QF-3' RINGS AND MORITA DUALITY

By

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In [2] we proved that a one-sided artinian ring is QF-3 if and only if its double dual functors preserve monomorphisms. Here with the aid of [3] we prove that the double dual functor over an arbitrary ring preserves monomorphisms of left modules if and only if it is a left QF-3' ring. In view of this theorem results in [3] and [4] provide an analogue for QF-3' rings of the Morita-Tachikawa representation theorem for QF-3 rings ([9], Chapter 5). Also we apply it to obtain a characterization of Morita duality between Grothendieck categories that serves to generalize Onodera's theorem [7] that cogenerator rings are self injective, by showing that injectivity is redundant in the classical bimodule characterization of Morita duality for categories of modules.

We denote both the dual functors Hom\(_R(\_, R_R)\) and Hom\(_R(\_, R\_R)\) by \((\_)^*\). Recall that there is a natural transformation \(\sigma: \text{Id}_{R-\text{Mod}} \longrightarrow (\_)^*\), defined via the usual evaluation maps \(\sigma_M: M \longrightarrow M^{**}\). An \(R\)-module \(M\) is called \(R\)-reflexive (\(R\)-torsionless) in case \(\sigma_M\) is an isomorphism (a monomorphism). Also recall that \(R\) is left QF-3' if the injective envelope \(E(R_R)\) of \(rR\) is \(R\)-torsionless.

1. **Theorem.** For any ring \(R\), the following are equivalent:

(a) \(R\) is left QF-3';

(b) The double dual functor \((\_)^*\) preserves monomorphisms in \(R\)-Mod;

(c) If \(i: R \longrightarrow E\) is the inclusion of \(R\) into its injective envelope \(E\) in \(R\)-Mod, then \(i^{**}\) is a monomorphism.

**Proof.** That (b) implies (c) is immediate, and (c) implies (a) is easy (see [3], Proposition 1.2). Assume that \(R\) is a left QF-3' ring. Since \(E = E(R_R)\) is torsionless there is a sequence

\[
R \overset{i}{\longrightarrow} E \overset{j}{\longrightarrow} R^x
\]

for some set \(X\) where \(i\) is the inclusion and \(j\) is a monomorphism.

Let \(p_x: R^x \longrightarrow R\) be the canonical projections and let \(b_x = p_x \circ j \circ i(1) \epsilon R\) for each...
Then if \( K = \sum b_x R : x \in X \) it follows that the left annihilator of \( K \) in \( R \) is zero. Now suppose \( \alpha : M \rightarrow N \) is a monomorphism in \( R\text{-Mod} \) and consider the induced sequence

\[
\xymatrix{ N^* \ar[r]^{\alpha^*} & M^* \ar[r]^{\beta} & \operatorname{Coker} \alpha^* \ar[r] & 0. }
\]

If \( f \in M^* \), then since \( E \) is injective there exists \( \tilde{f} \in \text{Hom}_R(N, E) \) such that \( \tilde{f} \circ \alpha = i \circ f \). Then considering the diagram

\[
\xymatrix{ 0 \ar[r] & M \ar[r]^\alpha & N \ar[r]^j \ar[d] & E \ar[r]^p_x \ar[d]^i & R \ar[r] & 0 }
\]

it follows easily that

\[
\alpha^*(p_x \circ j \circ \tilde{f}) = p_x \circ \tilde{f} \circ \alpha = p_x \circ j \circ f = j \circ \tilde{f} = j \circ f = f \circ \alpha.
\]

Hence \( M^*K \subseteq \text{Im } \alpha^* \) so \( (\text{Coker } \alpha^*)^* = (\text{Coker } \alpha^* K = (\beta(M^* K) = (\beta(M^* K) = 0 \). Thus if \( \phi \in (\text{Coker } \alpha^*)^* \) we have \( \phi \circ \alpha (\text{Coker } \alpha^*) K = \phi(K) \circ \alpha = \phi(K) = 0 \) so, since the left annihilator of \( K \) is zero, \( \phi = 0 \). But then since \( (\text{Coker } \alpha^*)^* = 0 \) we see that \( M^* \rightarrow N^* \) is monic.

The following theorem follows immediately from \([3], \text{Theorem 1.4}\) and Theorem 1.

2. **Theorem.** For any ring \( R \), the following are equivalent:

1. \( R \) is left QF-3' and its own maximal left quotient ring;
2. The double dual functor \( (\ )^{**} \) is left exact on \( R\text{-Mod} \);
3. If \( 0 \rightarrow R \rightarrow E_1 \rightarrow E_2 \) is exact with \( E_1 \) and \( E_2 \) injective in \( R\text{-Mod} \) then \( 0 \rightarrow R \rightarrow E_1^{**} \rightarrow E_2^{**} \) is also exact.

Let \( D : a \rightarrow a' : D' \) be a pair of contravariant functors between abelian categories \( a \) and \( a' \) that are adjoint on the right, i.e., there are isomorphisms

\[
\eta_{A, A'} : \text{Hom}_a(A, D'(A')) \rightarrow \text{Hom}_{a'}(A', D(A)),
\]

natural in \( A \in a \) and \( A' \in a' \). Associated with \( \eta_{A, A'} \) are the arrows of right adjunction \( \tau : 1_{A} \rightarrow D' D \) and \( \tau' : 1_{A'} \rightarrow D D' \) defined by \( \tau_A = \eta^{-1}_{A, A}, \nu_{A}(1_{D(A)}) \) and \( \tau'_{A'} = \eta_{D'(A'), A'}(1_{D'(A')}) \), respectively. These satisfy, for each \( A \in a \), \( A' \in a' \),

\[
D(\tau_A) \circ \nu_{A}(1_{D(A)}) = 1_{D(A)} \text{ and } D'(\tau'_{A'}) \circ \nu_{A'}(1_{D'(A')}) = 1_{D'(A')}. \]

We recall that any pair of such functors \( D : a \rightarrow a' : D' \) which are adjoint on the right are left exact \([8], \text{Corollary 3.2.3}\).

We call an object \( A \) of a \( (A' \text{ of } a') \) reflexive \( (\text{respectively, torsionless}) \) in case \( \tau_A (\tau'_A) \) is an isomorphism \( (\text{respectively, a monomorphism}) \); and we note that \( (\text{as in } [1], \text{Section 23}) \ D \) and \( D' \) define a duality between the full subcategories of
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reflexive objects \(a_0 \subseteq a\) and \(a'_0 \subseteq a'\). Then as in [3] we say that the pair \(D: a \to a': D'\) defines a Morita duality in case \(D\) and \(D'\) are exact and the subcategories \(a_0 \subseteq a\) and \(a'_0 \subseteq a'\), are closed under subobjects and quotient objects and contain sets of generators for \(a\) and \(a'\), respectively.

According to ([3], Proposition 2.3) the functors \(D\) and \(D'\) of a Morita duality are faithful as well as exact. We shall now show that these conditions imply the closure condition for reflexive objects (as is well known if \(a\) and \(a'\) are module categories).

3. **Lemma.** Let \(D: a \to a': D\) be a right adjoint pair of contravariant functors between abelian categories. Then \(D\) and \(D'\) are faithful if and only if all objects in \(a\) and \(a'\) are torsionless.

**Proof.** If \(D\) is faithful and \(0 \to K \to A \to D'(A)\) is exact, then since \(D(\tau_A; \tau'_{D(A)}) = 1_{D(A)}\), \(D(f) = 0\) so \(f = 0\) also. On the other hand, if all objects of \(A\) are torsionless and \(f \in \text{Hom}(A, B)\), \(f \neq 0\), then \(D'D(f) = \tau_A = \tau'_{B'} f \neq 0\) so \(D'D(f) \neq 0\), hence \(D(f) \neq 0\), so \(D\) is faithful.

4. **Proposition.** A right adjoint pair of contravariant functors \(D: a \to a': D'\) between abelian categories defines a Morita duality if and only if \(a\) and \(a'\) contain generating sets of reflexive objects and \(D\) and \(D'\) are faithful and exact.

**Proof.** From Lemma 3 and exactness we obtain a commutative diagram

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & A & B_0 & C & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & D'(A) & D'(B_0) & D'(C) & 0 \\
\downarrow & \\
0 & \\
\end{array}
\]

with exact rows and columns when \(B_0\) is reflexive. Thus the Five Lemma apples.

We don't know whether, in the presence of reflexive generating sets, the closure properties for \(a_0\) and \(a'_0\) imply that \(D\) and \(D'\) are exact and faithful. Of course they do if \(a\) and \(a'\) are the categories of modules over a pair of rings (or even if they are functor categories [10]).

We now turn to the general setting of contravariant functors \(D: a \to a': D'\), adjoint on the right, where \(a\) and \(a'\) are Grothendieck categories. With \(a_0\) and \(a'_0\) as above we assume that these contain generators \(V \in a_0\), \(V' \in a'_0\). Then
letting $U = V \oplus D'V'$ and $U' = V' \oplus DV$ (so that $DU \cong U'$ and $D'U' \cong U$), $R = \text{End}_R(U)$ ($\cong \text{End}_R(U')^\op$), $S = \text{Hom}_R(U, \_)$ and $S' = \text{Hom}_R(U', \_)$, we have, as in ([6], Theorem 8.1) and ([3], Theorem 3.1), functors

$$
\begin{array}{c}
R\text{-Mod} \\
\downarrow \text{S}
\end{array}
\xrightarrow{\begin{array}{c}
T \\
\downarrow \text{D}
\end{array}}
\begin{array}{c}
a \\
\downarrow \text{D'}
\end{array}
$$

 Modi-R
\xrightarrow{\begin{array}{c}
T' \\
\downarrow \text{S'}
\end{array}}
a'

where $T(T')$ is a left adjoint of $S(S')$, $T$ and $T'$ are exact, and $TS$ and $T'S'$ are equivalent to the identity functors on $a$ and $a'$, respectively. Also, as in ([3], Theorem 3.1), $S\circ D\circ T \cong ( \_ )^*$ and $S'\circ D'\circ T' \cong ( \_ )^*$ so $D\circ T \cong T' \circ ( \_ )^*$ and $D'\circ T' \cong T \circ ( \_ )^*$. Thus $\text{Ker } T \subseteq \text{Ker } ( \_ )^*$ and $\text{Ker } T' \subseteq \text{Ker } ( \_ )^*$.

5. Lemma. Let $D$, $D'$, $a$, $a'$, $U$, $U'$ and $R$ be as above. Then the following are equivalent:

(a) $D$ and $D'$ are faithful;
(b) $U$ and $U'$ are cogenerators in $a$ and $a'$, respectively;
(c) $\text{Ker } T = \text{Ker } ( \_ )^*$ and $\text{Ker } T' = \text{Ker } ( \_ )^*$.

Proof. If $a \in \text{Hom}_R(A, B)$, we have a commutative square

\[
\begin{array}{ccc}
\text{Hom}_R(A, U) & \xrightarrow{\cong} & \text{Hom}_R(U', DA) \\
\text{Hom}(a, U) & \uparrow & \uparrow \text{Hom}(U', D(a)) \\
\text{Hom}(B, U) & \xrightarrow{\cong} & \text{Hom}_R(U', DB)
\end{array}
\]

Now $D(a) \neq 0$ if and only if $\text{Hom}(U', D(a)) \neq 0$ (since $U'$ is a generator) if and only if $\text{Hom}(a, U) \neq 0$. It follows that (a) and (b) are equivalent. Now since $D\circ T \cong T' \circ ( \_ )^*$, and $D'\circ T' \cong T \circ ( \_ )^*$, it is clear that if $D$ and $D'$ are faithful, then $Ker T = Ker D\circ T = Ker D'\circ T' \subseteq Ker ( \_ )^*$ and similarly $Ker T' \subseteq Ker ( \_ )^*$. Thus (a) implies (c). Suppose $Ker T = Ker ( \_ )^*$. If $a \in \text{Hom}_R(A, B)$ and $D(a) = 0$, then $(S(a))^* \cong S'DTS(a) \cong S'(a) = 0$ so $a = TS(a) = 0$, also. Thus (c) implies (a).

We denote the full subcategories of $R\text{-Mod}$ and $\text{Mod-}R$ whose objects are the torsion modules, i.e., those modules $M$ with $M^* = 0$, by $R\text{-Tors}$ and $\text{Tors-}R$, respectively. Then, if $R$ is $\text{QF-3'}$, an $R$-module $M$ is torsion if and only if $\text{Hom}(M, B(R)) = 0$, and $R\text{-Tors}$ and $\text{Tors-}R$ are then localizing subcategories of $R\text{-Mod}$ and $\text{Mod-}R$, respectively (see [3], Proposition 1.1).

6. Theorem. Every right adjoint pair of contravariant faithful functors
$D : \alpha \rightarrow \alpha' : D'$ between Grothendieck categories with reflexive generators defines a Morita duality.

**Proof.** Suppose that $D : \alpha \rightarrow \alpha' : D', U, U', R, T$, and $T'$ are as in Lemma 5 and that $D$ and $D'$ are faithful. Then by Lemma 5, $U$ and $U'$ are generator-cogenerators, so by Morita's ([6], Theorems 8.3 and 5.6) $R$ is QF-3' and its own maximal quotient ring. (See also [8], Theorem 4.13.4 or [5], Proposition 4.3.1). Thus by Theorem 2 the $R$-double duals are left exact. But by Lemma 5 we also have $\text{Ker} T = \text{Ker}(\bigotimes^2)$ so by ([8], Theorem 4.4.9) we may identify $T : R\text{-Mod} \rightarrow \alpha'$ and $T' : \text{Mod}-R \rightarrow \alpha'$ with the canonical functors $T : R\text{-Mod} \rightarrow R\text{-Mod}/\text{R-Tors}$ and $T' : \text{Mod}-R \rightarrow \text{Mod}-R/\text{Tors-R}$ and conclude that $(D, D')$ define a Morita duality by ([3], Theorem 2.6).

Specializing Theorem 6 to the case of module categories yields the following generalization of Onodera's theorem that cogenerator rings are injective [7] and provides a new characterization of Morita duality between categories of modules.

7. **Corollary.** If $R$ and $S$ are rings and $\rho U_S$ is a bimodule with $S = \text{End}_R(U)$ and $R = \text{End}(U_S)$ and if $\rho U$ and $U_S$ are cogenerators, then $\rho U$ and $U_S$ are injective.

**Proof.** Apply Theorem 6 to the functors $D = \text{Hom}_R(\cdot, U)$ and $D' = \text{Hom}_S(\cdot, U)$.

8. **Remarks.**

(1) One can apply the technique used to prove Theorem 1 to the sequence $U \rightarrow E(\rho U) \rightarrow U^k$ to give a direct proof that if $\rho U_S$ a balanced bimodule and $\rho U$ and $U_S$ are cogenerators, then $\rho U$ is injective.

(2) In ([4], Theorem 1), conditions (i) and (ii) and the last part of (iii) easily imply that $R$ is QF-3' so by Theorem 1, as we speculated in ([4], Remark (a)), we can delete the first part of condition (iii) from the statement of that theorem; in view of Theorems 1 and 2 and ([3], Theorem 3.1) it now becomes an analogue for QF-3' rings of the Morita-Tachikawa representation theorems for QF-3 rings ([9], Theorems 5.3 and 5.8).

**References**


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