COHOMOLOGIES OF HOMOGENEOUS ENDOMORPHISM
BUNDLES OVER LOW DIMENSIONAL
KÄHLER C-SPACES

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1. Introduction

In this paper, we determine the infinitesimal deformations of an Einstein-Hermitian structure of a homogeneous vector bundle in several cases. In particular, we get the tangent space at the homogeneous structure of the moduli space of Einstein-Hermitian structures as the representation space of a compact Lie group.

A compact simply connected homogeneous Kähler manifold is called a Kähler C-space. Such a manifold can be written as $G/K$ where $G$ is a compact semisimple Lie group and $K$ is the centralizer of a toral subgroup of $G$ ([10]). Let $G^c$ be the complexification of $G$ and $K^c$ the complexification of $K$ in $G^c$. We denote by $L$ the parabolic subgroup of $G^c$ which contains $K^c$. $G/K$ is diffeomorphic to $G^c/L$. Thus $G/K$ admits a holomorphic structure from the holomorphic structure of $G^c/L$. Moreover it admits a $G$-invariant Kähler metric.

Let $(\rho, V)$ be a complex representation of $K$. Then $(\rho, V)$ can be extended to a holomorphic representation $(\rho_L, V)$ of $L$. The homogeneous vector bundle $E_{\rho}=G \times_{\rho} V$ is isomorphic to the homogeneous holomorphic vector bundle $E_{\rho_L}=G^c \times_{\rho_L} V$ as $C^\infty$-vector bundles. Thus the homogeneous vector bundle $E_{\rho}$ has a natural holomorphic structure from the holomorphic structure of $E_{\rho_L}$ ([3]). Moreover if $(\rho, V)$ is irreducible, then $E_{\rho}$ has a unique $G$-invariant Einstein-Hermitian structure up to a homothety ([8]).

An irreducible complex representation $(\rho, V)$ is determined by the highest weight. Then a homogeneous vector bundle $E_{\rho}$ is determined by the highest weight of $(\rho, V)$, if $(\rho, V)$ is irreducible. It is natural to ask how we describe the deformations of the holomorphic structure and the Einstein-Hermitian structure by the highest weight. Also we ask how we describe moduli spaces of holomorphic structures and Einstein-Hermitian structures by the highest weight.

Received January 17, 1994.
In the deformation theory of complex structures of complex manifolds, the complex structure of a Kähler C-space is locally stable ([3]). But the holomorphic structure of a homogeneous vector bundle generally is not locally stable. Here the local stability means that any deformation space is trivial. So there is a problem to find sufficient conditions for the local stability of the holomorphic structure of a homogeneous vector bundle.

Let \( \text{End}(E) \) be the endomorphism bundle of \( E \) and \( \mathfrak{sl}(E) \) the subbundle of \( \text{End}(E) \) which consists of trace free endomorphisms. It is well known that the Dolbeault cohomology group \( H^{0,1}(G/K, \text{End}(E)) \) is the tangent space of the moduli space of holomorphic structures if \( H^{0,0}(G/K, \mathfrak{sl}(E))=H^{0,0}(G/K, \mathfrak{sl}(E))=\{0\} \). The moduli space of Einstein-Hermitian structures is an open subset of the moduli space of holomorphic structures. Under the same conditions, \( H^{1,0}(G/K, \text{End}(E)) \) is also the tangent space of the moduli space of Einstein-Hermitian structures ([7], [6] and [9]). So we think that it is important to compute these cohomologies for our problems.

In this paper, for a first step of problems above we investigate Kähler C-spaces \( G/K \) with rank \( G=2 \). In §2 we shall review a construction of Kähler C-spaces and some properties of vector bundles over them. We shall state our results in §3 and prove them in §4. Our main results are Theorems 4, 5 and Corollary 6. In the case of rank \( G=2 \), we compute \( H^{0,1}(G/K, \text{End}(E)) \) and \( H^{1,0}(G/K, \mathfrak{sl}(E)) \) from the highest weight of \( (\rho, V) \) (Theorems 4 and 5). Then we get dimension of the moduli space of Einstein-Hermitian structures of a homogeneous vector bundle in several cases (Corollary 6). The following theorem is an immediate consequence of these results.

**Theorem 1.** Let \( G/K \) be a Kähler C-space where \( G \) is of type \( A_2 \) or \( B_2 \). Let \( (E, h) \) be an irreducible Einstein-Hermitian homogeneous vector bundle over \( G/K \) with rank \( E=r \). Then the dimension of the moduli space of irreducible Einstein-Hermitian structures of \( E \) is as follows:

1. If \( G \) is of type \( A_2 \) or \( B_2 \) and if \( K \) is a maximal torus, then the dimension of the moduli space is 0.
2. If \( G/K\cong SU(3)/S(U(1)\times U(2))\cong P_3 \mathbb{C} \), then the dimension of the moduli space is
   \[
   \frac{1}{2} \sum_{k=1}^{r-1} (2k+1)(k+2)(k-1).
   \]
3. If \( G/K\cong Sp(2)/(U(1)\times Sp(1))\cong P_3 \mathbb{C} \), then the dimension of the moduli space is
   \[
   \frac{1}{3} \sum_{k=1}^{r-1} (2k+3)(2k+1)(2k-1).
   \]
(4) If $G/K \equiv SO(5)/(U(1) \times SO(3)) \equiv Q_4(C)$, then the dimension of the moduli space is

$$\frac{1}{2} \sum_{k=1}^{r-1} (2k-1)(k+2)(k-1).$$

The author would like to express his gratitude to Professor Mitsuhiro Itoh and Professor Hiroyuki Tasaki for their valuable advices and encouragement.

2. Preliminaries

In this section, we review a construction of Kähler C-spaces and some properties of homogeneous bundles over them. We refer the reader to the books [1] and [5] for the representation theory of compact Lie group.

Following Wang ([10]), we call a compact simply connected homogeneous Kähler manifold a Kähler C-space. A vector bundle $E$ over a homogeneous space $G/K$ is said to be homogeneous if it is associated to the principal $K$-bundle $G \to G/K$.

Let $G$ be a compact simply connected semisimple Lie group. Let $T_0$ be a toral subgroup of $G$ and $K$ the centralizer of $T_0$ in $G$. Then $G/K$ is a compact simply connected homogeneous manifold. Let $T$ be a maximal torus of $G$ which contains $T_0$. Then $T$ is contained in $K$ and we put $l = \dim T$. We denote by $\Delta$ the set of nonzero roots of $G$ relative to $T$. Let $\Pi = \{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ be a fundamental system of $\Delta$. We may assume that $\Pi$ is the set of simple roots of $\Delta$ for a suitable order of the Lie algebra of $T$. In this order, we denote the set of positive roots by $\Delta^+$. Let $\Delta_{\Pi_1}$ be the set of nonzero roots of $K$ relative to $T$ and $\Pi_1 = \{\alpha_i, \alpha_{i_q}, \ldots, \alpha_{i_r}\}$ the subset of $\Pi$ which generates $\Delta_{\Pi_1}$. If we denote the set of positive roots of $\Delta_{\Pi_1}$ by $\Delta_{\Pi_1}^+$, then we have $\Delta_{\Pi_1}^+ = \Delta^+ \cap \Delta_{\Pi_1}^+$.

Let $G^c$ and $K^c$ be complexifications of $G$ and $K$, respectively. Let $L$ be the parabolic subgroup of $G^c$ such that its Lie algebra is generated by the Lie algebra of $K^c$ and $\{E_\alpha; \alpha \in \Delta^+ \setminus \Delta_{\Pi_1}^+\}$. Here $E_\alpha$ denotes the root vector of $\alpha \in \Delta$. Then we see

$$G/K \cong G^c/L$$

as $C^\infty$-manifolds. So $G/K$ is a homogeneous complex manifold by this holomorphic structure. Moreover it has a $G$-invariant Kähler metric ([2]). Thus we get a Kähler C-space $G/K$. Conversely every Kähler C-space can be described as above. Also we can construct the Kähler C-space from a pair $(\Pi, \Pi_1)$, where $\Pi$ is a fundamental system of roots and $\Pi_1$ is a subset of $\Pi$ ([10] and [2]).
Let \((\rho, V)\) be an irreducible finite dimensional complex representation of \(K\) with highest weight \(\rho\). We denote by \(\{\varpi_1, \varpi_2, \ldots, \varpi_i\}\) the system of fundamental weights of \(\Pi\). Then the highest weight \(\rho\) of \((\rho, V)\) can be written as follows:
\[
\rho = n_1\varpi_1 + n_2\varpi_2 + \cdots + n_i\varpi_i,
\]
where \(n_1, n_2, \ldots, n_i\) are integers and if \(\alpha_i \in \Pi\), then \(n_i \geq 0\).

In this case we can uniquely extend \((\rho, V)\) to a holomorphic representation \((\rho_L, V)\) of \(L\) ([3]). We put
\[
E_\rho = G \times_\rho V,
\]
\[
E_{\rho L} = G^c \times_{\rho L} V.
\]
Then
\[
E_\rho \cong E_{\rho L}
\]
as \(C^\infty\)-vector bundles. We regard \(E_\rho\) as a holomorphic vector bundle by the isomorphism above unless otherwise stated. Also if \((\rho, V)\) is irreducible, there is a unique \(G\)-invariant Hermitian structure \(h\) up to a homothety and \((E_\rho, h)\) is an irreducible Einstein-Hermitian vector bundle ([8]). Therefore we consider \(E_\rho\) as an irreducible Einstein-Hermitian vector bundle if \((\rho, V)\) is irreducible. For more details about an Einstein-Hermitian vector bundle, we refer the reader to [7].

By \(\text{End}(E_\rho)\) we denote the endomorphism bundle of \(E_\rho\). Let \(\mathfrak{sl}(E_\rho)\) be the subbundle of \(E_\rho\) which consists of trace free endomorphisms. By definition of \(\text{End}(E_\rho)\) and \(\mathfrak{sl}(E_\rho)\),
\[
\begin{align*}
\text{End}(E_\rho) &\cong G \times_{\rho \otimes \rho^*} \text{End}(V), \\
\mathfrak{sl}(E_\rho) &\cong G \times_{\rho \otimes \rho^*} \mathfrak{sl}(V),
\end{align*}
\]
where \(\text{End}(V)\) is the linear space of endomorphisms and \(\mathfrak{sl}(V)\) is the subspace of \(\text{End}(V)\) consisting of trace free endomorphisms. Thus
\[
\text{End}(V) = V \otimes V^*,
\]
where \(V^*\) is the dual space of \(V\). And \(K\) acts \(\text{End}(V)\) by the tensor product representation \((\rho \otimes \rho^*, V \otimes V^*)\) where \((\rho^*, V^*)\) is the dual representation of \((\rho, V)\). By the way \(\mathfrak{sl}(E)\) is invariant by \(K\). Thus \(K\) acts \(\mathfrak{sl}(V)\) via the action for \(\text{End}(V)\).

Finally we denote the Dolbeault cohomology groups of \(\text{End}(E_\rho)\) and \(\mathfrak{sl}(E_\rho)\) by \(H^{p,q}(G/K, \text{End}(\rho))\) and \(H^{p,q}(G/K, \mathfrak{sl}(\rho))\), respectively. We set
\[
\begin{align*}
h^{p,q}(\text{End}(E_\rho)) &= \dim H^{p,q}(G/K, \text{End}(E_\rho)), \\
h^{p,q}(\mathfrak{sl}(E_\rho)) &= \dim H^{p,q}(G/K, \mathfrak{sl}(E_\rho)).
\end{align*}
\]
3. Main Results

We continue with the notation and the situation in §2. Let $G/K$ be a Kähler C-space where $G$ is a compact simply connected semisimple Lie group and $K$ is the centralizer of a toral subgroup of $G$.

**Lemma 2.** Let $(\rho, V)$ be an irreducible complex representation of $K$. Then the restriction of $(\rho \otimes \rho^*, V \otimes V^*)$ to the center of $K$ is trivial.

**Proof.** Let $Z_K$ be the center of $K$ and $K'$ the semisimple part of $K$. Then

$$\varphi: Z_K \times K' \to K, \quad (z, k) \mapsto zk$$

is a Lie group homomorphism with kernel $Z_K \cap K'$. Then $(\rho \circ \varphi, V)$ is a representation of the direct product Lie group $Z_K \times K'$ on $V$. We note that

$$\rho \circ \varphi \mid_{Z_K} = \rho \mid_{Z_K}, \quad \rho \circ \varphi \mid_{K'} = \rho \mid_{K'}.$$

Because of irreducibility of $(\rho, V)$, $(\rho \circ \varphi, V)$ is also irreducible. So there are irreducible representations $(\rho_{Z_K}, V_{Z_K})$ of $Z_K$ and $(\rho_{K'}, V_{K'})$ of $K'$ such that

$$(\rho_{Z_K} \otimes \rho_{K'}, V_{Z_K} \otimes V_{K'}) \cong (\rho \circ \varphi, V),$$

where $(\rho_{Z_K} \otimes \rho_{K'}, V_{Z_K} \otimes V_{K'})$ denotes the exterior tensor product representation of $(\rho_{Z_K}, V_{Z_K})$ and $(\rho_{K'}, V_{K'})$. By irreducibility of $(\rho_{Z_K}, V_{Z_K})$, $V_{Z_K}$ is a one dimensional space. Then the tensor product $(\rho_{Z_K} \otimes \rho_{K'}, V_{Z_K} \otimes V_{K'})$ is isomorphic to the trivial representation. Q.E.D.

**Corollary 3.** Let $G$ be a compact semisimple Lie group and $K=\mathbb{T}$ be a maximal torus of $G$. Let $(\rho, V)$ be an irreducible complex representation of $\mathbb{T}$. Then

$$\text{End}(E_\rho) \cong G/T \times \mathcal{C},$$

$$\text{sl}(E_\rho) \cong G/T \times \{0\}.$$

In particular,

$$h^{0, \nu}(\text{End}(E_\rho)) = \begin{cases} 1, & \text{for } \nu = 0 \\ 0, & \text{for } \nu \geq 1, \end{cases}$$

$$h^{0, \nu}(\text{sl}(E_\rho)) = 0, \text{ for } \nu \geq 0.$$

**Proof.** We note that any irreducible complex representation space of a torus is one dimensional. From Lemma 1 it is easy to see that $\text{End}(E_\rho)$ is trivial. And also it is easy to see that Hodge numbers of $\text{End}(E_\rho)$ and $\text{sl}(E_\rho)$ are as stated above (for example, by means of Bott's generalized Borel-Weil
Next we consider the case of rank $G=2$. In this case the fundamental system of roots $\Pi$ is $\{\alpha_1, \alpha_2\}$. And if $G/K$ is a Kähler $C$-space then the corresponding $\Pi_1$ as in section 2 is $\{\alpha_1, \alpha_2\}$, $\{\alpha_1\}$, $\{\alpha_2\}$ or $\phi$. Furthermore a compact simple Lie group $G$ is of type $A_2$, $B_2$, or $G_2$ in this case. If $G$ is of classical type then corresponding Kähler $C$-spaces are

$$
G/K \cong SU(3)/S(U(2) \times U(1)) \cong P_3C \text{ if } G = A_2 \text{ and } \Pi_1 = \{\alpha_1\},
$$

$$
G/K \cong SU(3)/S(U(1) \times U(2)) \cong P_3C \text{ if } G = A_2 \text{ and } \Pi_1 = \{\alpha_2\},
$$

$$
G/K \cong Sp(2)/(U(1) \times Sp) \cong P_3C \text{ if } G = B_2 \text{ and } \Pi_1 = \{\alpha_1\},
$$

$$
G/K \cong SO(5)/(U(1) \times SO(3)) \cong Q_3(C) \text{ if } G = B_2 \text{ and } \Pi_1 = \{\alpha_2\}.
$$

It is clear that the first case is isomorphic to the second one in the above. And we note that if $\Pi_1 = \{\alpha_1, \alpha_2\}$ then $K = T$ is a maximal torus of $G$, i.e., it is the case of Corollary 3. Also if $\Pi = \phi$ then $K = G$, i.e., $G/K$ consists of a one point $\{\sigma\}$.

Then we state the main theorem.

**Theorem 4.** Let $G$ be a compact simply connected simple Lie group with rank $G=2$. Let $\Pi = \{\alpha_1, \alpha_2\}$ be a fundamental system of roots relative to a maximal torus $T$ of $G$. We put $\Pi_i = \{\alpha_i\}$ ($i=1, 2$). We denote by $K$ the analytic subgroup of $G$ with maximal rank which corresponds to $\Pi_i$. Let $(\rho, V)$ be an irreducible complex representation of $K$ with highest weight $\hat{\rho} = n_1 \varpi_1 + n_2 \varpi_2$, where $\{\varpi_1, \varpi_2\}$ is the system of fundamental weights. Then the Hodge number $h^{\rho, p}(\text{End}(E_\rho))$ is as follows:

(I) If $G$ is of type $A_2$ and $\Pi_1 = \{\alpha_2\}$ then

$$
h^{\rho, p}(\text{End}(E_\rho)) = \begin{cases} 
1, & \text{if } p=0, \\
\frac{1}{2} \sum_{k=1}^{n_2} (2k+1)(k+2)(k-1), & \text{if } p=1, \\
0, & \text{if } p \geq 2.
\end{cases}
$$

(II) If $G$ is of type $B_2$ and $\Pi_1 = \{\alpha_1\}$ then

$$
h^{\rho, p}(\text{End}(E_\rho)) = \begin{cases} 
1, & \text{if } p=0, \\
\frac{1}{3} \sum_{k=1}^{n_1} (2k+3)(2k+1)(2k-1), & \text{if } p=1, \\
0, & \text{if } p \geq 2.
\end{cases}
$$
(III) If $G$ is of type $B_2$ and $\Pi_1 = \{\alpha_2\}$ then

$$h^{p, q}(\text{End}(E_\rho)) = \begin{cases} 
1, & \text{if } p = 0, \\
1/2 \sum_{k=1}^{n_2} (2k-1)(k+2)(k-1), & \text{if } p = 1, \\
0, & \text{if } p \geq 2.
\end{cases}$$

(IV) If $G$ is of type $G_2$ and $\Pi_1 = \{\alpha_1\}$ then

$$h^{p, q}(\text{End}(E_\rho)) = \begin{cases} 
1, & \text{if } p = 0, \\
7, & \text{if } p = 1 \text{ and } n_1 = 2, \\
21, & \text{if } p = 1 \text{ and } n_1 \geq 3, \\
1/40 \sum_{k=3}^{n_1} (2k+1)(k+5)(k+2)(k-1)(k-4), & \text{if } p = 2, \\
0, & \text{otherwise}.
\end{cases}$$

(V) If $G$ is of type $G_2$ and $\Pi_1 = \{\alpha_2\}$ then

$$h^{p, q}(\text{End}(E_\rho)) = \begin{cases} 
1, & \text{if } p = 0, \\
2261, & \text{if } p = 1, \\
1/24 \sum_{k=3}^{n_2} (21k-2)(18k-1)(13k-1)(8k-1)(3k-1)k, & \text{if } p = 2, \\
0, & \text{if } p \geq 3.
\end{cases}$$

**Theorem 5.** Under the same assumption of Theorem 4, the Hodge number $h^{p, q}(\mathfrak{g}(E_\rho))$ is equal to

$$h^{p, q}(\mathfrak{g}(E_\rho)) = \begin{cases} 
0, & \text{if } p = 0, \\
h^{p, q}(\text{End}(E_\rho)), & \text{if } p \geq 1,
\end{cases}$$

for every cases (I)~(V) in Theorem 4.

We shall prove these theorems in the next section. We state some consequences of these results here. If $h^{0, 0}(\mathfrak{g}(E_\rho)) = h^{0, 2}(\mathfrak{g}(E_\rho)) = 0$, then we can identify $H^{0, 1}(G/K, \text{End}(E_\rho))$ with the tangent space at the homogeneous structures of the moduli space of $E_\rho$ ([6], [9] and [7, Chapter VII]). Then we get the following corollary.

**Corollary 6.** Under the same assumption of theorems above, the dimension of the moduli space of irreducible Einstein-Hermitian structures is as follows:

1. If $G$ is of type $A_2$ and $\Pi_1 = \{\alpha_2\}$ then

$$\frac{1}{2} \sum_{k=1}^{n_2} (2k+1)(k+2)(k-1).$$
II) If \( G \) is of type \( B_2 \) and \( \Pi_1 = \{ \alpha_1 \} \) then
\[
\frac{1}{3} \sum_{k=1}^{n_1} (2k+3)(2k+1)(2k-1).
\]

III) If \( G \) is of type \( B_2 \) and \( \Pi_1 = \{ \alpha_2 \} \) then
\[
\frac{1}{2} \sum_{k=1}^{n_2} (2k-1)(k+2)(k-1).
\]

IV) If \( G \) is of type \( G_2 \) and \( \Pi_1 = \{ \alpha_1 \} \) then
\[
\begin{cases}
0, & \text{for } n_1 = 0 \text{ or } 1, \\
7, & \text{for } n_1 = 2, \\
21, & \text{for } n_1 = 3 \text{ or } 4.
\end{cases}
\]

(V) If \( G \) is of type \( G_2 \) and \( \Pi_1 = \{ \alpha_2 \} \) then
\[
\begin{cases}
0, & \text{for } n_2 = 0, \\
261, & \text{for } n_2 = 1.
\end{cases}
\]

Because of Theorem 5, we see that if \( G \) is of classical type, then
\[
h^\rho,\gamma(\mathfrak{g}(E_\rho)) = h^{\rho,\gamma}(\mathfrak{sl}(E_\rho)) = 0.
\]
Moreover we note that if \( \Pi_1 = \{ \alpha_i \} \) then \( \text{rank } E_\rho = n_i + 1 \) in theorems and corollary above. Also under the same condition, we note that the dimension of the moduli space of \( E_\rho \) depends only on \( n_i \). Then we get Theorem 1 from Corollaries 3 and 6.

4. Proof of Theorems

In this sections, we prove Theorems 4 and 5. The two theorems 4 and 5 are proved at the same time. First we note equations (1) and (2) in §2. So \( \text{End}(E_\rho) \) and \( \mathfrak{sl}(E_\rho) \) are defined by representations \( (\rho \otimes \rho^*, V \otimes V^*) \) and \( (\rho \otimes \rho^*, \mathfrak{sl}(V)) \), respectively. Because of Lemma 2, \( \rho \otimes \rho^* \) is trivial on the center of \( K \). The semisimple part of \( K \) is of type \( A_1 \) in these cases. Thus we can apply the Clebsch-Gordan theorem to the representation \( \rho \otimes \rho^* \). Therefore we see that if \( \Pi_1 = \{ \alpha_i \} \) and the highest weight of \( \rho \) is given by \( \rho = n_1 \omega + n_2 \omega' \), then the highest weight which corresponds to each irreducible component of \( (\rho \otimes \rho^*, \text{End}(V)) \) are given by
\[
(4) \quad n_1 \alpha_i, (n_1-1) \alpha_i, (n_1-2) \alpha_i, \ldots, \alpha_i, 0.
\]

Also under the same assumption the highest weight which corresponds to each irreducible component of \( (\rho \otimes \rho^*, \mathfrak{sl}(V)) \) are given by
\[
(5) \quad n_1 \alpha_i, (n_1-1) \alpha_i, (n_1-2) \alpha_i, \ldots, 2 \alpha_i, \alpha_i.
\]
Let \( \rho_k, V_k \) be the complex irreducible representation of \( K \) with highest weight \( \tilde{\rho}_k = k \alpha_i \) and we put \( E_{\rho_k} = G \times_{\rho_k} V_k \). Then (4) and (5) imply the following, respectively:

\[
H^{\alpha, p}(G/K, \text{End}(E_{\rho})) = \bigoplus_{k=0}^{n_k} H^{\alpha, p}(G/K, E_{\rho_k}),
\]

\[
H^{\alpha, p}(G/K, \mathfrak{sl}(E_{\rho})) = \bigoplus_{k=1}^{n_k} H^{\alpha, p}(G/K, E_{\rho_k}).
\]

Next we compute the cohomology one by one. We denote by \( \delta \) the half of the sum of the positive roots, i.e.,

\[
\delta = \varpi_1 + \varpi_2
\]

in these cases. And we denote by \( S_\alpha \) the reflection with respect to \( \alpha \in \Delta \). We use the tables of root systems in [4] for the following.

**Case (1)** In this case \( G \) is of type \( A_2 \) and \( \Pi_1 = \{ \alpha_i \} \). Then we see

\[
\delta + \tilde{\rho}_k = \delta + k \alpha_i
\]

\[
= -(k-1)\varpi_1 + (2k+1)\varpi_2.
\]

From this, we see that

\[
\delta + \tilde{\rho}_k \begin{cases} \text{regular} & \text{if } k \neq 1, \\ \text{singular} & \text{if } k = 1 \end{cases}
\]

in the sense of [3]. Also we see that

\[
\text{the index of } \delta + \tilde{\rho}_k = \begin{cases} 0, & \text{if } k = 0, \\ 1, & \text{if } k \neq 0, 1. \end{cases}
\]

By the way, we have

\[
S_{\alpha_1}(\delta + \tilde{\rho}_k) = -(k-1)\varpi_1 + (k+2)\varpi_2.
\]

This implies that \( S_{\alpha_1}(\delta + \tilde{\rho}_k) \) is contained in the fundamental Weyl chamber. We put

\[
\lambda_k = -(k-2)\varpi_1 + (k+1)\varpi_2
\]

and \( V_{\lambda_k} \) denotes the complex irreducible representation space of \( G \) with highest weight \( \lambda_k \). By means of Bott’s generalized Borel-Weil theorem ([3, Theorem IV']), we get

\[
H^{\alpha, p}(G/K, E_{\rho_k}) \cong \begin{cases} V_{\lambda_k}, & \text{for } k \geq 1 \text{ and } p=1, \\ 0, & \text{for } k \geq 1 \text{ and } p \neq 1, \end{cases}
\]

and
as complex $G$-spaces. We have

\[
H^{0,p}(G/K, E_{\rho_k}) \cong \begin{cases} 
C, & \text{for } k=0 \text{ and } p=0, \\
\{0\}, & \text{for } k=0 \text{ and } p \geq 1.
\end{cases}
\]

from equations (6), (8) and (9) and

\[
H^{0,p}(G/K, \text{End}(E_{\rho})) \cong \bigoplus_{k=1}^{n_3} V_{\lambda_k}, \quad \text{for } p=1,
\]
\[
\{0\}, \quad \text{for } p \geq 2,
\]

from equations (7), (8) and (9) as complex $G$-spaces. We can compute $h^{0,p}(\text{End}(E_{\rho}))$ and $h^{0,p}(\text{End}(E_{\rho}))$ by Weyl's dimension formula for a complex irreducible representation space. Then we obtain theorems in the case (1).

**Case (II)** In this case $G$ is of type $B_2$ and $\Pi = \{ \alpha_i \}$. Then we see

\[
\delta + \hat{\rho}_k = \delta + k \alpha_k
\]
\[
= (2k+1)\sigma_1 - (2k-1)\sigma_2.
\]

From this, we see $\delta + \hat{\rho}_k$ is regular for any $k$. Also we see

\[
\text{the index of } \delta + \hat{\rho}_k = \begin{cases} 
0, & \text{for } k=0, \\
1, & \text{for } k \neq 0.
\end{cases}
\]

By the way, we have

\[
S_{\alpha_2}(\delta + \hat{\rho}_k) = 2\sigma_1 + (2k-1)\sigma_2.
\]

This implies that $S_{\alpha_2}(\delta + \hat{\rho}_k)$ is contained in the fundamental Weyl chamber. In the same way as in the case (1), we get

\[
H^{0,p}(G/K, E_{\rho}) \cong \bigoplus_{k=1}^{n_1} V_{\lambda_k}, \quad \text{for } p=1,
\]
\[
\{0\}, \quad \text{for } p \geq 1,
\]

as complex $G$-spaces. Here

\[
\lambda_k = (2k-1)\sigma_1 + (2k-2)\sigma_2
\]

and $V_{\lambda_k}$ is the irreducible complex representation space of $G$ which corresponds
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We get $h^{a,p}(\text{End}(E_p))$ and $h^{a,p}(\mathfrak{g}(E_p))$ by Weyl's dimension formula as before.

**Case (III)** In this case $G$ is of type $B_\alpha$ and $II = \{\alpha\}$. Then we see

$$\delta + \rho_k = \delta + k\alpha$$

$$= -(k-1)\varpi_1 + (2k+1)\varpi_2.$$

From this, we see

$$\delta + \rho_k$$

is

$$\begin{cases} 
\text{regular,} & \text{if } k \neq 1, \\
\text{singular,} & \text{if } k = 1,
\end{cases}$$

and

$$\text{the index of } \delta + \rho_k = \begin{cases} 
0, & \text{for } k = 0, \\
1, & \text{for } k \neq 0, 1.
\end{cases}$$

By the way, we have

$$S_{\alpha}(\delta + \rho_k) = (k-1)\varpi_1 + 3\varpi_2.$$

This implies that $S_{\alpha}(\delta + \rho_k)$ is in the fundamental Weyl chamber. As before, we get

$$H^{a,p}(G/K, \text{End}(E_p)) \cong \begin{cases} 
C, & \text{for } p = 0, \\
\bigoplus_{k=1}^{n_2} V_{\lambda_k}, & \text{for } p = 1, \\
\{0\}, & \text{for } p \geq 2,
\end{cases}$$

and

$$H^{a,p}(G/K, \mathfrak{g}(E_p)) \cong \begin{cases} 
\bigoplus_{k=1}^{n_2} V_{\lambda_k}, & \text{for } p = 1, \\
\{0\}, & \text{for } p \neq 1
\end{cases}$$

as complex $G$-spaces. Here

$$\lambda_k = (k-2)\varpi_1 + 2\varpi_2$$

and $V_{\lambda_k}$ is the irreducible complex representation space of $G$ which corresponds to $\lambda_k$. We get $h^{a,p}(\text{End}(E_p))$ and $h^{a,p}(\mathfrak{g}(E_p))$ by Weyl's dimension formula as before.

**Case (IV)** In this case $G$ is of type $G_2$ and $II = \{\alpha\}$. Then we see

$$\delta + \rho_k = \delta + k\alpha$$

$$= (2k+1)\varpi_1 -(k-1)\varpi_2.$$

From this, we see that

$$\delta + \rho_k$$

is

$$\begin{cases} 
\text{regular,} & \text{if } k \neq 1 \text{ and } 4, \\
\text{singular,} & \text{if } k = 1 \text{ or } 4,
\end{cases}$$

and
the index of $\delta + \hat{\rho}_k = \begin{cases} 0, & \text{for } k = 0, \\ 1, & \text{for } k = 1, 2, 3 \text{ or } 4, \\ 2, & \text{for } k \geq 5. \end{cases}$

By the way we have,

$$S_{\alpha_1} (\delta + \hat{\rho}_k) = -(k - 4) \omega_1 + (k - 1) \omega_2$$

and

$$S_{\alpha_1} S_{\alpha_2} (\delta + \hat{\rho}_k) = (k - 4) \omega_1 + 3 \omega_2.$$

This implies that $S_{\alpha_1} (\delta + \hat{\rho}_k)$ is contained in the fundamental Weyl chamber if $k = 2, 3$ and $S_{\alpha_1} S_{\alpha_2} (\delta + \hat{\rho}_k)$ is contained in the fundamental Weyl chamber if $k \geq 5$. As before, we get

$$H^{0, p}(G/K, \text{End}(E_\rho)) \cong \begin{cases} C, & \text{if } p = 0, \\ \{0\}, & \text{if } p = 1 \text{ and } n_1 = 0, 1, \\ V_{\lambda_2}, & \text{if } p = 1 \text{ and } n_1 = 2, \\ V_{\lambda_2} \oplus V_{\lambda_3}, & \text{if } p = 1 \text{ and } n_1 \geq 3, \\ \bigoplus_{k=5}^{n_1} V_{\lambda_k}, & \text{if } p = 2, \\ \{0\}, & \text{if } p \geq 3, \end{cases}$$

and

$$H^{0, p}(G/K, \mathfrak{g}(E_\rho)) \cong \begin{cases} \{0\}, & \text{if } p = 0, \\ \{0\}, & \text{if } p = 1 \text{ and } n_1 = 0, 1, \\ V_{\lambda_2}, & \text{if } p = 1 \text{ and } n_1 = 2, \\ V_{\lambda_2} \oplus V_{\lambda_3}, & \text{if } p = 1 \text{ and } n_1 \geq 3, \\ \bigoplus_{k=5}^{n_1} V_{\lambda_k}, & \text{if } p = 2, \\ \{0\}, & \text{if } p \geq 3 \end{cases}$$

as complex $G$-spaces. Here

$$\lambda_k = \begin{cases} -(k - 3) \omega_1 + (k - 2) \omega_2, & \text{for } k = 2, 3, \\ (k - 5) \omega_1 + 2 \omega_2, & \text{for } k \geq 5. \end{cases}$$

And $V_{\lambda_k}$ is the irreducible complex representation space of $G$ which corresponds to $\lambda_k$. We can compute $h^{0, p}(\text{End}(E_\rho))$ and $h^{0, p}(\mathfrak{g}(E_\rho))$ by Weyl's dimension formula as before.

**Case (V)** In this case $G$ is of type $G_2$ and $\Pi_1 = \{\alpha_3\}$. Then we see

$$\delta + \hat{\rho}_k = \delta + k \alpha_3$$

$$= -(3k - 1) \omega_1 + (2k + 1) \omega_2.$$

From this, we see that
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\[ \delta + \rho_k \begin{cases} \text{regular,} & \text{if } k \neq 2, \\ \text{singular,} & \text{if } k = 2. \end{cases} \]

and

\[ \text{the index of } \delta + \rho_k = \begin{cases} 0, & \text{for } k = 0, \\ 1, & \text{for } k = 1, \\ 2, & \text{for } k \geq 2. \end{cases} \]

By the way, we have

\[ S_{\alpha_1}(\delta + \rho_k) = (3k - 1)\varpi_1 + 5k\varpi_2. \]

This implies that \( S_{\alpha_1}(\delta + \rho_k) \) is contained in the fundamental Weyl chamber. As before, we get

\[ H^{n_p}(G/K, \text{End}(E_\rho)) \cong \begin{cases} C, & \text{for } p = 0, \\ \{0\}, & \text{for } p = 1 \text{ and } n_2 = 0, \\ V_{\lambda_1}, & \text{for } p = 1 \text{ and } n_2 \geq 1, \\ \bigoplus_{k=2}^{n_2} V_{\lambda_k}, & \text{for } p = 2, \\ \{0\}, & \text{for } p \geq 3, \end{cases} \]

and

\[ H^{n_p}(G/K, \mathfrak{sl}(E_\rho)) \cong \begin{cases} \{0\}, & \text{for } p = 0, \\ \{0\}, & \text{for } p = 1 \text{ and } n_2 = 0, \\ V_{\lambda_1}, & \text{for } p = 1 \text{ and } n_2 \geq 1, \\ \bigoplus_{k=2}^{n_2} V_{\lambda_k}, & \text{for } p = 2, \\ \{0\}, & \text{for } p \geq 3 \end{cases} \]

as complex \( G \)-spaces. Here

\[ \lambda_k = (3k - 2)\varpi_1 + (5k - 1)\varpi_2, \]

and \( V_{\lambda_k} \) is the irreducible complex representation space of \( G \) which corresponds to \( \lambda_k \). We can compute \( h^{n_p}(\text{End}(E_\rho)) \) and \( h^{n_p}(\mathfrak{sl}(E_\rho)) \) by Weyl's dimension formula as before.

Now we complete the proof of Theorems 4 and 5.

**Remark.** We only write the dimensions in the statement of Theorems and Corollary, but we get the cohomology groups and the tangent spaces of moduli spaces as the representation space of \( G \). Indeed, we have determined the highest weight of each irreducible component in the proof.
References


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