^\infty(\Omega)}$ for $t \in [0, T_1]$. These two estimates are in order simple consequence of the one given in [12], p. 156:
where $k > 0$ and $\sigma \in [0, 1)$ are constants independent of $u$ (dependent on the special form (27) of $F$, $k$ also on $\|\nabla u_0\|_{L^2(\Omega)}$). We have:

**Lemma 5.** For a sufficiently regular solution of the Cahn-Hilliard equation $(n \leq 3)$ the two a priori estimates are valid:

\[ \|u(t, \cdot) - \bar{u}\|_{L^\infty(\Omega)} \leq C(\|\Delta u_0\|_{L^2(\Omega)} + m)^{1/9}, \]  

with $\bar{u} = |\Omega|^{-1}\int_\Omega u_0(x)dx$, also

\[ \|\Delta u(t, \cdot)\|_{L^\infty(\Omega)} \leq C(\|u_0\|_{W^{4,1}(\Omega)}, T_1) \]

where $C$ is a positive function increasing with respect to both arguments.

**Proof.** We start with the proof of (30). Because of (3), integrating (26) over $\Omega$ we find that

\[ \frac{d}{dt}\int_\Omega u(t, x)dx = 0, \]

hence the mean value $\bar{u}$ is preserved in time. Multiplying (26) by $\Delta^2 u$ and integrating over $\Omega$ we get:

\[ \frac{1}{2} \frac{d}{dt}\int_\Omega (\Delta u)^2 dx = -\varepsilon^2 \int_\Omega (\Delta^2 u)^2 dx + \int_\Omega \Delta(F(u)) \Delta^2 u dx \]

\[ \leq \left( -\varepsilon^2 + \frac{\varepsilon^2}{2} \right) \int_\Omega (\Delta^2 u)^2 dx + \frac{1}{2\varepsilon^2} \int_\Omega [\Delta(F(u))]^2 dx \]

\[ \leq -\varepsilon^2 \int_\Omega (\Delta^2 u)^2 dx + \frac{k}{2\varepsilon^2} \left[ 1 + \int_\Omega (\Delta^2 u)^2 dx \right], \]

where (29) was also used. The right side of (32) is a function of $z := \int_\Omega (\Delta^2 u)^2 dx$, having the form $(-\varepsilon^2 z + (k/\varepsilon^2)z^2 + (k/\varepsilon^3))$ and therefore must be bounded from above, say by $m$, for $z \geq 0$. Hence:

\[ \int_\Omega (\Delta u)^2 dx \leq \int_\Omega (\Delta u_0)^2 dx + 2mt. \]

Since, for $n \leq 3$, as a consequence of (7) and (5)

\[ \|u(t, \cdot) - \bar{u}\|_{L^\infty(\Omega)} \leq C\|\Delta u(t, \cdot)\|_{L^2(\Omega)}, \]

we have (30). Note the slow growth of the right side of (30) of the order $t^{1/9}$.

To obtain (31) we shall consider first $u_\varepsilon$ in $L^3(\Omega)$. Formally we proceed as in the proof of Lemma 3, but now without using implicit Lipschitz constants.
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\[
\frac{1}{2} \int_{\Omega} u_t^2 \, dx = -\varepsilon^2 \int_{\Omega} (\Delta u_h)^2 \, dx + \int_{\Omega} [\Delta(F(u))]_h u_h \, dx
\]

\[
= -\varepsilon^2 \int_{\Omega} (\Delta u_h)^2 \, dx + \int_{\Omega} (F(u))_h \Delta u_h \, dx
\]

\[
\leq -\varepsilon^2 \int_{\Omega} (\Delta u_h)^2 \, dx + \int_{\Omega} F'(\bar{u}) u_h \Delta u_h \, dx.
\]

As a consequence of (30), \(|F'(u)| \leq K\), hence

\[
\int_{\Omega} u_t(t, x) \, dx \leq \int_{\Omega} [-\varepsilon^2 \Delta u + \Delta(F(u))]^2 \, dx \exp \left( (K/\varepsilon)^t \right).
\]

Finally, from (26)

\[
\varepsilon^2 \Delta^2 u = -u_t + F'(u) \Delta u + F''(u) \nabla u \cdot \nabla u,
\]

where from (30), \(F'(u)\) and \(F''(u)\) are in \(L^\infty([0, T_1] \times \bar{\Omega})\), \(\Delta u\) is in \(L^\infty(0, T_1; L^2(\Omega))\) as a result of (33), \(u_t\) is in \(L^\infty(0, T_1; L^2(\Omega))\) as follows from (35). Hence, as a consequence of the Sobolev inequality and (5)

\[
\|\nabla u\|_{L^2(\Omega)} \leq \text{const.} \left( \|u\|_{L^2(\Omega)} + |\bar{\Omega}| \right), \quad n \leq 3,
\]

also \(|\nabla u|^2 \in L^\infty(0, T_1; L^2(\Omega))\). We have now verified that the right side of (36) belongs to \(L^\infty(0, T_1; L^2(\Omega))\), thus \(\Delta^2 u \in L^\infty(0, T_1; L^2(\Omega))\), which from (7), (4) for \(n \leq 3\) means that \(\Delta u \in L^\infty([0, T_1] \times \bar{\Omega})\). Also \(|\nabla u|\) is bounded in \([0, T_1] \times \bar{\Omega}\). The proof of Lemma 5 is completed.

For \(n \leq 3\) we have thus verified existence of the global Hölder solution to (26), (2), (3).

**Remark 1.** The polynomial form of \(F\) in [12] is rather restricting. Under a weak assumption only;

\[
\exists \gamma > 0 \forall \gamma \in \mathbb{R} - \int_0^\gamma F(z) \, dz \leq M,
\]

evidently satisfied by any \(F\) admitted by other authors [2, 3], we have the time independent estimate

\[
\|u(t, \cdot) - \bar{u}\|_{L^2(\Omega)} \leq c_\varepsilon \|\nabla u(t, \cdot)\|_{L^2(\Omega)} \leq \text{const.}
\]

\[
= c_\varepsilon \left\{ \|\nabla u_0\|_{L^2(\Omega)} + \frac{2}{\varepsilon^2} \left[ \int_0^\gamma x^2 F(z) \, dz \, dx + M |\Omega| \right] \right\},
\]
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c, being a constant in the Poincaré inequality. Estimate (38) is a simple consequence of (37) and the existence of a Liapunov functional for the solution of (26), (2), (3);

\begin{equation}
\frac{d}{dt} \left[ \frac{s^2}{2} \sum \phi_i(t, x) + \int \phi(z) \, dz \right] \leq 0.
\end{equation}

7. Kuramoto-Sivashinsky equation.

Considering [12], p. 137, let us study the problem

\begin{equation}
\tag{40}
\frac{\partial u}{\partial t} = -u_{xxxx} - \frac{1}{2} (u_x)^2,
\end{equation}

t > 0, x \in [-L/2, L/2], equipped by the space-periodic boundary conditions

\begin{equation}
\tag{41}
\left. \frac{\partial^j u}{\partial x^j} \right|_{-L/2} = \left. \frac{\partial^j u}{\partial x^j} \right|_{L/2}, \quad j = 0, 1, 2, 3,
\end{equation}

\begin{equation}
\tag{42}
\left. u \right|_{x=0} = \left. u \right|_{x=L} \quad \text{for} \quad x \in [-L/2, L/2].
\end{equation}

We note that as a consequence of (41) (all the unspecified integrals here are taken over \([-L/2, L/2]\));

\begin{equation}
\int u_x(t, x) \, dx = \int u_{xx}(t, x) \, dx = \int u_{xxx}(t, x) \, dx = \int u_{xxxx}(t, x) \, dx = 0
\end{equation}

since, e.g.

\begin{equation}
\int u_x(t, x) \, dx = u(t, L/2) - u(t, -L/2) = 0.
\end{equation}

With this observation it is easy to check that the expression

\begin{equation}
\left[ \int (\varphi^{(k)}(x))^2 \, dx + \left| \int \varphi(x) \, dx \right| \right]^{1/2}, \quad k = 1, 2, 3, 4
\end{equation}

define equivalent norms in \(H^k(-L/2, L/2)\) for functions satisfying (41) (or first \(k\) conditions in (41) when \(k < 4\)). For space-periodic boundary conditions (41) the last observation replaces the Calderon-Zygmund estimates (4), (5).

For the problem (40)-(42) the term \(f\) has the form:

\begin{equation}
\tag{45}
f(t, x, u, u_x, u_{xx}) = -u_{xx} - \frac{1}{2} (u_x)^2,
\end{equation}

hence, to show global existence of the solution, we shall find a global in time \(L^\infty\) a priori estimate of \(u_x\). This estimate will be obtained in two steps.

First step. Estimate of \(\int (u_x)^2 \, dx\).

Multiplying (40) by \(u_{xx}\) and integrating over \([-L/2, L/2]\) we find that:
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\[- \frac{1}{2} \frac{d}{dt}\int (u_x)^2 \, dx = \nu \int (u_{xxx})^2 \, dx - \int (u_{xx})^2 \, dx - \frac{1}{2} \int (u_x)^2 u_{xx} \, dx,\]

but

\[\int (u_x)^2 u_{xx} \, dx = \frac{1}{3} \int [(u_x)^2]_x \, dx = 0\]

because of (41), hence applying (6) we obtain

\[\frac{d}{dt}\int (u_x)^2 \, dx = -2\nu \int (u_{xxx})^2 \, dx - \int (u_{xx})^2 \, dx \leq (-2\nu + 2\nu) \int (u_{xxx})^2 \, dx + 2C \int (u_x)^2 \, dx\]

or

\[\int (u_x)^2(t, x) \, dx \leq \int (u_{xx})^2 \, dx \exp (2C_t) \leq \int (u_{xx})^2 \, dx \exp (2C T_1) =: m_0.\]

Second step. Estimate of \(\int (u_{xx})^2 \, dx\).

Multiplying (40) by \(u_{xxxx}\) and integrating over \([-L/2, L/2]\) we find:

\[\frac{1}{2} \frac{d}{dt}\int (u_{xx})^2 \, dx = -\nu \int (u_{xxxx})^2 \, dx + \int (u_{xx})^2 \, dx - \frac{1}{2} \int (u_x)^2 u_{xxxx} \, dx,\]

next, using (46) and the Poincaré inequality we have

\[\left| \int (u_x)^2 u_{xxxx} \, dx \right| \leq \|u_x\|_L^2 \|u_{xx}\|_L^2 \|u_{xxxx}\|_L^2 \leq m_0 \left( \frac{\delta}{2} \|u_{xxxx}\|_{L^2} + \frac{c_\nu}{\delta^2} \|u_{xx}\|_{L^2} \right).\]

Choosing \(m_0(\delta/2) = \nu\) (hence \((m_0 c_\nu/2\delta) = (\nu c_\nu/\delta^2)\)), and using (6) to estimate the third derivative in (47), we obtain

\[\frac{1}{2} \frac{d}{dt}\int (u_{xx})^2 \, dx \leq \left( -\nu + \nu^2 \frac{\nu}{2} \right) \int (u_{xxxx})^2 \, dx + \int \left[ C_{\nu/2} + \frac{\nu c_\nu}{2\delta^2} \right] (u_{xx})^2 \, dx,\]

which together with (46) and the inequality following from (7) and (43)

\[\|u_x(t, \cdot)\|_L^2 \leq c \int (u_{xx})^2(t, x) \, dx \quad (n = 1)\]

justify the required \(L^\infty([0, T_1] \times [-L/2, L/2])\) estimate of \(u_x\). From our general result it is clear that there exists a global Hölder solution of the problem (40)-(42). Our considerations are completed.
Part III. Appendix.

8. Proof of Lemma 1.

Since in fact the proof of (11) coincides with that of

\[ \|u(t, \cdot)\|_{W^{1,\infty}(\Omega)} \leq C\left( \int_{\Omega} u_0^2 \, dx + N^2 |\Omega| \right) + C\int_{\Omega} u^2 \, dx, \]

we will present only the first proof. For \( w := u_{x_i x_j} \), as a consequence of (7) with \( p=4, \ell=1, n \leq 3 \):

\[ \|w\|_{L^{\infty}(\Omega)} \leq C \|w\|_{W^{1,\infty}(\Omega)} \leq C C' \|w\|_{W^{1,\infty}(\Omega)} \|w\|_{L^4(\Omega)}, \]

where the inequality (8) has also been used. Now, from the Young inequality

\[ \|w\|_{L^{\infty}(\Omega)} \leq \frac{\delta}{2} \|w\|_{W^{1,\infty}(\Omega)} + C(\delta) \|w\|_{L^4(\Omega)} \]

(with \( C(\delta) = \text{const.} \delta^{-4} \)), hence from (6) we may claim

\[ \|u_{x_i x_j}\|_{L^{\infty}(\Omega)} \leq \delta \|u\|_{W^{1,\infty}(\Omega)} + C_\delta \|u\|_{L^4(\Omega)} \quad (n \leq 3). \]

As a consequence of (1), when \( u \) remains in \( X \)

\[ \int_{\Omega} (\Delta u)^2 \, dx = \int_{\Omega} [u_t - f(t, x, u, u_x, u_{xx})^2] \, dx \]

\[ \leq 3\varepsilon^{-4} \int_{\Omega} [u_t^2 + f^4(t, x, 0, 0, 0) + (f(t, x, 0, 0, 0) - f(t, x, u, u_x, u_{xx}))^2] \, dx \]

\[ \leq 3\varepsilon^{-4} \int_{\Omega} [u_t^2 + N^2] \, dx + c_4 \varepsilon^{-4} \|u\|_{W^{1,\infty}(\Omega)}, \]

where \( c_4 = c_4(L_3, L_4, L_6) \). As a result of (4), (51)

\[ \|u_{x_i x_j}\|_{L^{\infty}(\Omega)} \leq 2\delta^3 \|u\|_{W^{1,\infty}(\Omega)} + 2(C_\delta)^3 \|u\|_{L^4(\Omega)} \]

\[ \leq 2c^3 \delta^3 (\|\Delta u\|_{L^2(\Omega)} + \|u\|_{L^4(\Omega)}) + 2(C_\delta)^3 \|u\|_{L^4(\Omega)}. \]

Next, from (52)

\[ \|u_{x_i x_j}\|_{L^{\infty}(\Omega)} \leq 12\varepsilon^{-4} c^3 \delta^3 \left[ \int_{\Omega} (u_t^2 + N^2) \, dx + \frac{c_4}{3} \|u\|_{W^{1,\infty}(\Omega)} \right] \]

\[ + (4c^3 \delta^3 + 2(C_\delta)^3) \|u\|_{L^4(\Omega)}. \]

As a consequence of (7) with \( p=n+1, r=2 \), we may show that

\[ \|u\|_{W^{1,\infty}(\Omega)} \leq c(\|\Delta u\|_{L^{n+1}(\Omega)} + \|u\|_{L^4(\Omega)}) \]

\[ \leq c_6 (\|\Delta u\|_{L^{\infty}(\Omega)} + \|u\|_{L^4(\Omega)}). \]
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Summing (53) with respect to \(i, j\) or with respect to \(i, i\) to get the bound for \(\sum_{i,j} \|u_{x_i x_j}\|_{L^\infty(\Omega)}\) or \(\|\Delta u\|_{L^\infty(\Omega)}\), respectively, we finally have

\[
\|u\|_{L^\infty(\Omega)} \leq (n^2 + c_\delta n) \cdot \text{(right side of (53))} + c_\delta \|u\|_{L^1(\Omega)},
\]

which, for \(\nu := 12e^{-\delta}(n^2 + c_\delta n)\) and \(\delta\) taken so small that

\[
\frac{12e^{-\delta}(n^2 + c_\delta n)\epsilon}{3} |\Omega| \leq \frac{1}{2},
\]

gives (10). Condition (54) defines the value \(\nu_0\) mentioned in Lemma 1 (\(\nu \in (0, \nu_0]\)) in such a way, that

\[
\frac{1}{3} \nu_0 \epsilon |\Omega| = 1.
\]

The proof of (11) is similar to that of (10) with one exception, instead of (49) our starting point is an estimate (valid for \(n \geq 4\));

\[
\|u\|_{L^\infty(\Omega)} \leq C \|u\|_{L^1(\Omega)} \leq C C' \|u\|_{L^\infty(\Omega)} \|w\|_{L^\infty(\Omega)},
\]

used for \(w = u_{x_i x_j}\) as previously. The proof of Lemma 1 is completed.

9. Space dimensions \(n > 3\).

We have now complete information necessary to obtain the a priori estimates of \(u\) in \(W^{2,\infty}(\Omega)\) for arbitrary \(n\). To simplify notation we denote by \(T_\delta\) a positive time such that

\[
\|u\|_{L^\infty(0, T_\delta; W^{2,\infty}(\Omega))} \leq R,
\]

which is equivalent to saying that \(u\) remains in \(X\) until a time \(T_\delta\) (such \(T_\delta > 0\) exists due to continuity of the Hölder solution and (12); we need to estimate it). The key idea of our further proof is that estimates obtained for \(u\) will be valid as well for \(u_t\) solving the equation

\[
\epsilon^2 \Delta u + f_t + f u_t + \sum_{i,j} f_{x_i x_j} u_{x_i x_j} + \sum_{i,j} f_{x_i x_j} u_{x_i x_j},
\]

From (11) and Lemmas 2, 3 we have

\[
u \in L^\infty(0, T_\delta; W^{2,2n/n-2}(\Omega)),
\]

and from an estimate similar to (11), valid for \(u_t\) (we need our supplementary assumptions on \(f, u_0\) to justify it):

\[
u_t \in L^\infty(0, T_\delta; W^{2,2n/n-2}(\Omega)),
\]

and, as a consequence of (1), (58) and (59)

\[
\epsilon^2 \Delta u = -u_t + f(t, x, u, u_x, u_{xx}) \in L^\infty(0, T_\delta; L^{2n/n-2}(\Omega)),
\]
Then from the elliptic regularity theory [7, 11]:
\begin{equation}
    u \in L^\infty(0, T; W^{4, \frac{2n}{n-4}}(\Omega)).
\end{equation}

For \( n \leq 5 \), as a consequence of (7)
\[ W^{2, \infty}(\Omega) \subset W^{4, \frac{2n}{n-4}}(\Omega), \]
thus using (60) we have verified (57). At this point we will fix the time \( T_n \) (for \( n=4, 5 \)) in a similar way as previously for \( n \leq 3 \) in considerations following (19).

Next, for \( u=6, \ldots, 9 \), using (60), the analogous estimate for \( u_t \);
\begin{equation}
    u_t \in L^\infty(0, T; W^{4, \frac{2n}{n-4}}(\Omega))
\end{equation}
(requiring new assumptions on \( f, u_0 \)) and (1) we justify that
\[ u \in L^\infty(0, T; W^{4, \frac{2n}{n-4}}(\Omega)) \subset L^\infty(0, T; W^{2, \infty}(\Omega)). \]
We shall continue this procedure for larger \( n \).

**Remark 2.** In spite of certain technical complications involved in our proofs, the general idea of Theorem 1 is simple. It is based on Lemmas 1, 2, 3 giving a priori estimates and on the theory of linear problems known in literature. Moreover, our a priori estimates technique offers the possibility of effective estimates (as in [6]) of the life time of solutions. As a competitive technique we should mention the semigroups theory and its generalizations; compare e.g. [7, 4, 13].

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