A DIFFERENTIAL GEOMETRIC CHARACTERIZATION
OF HOMOGENEOUS SELF-DUAL CONES

(Dedicated to Professor K. Murata on his sixtieth birthday)

By

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In this note we give a differential geometric characterization of self-dual cones among affine homogeneous convex domains not containing any full straight line. Let $\mathcal{Q}$ be an affine homogeneous convex domain in an $n$-dimensional real vector space $V^n$. Then $\mathcal{Q}$ admits an invariant volume element

$$v = \phi dx^1 \wedge \cdots \wedge dx^n$$

and the canonical bilinear form defined by

$$g = \sum_{i,j} \frac{\partial^2 \phi}{\partial x^i \partial x^j} dx^i dx^j$$

is positive definite and so gives an invariant Riemannian metric on $\mathcal{Q}$, where \{x^1, \ldots, x^n\} is an affine coordinate system on $V^n$ [5]. In an affine coordinate system \{x^1, \ldots, x^n\} the components of the Riemannian connection $\Gamma$ and the Riemannian curvature tensor $R$ for $g$ are expressed as follows

$$\Gamma_{jk}^i = \frac{1}{2} \sum_p g^{ip} \frac{\partial^3 \phi}{\partial x^j \partial x^k \partial x^p},$$

$$R_{jkl}^i = \sum_p (\Gamma_{jk}^p \Gamma^i_{pl} - \Gamma^i_{pl} \Gamma^p_{jk}),$$

where $g_{ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j}$ and $\sum_p g^{ip} g_{pj} = \delta^i_j$ (Kronecker's delta). Since $\frac{1}{2} \sum_p g^{ip} \frac{\partial^3 \phi}{\partial x^j \partial x^k \partial x^p}$ defines a tensor field on $\mathcal{Q}$, we denote this tensor field by the same letter $\Gamma$.

An open convex set $\mathcal{Q}$ in $V^n$ is called a cone with vertex $o$ if $o + \lambda(x-o) \in \mathcal{Q}$ for all $x \in \mathcal{Q}$ and $\lambda > 0$. An open convex cone $\mathcal{Q}$ with vertex $o$ is said to be a self-dual cone if $V^n$ admits an inner product $\langle , \rangle$ such that

(i) $\langle x-o, y-o \rangle > 0$ for all $x, y \in \mathcal{Q}$;

(ii) if $x \in V^n$ is a vector such that $\langle x-o, y-o \rangle \geq 0$ for all $y \in \bar{\mathcal{Q}}$ then $x \in \bar{\mathcal{Q}}$, where $\bar{\mathcal{Q}}$ is the closure of $\mathcal{Q}$ in $V^n$.

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Theorem 1. A homogeneous convex cone $\Omega$ not containing any full straight line is a self-dual cone if and only if $R$ is parallel with respect to $\Gamma$.

Theorem 2. A homogeneous convex domain $\Omega$ not containing any full straight line is a self-dual cone if and only if $\Gamma$ is parallel with respect to $\Gamma$.

The necessary conditions of these theorems have been proved by O.S. Rothaus [2].

If $\Gamma$ is parallel with respect to $\Gamma$, then by (4) $R$ is parallel with respect to $\Gamma$. The converse is not true. For example, in the case of the interior of a paraboloid; $x^2 + \frac{1}{2} (x^3)^2 > 0$, $R$ is parallel but $\Gamma$ is not parallel with respect to $\Gamma$.

We denote by $\mathcal{F}_X$ and $L_X$ the covariant differentiation for $\Gamma$ and the Lie differentiation in the direction of a vector field $X$ respectively. We set

$$A_x = L_x - \mathcal{F}_x.$$  

Then $A_x$ is a derivation of the algebra of tensor fields and for a vector field $Y$ we have

$$A_x Y = - \mathcal{F}_X Y.$$  

If $X$ and $Y$ are Killing vector fields, then we know [1]

$$R(X, Y) = [A_x, A_Y] - A_{(X,Y)}.$$  

For vector fields $X = \sum_i \xi_i \frac{\partial}{\partial x^i}$ and $Y = \sum_i \eta^i \frac{\partial}{\partial x^i}$ we define a vector field $X \circ Y$ by

$$X \circ Y = - \sum_{i,j,k} \iota^i_{jk} \xi^j \eta^k \frac{\partial}{\partial x^i},$$  

and we put

$$S_x Y = X \circ Y.$$  

The condition that $\Gamma$ is parallel with respect to $\Gamma$ is equivalent to

$$\mathcal{F}_x(X \circ Y) = (\mathcal{F}_x X) \circ Y + X \circ (\mathcal{F}_x Y).$$  

We shall now recall the construction of clans from affine homogeneous convex domains [5]. It is known that a homogeneous convex domain $\Omega$ not containing any full straight line admits a simply transitive triangular affine Lie group $T$. Let $t$ denote the Lie algebra of $T$. We fix a point $e \in \Omega$ and choose an affine coordinate system $(x^1, \ldots, x^n)$ such that $x^1(e) = \cdots = x^n(e) = 0$. Identifying $X \mathcal{E} t$ with the vector field induced by the one parameter group of transformations $\exp(-tX)$, $X$  

(*) T. Tsuji obtained the same result independently [7].
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has an expression \( X = \sum_{i} (\sum_{j} a_{i} x_{j} + a_{i}) \frac{\partial}{\partial x_{i}} \), where \( a_{i} \) and \( a_{i} \) are constants. Let \( V \) denote the tangent space of \( \Omega \) at \( e \). Since \( T \) acts simply transitively on \( \Omega \), for each \( a \in V \) there exists a unique element \( X_{a} \in T \) such that the values of \( X_{a} \) at \( e \) is equal to \( a \). For \( a, b \in V \) we define a multiplication \( a \odot b \) in \( V \) by

\[
(a \odot b) = \sum_{i} (\sum_{j} a_{i} b_{j}^{i}) \frac{\partial}{\partial x_{i}}
\]

where \( a_{i}^{j} \) and \( b_{i}^{j} \) are constants given by

\[
X_{a} = \sum_{i} (\sum_{j} a_{i}^{j} x_{j} + a_{i}) \frac{\partial}{\partial x_{i}} \quad \text{and} \quad X_{b} = \sum_{i} (\sum_{j} b_{i}^{j} x_{j} + b_{i}) \frac{\partial}{\partial x_{i}}.
\]

Then we have

\[
[X_{a}, X_{b}] = X_{a \odot b - a \odot b}.
\]

Denoting by \( L_{a} \) the left multiplication by \( a \in V \);

\[
L_{a} b = a \odot b,
\]

we have

\[
[L_{a}, L_{b}] = L_{a \odot b - b \odot a}.
\]

Let \( \langle , \rangle \) denote the inner product on \( V \) induced by \( g \) and we put

\[
s(a) = \text{Tr} L_{a}.
\]

Then we know

\[
\langle a, b \rangle = s(a \odot b).
\]

The algebra \( V \) together with the linear form \( s \) is said to be the clan of \( \Omega \) with respect to \( e \in \Omega \) and the simply transitive triangular group \( T \) and is denoted by \( V(\Omega) \).

**Proposition 1.** For \( a, b \in V \) we denote by \( S_{a}, A_{a}, \) and \( R(a, b) \) the values of \( S_{X_{a}}, A_{X_{a}}, \) and \( R(X_{a}, X_{b}) \) at \( e \) respectively. Then we have

(i) \( S_{a} = \frac{1}{2} (L_{a} + L_{a}^{t}) \), \quad \( S_{a} b = S_{b} a \),

(ii) \( A_{a} = -\frac{1}{2} (L_{a} - L_{a}^{t}) \),

(iii) \( R(a, b) = -[S_{a}, S_{b}] \),

where \( L_{a}^{t} \) is the transpose of \( L_{a} \) with respect to \( \langle , \rangle \).

**Proof.** We may assume \( \phi(e) = 1 \). Since \( v = \phi dx^{1} \wedge \cdots \wedge dx^{n} \) is invariant under
the one parameter group of transformations $\text{Exp} \, t \, X_a$ generated by $X_a$, we have

$$\phi((\text{Exp} \, t \, X_a)e) = \exp(-t \, \text{Tr} \, L_a)$$

and so

$$(16) \quad \log \phi((\text{Exp} \, t \, X_a)e) = -ts(a).$$

Expanding the left side in a power series of $t$ and evaluating the terms of the first, the second and the third orders, we have

$$(17) \quad \sum_t \frac{\partial \log \phi}{\partial x^i}(e)a^i = -s(a),$$

$$(18) \quad \sum_{i,j} \frac{\partial^2 \log \phi}{\partial x^i \partial x^j}(e)a^ia^j = \langle a, a \rangle = s(a \triangle a),$$

$$(19) \quad \sum_{i,j,k} \frac{\partial^3 \log \phi}{\partial x^i \partial x^j \partial x^k}(e)a^ia^ja^k = -2\langle a, a \triangle a \rangle = -2s(a \triangle (a \triangle a)),$$

where $a = \sum_i a^i \left(\frac{\partial}{\partial x^i}\right)$. Taking $a + b$ and $a + b + c$ instead of $a$ in the formulae (18) and (19) respectively we obtain

$$(18') \quad \sum_{i,j} \frac{\partial^2 \log \phi}{\partial x^i \partial x^j}(e)a^ib^j = \langle a, b \rangle = s(a \triangle b),$$

$$(19') \quad 3 \sum_{i,j,k} \frac{\partial^3 \log \phi}{\partial x^i \partial x^j \partial x^k}(e)a^ib^ja^k = -\langle a, b \triangle c \rangle + \langle a, c \triangle b \rangle + \langle b, a \triangle c \rangle + \langle b, c \triangle a \rangle + \langle c, a \triangle b \rangle + \langle c, b \triangle a \rangle.$$  

By (14) and (18') we have

$$(20) \quad \langle a \triangle b, c \rangle + \langle b, a \triangle c \rangle = \langle b \triangle a, c \rangle + \langle a, b \triangle c \rangle.$$  

Using this we get

$$(19'') \quad \sum_{i,j,k} \frac{\partial^3 \log \phi}{\partial x^i \partial x^j \partial x^k}(e)a^ib^jc^k = -\langle a \triangle b, c \rangle - \langle b, a \triangle c \rangle$$

$$= -\langle \langle L_a + L_a \rangle b, c \rangle.$$  

On the other hand it follows from (3) (8) (9) that

$$\sum_{i,j,k} \frac{\partial^3 \log \phi}{\partial x^i \partial x^j \partial x^k}(e)a^ib^jc^k = 2 \sum_{i,j,k} \langle g_{kl}^{(ij)}e \rangle a^ib^jc^k$$

$$= -2\langle S_a b, c \rangle.$$  

Thus we have

$$S_a b = \frac{1}{2} \langle L_a + L_a \rangle b$$

and (i) is proved.
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For $X_a = \sum_j (\sum_j a_j x^j + a^j) \frac{\partial}{\partial x^j}$ and $X_b = \sum_j (\sum_j b_j x^j + b^j) \frac{\partial}{\partial x^j}$ we have

$$F_{X_a}X_b = \sum_j \left( \sum_p b_p (\sum_q a_{pq} x^q + a^q) + \sum_r \left( \sum_s a_{rs} x^s + a^s \right) \frac{\partial}{\partial x^r} \right) \frac{\partial}{\partial x^j}.$$

Since $x^j(\epsilon) = 0$, by (8), (11) and (i) the value $(F_{X_a}X_b)e$ of $F_{X_a}X_b$ at $\epsilon$ is reduced to

$$(F_{X_a}X_b)e = \sum \left( \sum_p b_p a_p + \sum_r \left( \sum_s a_{rs} \epsilon^s \right) \frac{\partial}{\partial x^r} \right) \epsilon$$

$$= b \triangle a - b \bigtriangleup a$$

$$= \frac{1}{2} (L_a - L_b) a.$$

Therefore by (6) we get

$$A_ab = (A_{X_a}X_b)e = -(F_{X_a}X_b)e = -\frac{1}{2} (L_a - L_b) b,$$

and (ii) is proved.

By (7) we have

$$R(X_a, X_b) = [A_{X_a}, A_{X_b}] - A(X_a, X_b).$$

Using (12), (14), (i) and (ii) we obtain

$$R(a, b) = [A_a, A_b] - A_{b \triangle a - a \bigtriangleup b}$$

$$= \left[ -\frac{1}{2} (L_a - L_b), -\frac{1}{2} (L_b - L_a) \right] + \frac{1}{2} (L_{b \triangle a - a \bigtriangleup b} - L_{a \bigtriangleup b - b \triangle a})$$

$$= \frac{1}{4} [L_a - L_b, L_b - L_a] - \frac{1}{2} ([L_a, L_b] - [L_a, L_b])$$

$$= -[S_a, S_b],$$

and so (iii) is proved.

Q. E. D.

**Proposition 2.**

(i) If $\Gamma$ is parallel with respect to $\Gamma$, then we have

$$[A_a, S_b] = S_{A_a b}.$$

(ii) If $R$ is parallel with respect to $\Gamma$, then we have

$$[A_a, [S_b, S_c]] = [S_{A_a b}, S_c] + [S_b, S_{A_a c}].$$

**Proof.** Since $X_a = \sum_i \xi_i \frac{\partial}{\partial x^i}$ is an infinitesimal affine transformation with respect to $\Gamma$, we have
where $L_{X_a} \Gamma^i_{jk}$ is the Lie derivative of the tensor field $\Gamma^i_{jk}$ by $X_a$. Since $\xi^i = \sum_j x^j \xi^i_j + a^i$, we get

$$L_{X_a} \Gamma^i_{jk} = 0.$$  \hspace{1cm} (21)

From this we have

$$L_{X_a}(X_b \square X_c) = (L_{X_a} X_b) \square X_c + X_b \square (L_{X_a} X_c).$$  \hspace{1cm} (22)

Therefore by (5), (10) and (22) the condition $F_{X_a} \Gamma = 0$ is equivalent to

$$A_{X_a}(X_b \square X_c) = (A_{X_a} X_b) \square X_c + X_b \square (A_{X_a} X_c).$$

This implies (i). By (4) and (21) it follows

$$L_{X_a} R = 0.$$  \hspace{1cm} (i)

Thus by (5) the condition $F_{X_a} R = 0$ is equivalent to $A_{X_a} R = 0$. Since $A_{X_a}$ is a derivation of the algebra of tensor fields, we have

$$((A_{X_a} R)(X_b, X_c)) X_d = A_{X_a} (R(X_b, X_c) X_d) - R(A_{X_a} X_b, X_c) X_d$$

$$- R(X_b, A_{X_a} X_c) X_d - R(X_b, X_d) A_{X_a} X_d$$

$$= ([A_{X_a}, R(X_b, X_c)] - R(A_{X_a} X_b, X_c) - R(X_b, A_{X_a} X_c)) X_d$$

and so by Proposition 1 (iii)

$$((A_a R)(b, c)) d = ([A_a, R(b, c)] - R(A_a b, c) - R(b, A_a c)) d$$

$$= ([A_a, [S_b, S_c] + [S_{A_a b}, S_c] + [S_b, S_{A_a c}]] d$$

This proves (ii). Q. E. D.

**Lemma 1.** If $\Gamma$ is parallel with respect to $\Gamma$, then $\Omega$ is a cone.

**Proof.** It is known that if the clan $V(\Omega)$ of $\Omega$ has a unit element then $\Omega$ is a cone [5]. Therefore by Proposition 2 (i) it suffices to show that if $[A_a, S_b] = S_{A_a b}$ holds for all $a, b \in V(\Omega)$, then $V(\Omega)$ has a unit element. Let $\mathfrak{u}$ be the principal idempotent of the clan $V(\Omega)$, i.e., $\mathfrak{u}$ is an element in $V(\Omega)$ determined by $\langle \mathfrak{u}, a \rangle = s(a)$ for all $a \in V(\Omega)$. Then we get the principal decomposition

$$V(\Omega) = V_0 + N,$$

where $V_0 = \{ a \in V(\Omega) ; a \cap a = a \}$ and $N = \{ a \in V(\Omega) ; a \cap a = \frac{1}{2} a \}$. The principal de-
composition $V(\Omega) = V_0 + N$ is orthogonal with respect to the inner product $\langle , \rangle$ and the following relations hold [5]
\[
\begin{align*}
V_0 \wedge V_0 &\subset V_0, \quad V_0 \wedge N \subset N, \\
N \wedge V_0 &\subset\{0\}, \quad N \wedge N \subset V_0.
\end{align*}
\]
(23)

Let $p$ be an element in $N$. By our assumption and Proposition 1 we have
\[
0 = \langle (A_p, S_p) - S_p^p, p, p \rangle = \langle A_p S_p p - 2S_p A_p p, p \rangle
\]
\[
= -3 \langle S_p p, A_p p \rangle = \frac{3}{4} \langle (L_p + L_p^p)p, (L_p - L_p^p)p \rangle
\]
\[
= \frac{3}{4} \langle L_p p, L_p^p p \rangle - \langle L_p^p p, L_p^p p \rangle.
\]

Therefore by the orthogonality of the principal decomposition and by (23) we get
\[
\langle p \wedge p, p \wedge p \rangle = \langle L_p p, L_p^p p \rangle = \langle p, L_p^p L_p p \rangle = 0.
\]

This means $p \wedge p = 0$, $\langle p, p \rangle = s(p \wedge p) = 0$ and so $p = 0$. Thus we have $V(\Omega) = V_0$ and


$n$ is a unit element of the clan $V(\Omega)$.

Q.E.D.

We shall now recall the notion of $T$-algebras [5] [6].

A matrix algebra with involution is an algebra over the real number field $\mathbb{R}$ which is bigraded by the subspaces $\mathfrak{U}_{ij}$ $(i, j=1, \cdots, m)$ and provided with an involutive anti-automorphism $*$ in such a way that

$\mathfrak{U}_{ij} \mathfrak{U}_{jk} \subset \mathfrak{U}_{ik}$,

$\mathfrak{U}_{ij} \mathfrak{U}_{ik} = \{0\}$ if $j \neq k$,

$\mathfrak{U}_{ij}^* = \mathfrak{U}_{ji}$.

The general element of $\mathfrak{U}_{ij}$ will be denoted by $a_{ij}, b_{ij}$, etc..

A matrix algebra with involution is said to be a $T$-algebra if the following axioms are satisfied:

(T.1) For any $i$ the algebra $\mathfrak{U}_{ii}$ is one-dimensional and admits an isomorphism $\rho : \mathfrak{U}_{ii} \rightarrow \mathbb{R}$ with the following properties.

(T.2) $a_{ij}b_{ij} = \rho(a_{ii})b_{ij}$;

(T.3) $n_i \rho(a_{ij} b_{ji}) = n_j \rho(b_{ji} a_{ij})$, where $n_i = 1 + \frac{1}{2} \sum \dim \mathfrak{U}_{is}$;

(T.4) $\rho(a_{ij} a_{ji}^*) > 0$ if $a_{ij} \neq 0$;

(T.5) $a_{ij}(b_{jk} c_{ki}) = (a_{ij} b_{jk}) c_{ki}$;

(T.6) $a_{ij}(b_{jk} c_{ki}) = (a_{ij} b_{jk}) c_{ki}$ if $i < j < k$ and $j < l$;

(T.7) $a_{ij}(b_{jk} b_{ki}^*) = (a_{ij} b_{jk}) b_{ki}^*$ if $i < j < k$.

Let $\mathfrak{X}$ denote the space of hermitian matrices in the $T$-algebra $\mathfrak{U}$; $\mathfrak{X} = \{a \in \mathfrak{U} : a^* = a\}$;
$a^*=a$. For each $a=\sum_{i,j} a_{ij} \in \mathfrak{A}$ we put

$$\hat{a}=\frac{1}{2} \sum_i a_{ii} + \sum_{i,j} a_{ij},$$

$$a=\frac{1}{2} \sum_i a_{ii} + \sum_{i,j} a_{ij},$$

We define a multiplication $L_n b = a \cdot b$ in $\mathfrak{A}$ by the formula

$$a \cdot b = \hat{a}b + b\hat{a},$$

Then $\mathfrak{A}$ is a clan with unit element and we denote this clan by $\mathfrak{A}(\mathfrak{A})$. Let $\Omega(\mathfrak{A})$ be the set of matrices which are expressible in the form $tt^*$, where $t$ is an upper triangular matrix with positive elements on the diagonal. Then $\Omega(\mathfrak{A})$ is a homogeneous convex cone in $\mathfrak{A}(\mathfrak{A})$. For every homogeneous convex cone $\Omega$ there exists a $T$-algebra $\mathfrak{A}$ such that $\Omega$ is isomorphic to $\Omega(\mathfrak{A})$ and a clan $V(\Omega)$ of $\Omega$ is isomorphic to $\mathfrak{A}(\mathfrak{A})$.

Now we return to the proof of our theorems. By the above fact we may assume $V(\Omega) = \mathfrak{A}(\mathfrak{A})$. Then it is known that for $a, b \in \mathfrak{A}(\mathfrak{A})$

$$\text{Tr } L_a = \text{Spur } a,$$

$$\langle a, b \rangle = \text{Spur } ab,$$

where Spur $a = \sum_t n_t \rho(a_{tt})$. It is easy to see

$$^tL_a b = gb + ba.$$

Therefore we get

$$(24) \quad S_a b = \frac{1}{2} (ab + ba),$$

$$(25) \quad A_a b = -\frac{1}{2} [(a-g)b - b(a-g)].$$

Let $n_{ij}$ denote the dimension of $\mathfrak{A}_{ij}$. We define inductively an equivalence relation $\bar{R}$ in the set $\{1, \cdots, m\}$ of indices:

1. $i \equiv i \pmod{\bar{R}}$ for all $i$,
2. if we have already determined whether the $i, j$ such that $|i-j| < r$ are comparable modulo $\bar{R}$ or not, then for $|i-j|=r$ we define $i \equiv j \pmod{\bar{R}}$ if and only if
   (i) $n_{ij} \neq 0$, (ii) $n_{ik} = n_{jk}$ for all $k \neq i, j$, and (iii) for all $k$ lying between $i$ and $j$ (except $i$ and $j$) either $n_{ik} = n_{kj} = 0$ or $i \equiv k \pmod{\bar{R}}$ and $k \equiv j \pmod{\bar{R}}$.

We put

$$\mathfrak{A}_{ij} = \begin{cases} \mathfrak{A}_{ij} & \text{if } i \equiv j \pmod{\bar{R}} \\ \{0\} & \text{if } i \not\equiv j \pmod{\bar{R}}. \end{cases}$$
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\[ \mathcal{V} = \sum_{i,j} \mathcal{A}_{ij}. \]

Then \( \mathcal{V} \) is a \( T \)-algebra and the homogeneous convex cone \( \mathcal{O}(\mathcal{V}) \) corresponding to \( \mathcal{V} \) is self-dual.

**Lemma 2.** If the clan \( \mathcal{A}(\mathcal{V}) \) corresponding to a \( T \)-algebra \( \mathcal{V} \) satisfies the condition

\[ [A_a, [S_b, S_c]] = [S_{A_{ab}}, S_c] + [S_b, S_{A_{ac}}], \]

then we have \( \mathcal{V} = \mathcal{V}' \).

**Proof.** By the condition we have

\[ [[A_a, S_b] - S_{A_{ab}}, S_c] = [[A_a, S_c] - S_{A_{ac}}, S_b]. \]

Let \( a_{ij} \in \mathcal{A}_{ij}, b_{jk} \in \mathcal{A}_{jk} \) and \( e_i \in \mathcal{A}_{ii} \), where \( i < j, k \neq i, j \) and \( \rho(e_i) = 1 \). We put \( a = a_{ij} + a_{*j}^*, b = b_{jk} + b_{*k}^* \) and \( c = e_i \) and calculate the following formula

\[ [[A_a, S_b] - S_{A_{ab}}, S_c] b = [[A_a, S_c] - S_{A_{ac}}, S_b] b \]

Using (24) and (25), the left side is equal to

\[ \frac{1}{4} \{ a_{ij} (b_{jk} b_{*k}^*) - (a_{ij} b_{jk}) b_{*k}^* + (b_{jk} b_{*k}^*) a_{*j}^* - b_{jk} (b_{*k}^* a_{*j}^*) \} \]

and the right side is reduced to 0. Considering \( \mathcal{A}_{ij} \)-component, by (T.2) we get

\[ (a_{ij} b_{jk}) b_{*k}^* = a_{ij} (b_{jk} b_{*k}^*) = \rho(b_{jk} b_{*k}^*) a_{ij}. \]

Multiplying both sides on the right by \( a_{*j}^* \) we obtain

\[ (a_{ij} b_{jk}) b_{*k}^* a_{*j}^* = \rho(b_{jk} b_{*k}^*) a_{ij} a_{*j}^*. \]

Therefore, by (T.2) we have

\[ \rho(a_{ij} b_{jk}) (a_{ij} b_{jk})^* = \rho((a_{ij} b_{jk}) b_{*k}^*) = \rho(b_{jk} b_{*k}^*) \rho(a_{ij} a_{*j}^*). \]

Assume \( a_{ij} \neq 0 \). For \( a_{ij} \neq 0 \), by (26) the linear mapping given by \( \mathcal{A}_{jk} \ni b_{jk} \to a_{ij} b_{jk} \in \mathcal{A}_{ik} \) is injective and so \( n_{jk} \leq n_{ik} \). In the same way we have \( n_{ik} \leq n_{jk} \). Therefore we have \( n_{ik} = n_{jk} \) for all \( k \neq i, j \). This implies that \( i \equiv j \) (mod \( R \)) if \( n_{ij} \neq 0 \). Thus we have \( \mathcal{V} = \mathcal{V}' \).

Q. E. D.

Since the homogeneous convex cone \( \mathcal{O}(\mathcal{V}) \) determined by \( \mathcal{V} \) is self-dual, in view of Proposition 2, Lemma 1 and 2 the sufficient conditions of our theorems are proved.
References


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