ON CERTAIN CURVES OF GENUS THREE WITH MANY AUTOMORPHISMS

By

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Introduction.

Let $k$ be an algebraically closed ground field. When $C$ is a complete nonsingular curve of genus $g$ and $G$ is a subgroup of its automorphism group $\text{Aut}(C)$, we call the pair $(C, G)$ an AM curve of genus $g$ ("AM stands for "automorphism").

In Part I, we consider the AM curve $(K, \text{Aut}(K))$, where $K$ is the plane curve defined by $x_1^4 + x_2^5 + x_3^4$ (in $\text{char}(k)\neq 7$). It is known [7] that $\sharp \text{Aut}(K)$ attains the Hurwitz's bound: $84(g-1)$ with $g=3$, in case $\text{char}(k) > g+1$ with $g=3$. To determine $(K, \text{Aut}(K))$, we use the fact that $\text{Aut}(C)$ of a nonsingular quartic plane curve $C$ is canonically identified with a subgroup of $\text{PGL}(3, k)$. We shall show in particular that when $\text{char}(k)=3$, $(K, \text{Aut}(K))$ is isomorphic to the AM curve $(K_4, \text{PSU}(3, 3^2))$, where $K_4$ is defined by $x_1^4 + x_2^4 + x_3^4$ and $\text{PSU}(3, 3^2)$ is a simple subgroup of $\text{PGL}(3, k)$ of order 6048. We note that it is the maximum order among the automorphism groups of (complete nonsingular) curves of genus 3 [8].

In Part II we consider the families of AM curves $(C, G)$ of genus 3, where $G$ is isomorphic to the symmetric group of degree 4, $\Sigma_4$. (We note that $\text{Aut}(K)$ contains such subgroups.) In §1, we shall determine "normal forms" of such AM curves. In §2 we shall determine the isomorphism classes in the above normal forms. In §3, using these results, we explain the relations between the subgroups of Teichmüller modular group $\text{Mod}(3)$ which are isomorphic to $\Sigma_4$ and their representations on the spaces of holomorphic differentials. In fact, for an AM Riemann surface $(W, G)$ (similarly defined as in the case of AM curves), we obtain naturally a subgroup (denoted by $M(W, G)$) of the Teichmüller modular group $\text{Mod}(3)$, which is isomorphic to $G$. Also we obtain a subgroup (denoted by $\rho(W, G)$) of $\text{GL}(3, C)$ which is the image of the representation of $G$ on the space of holomorphic differentials. We shall prove:

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Theorem. Let \((W, G)\) be an AM Riemann surface of genus three. Assume that \(G\) is isomorphic to \(\mathfrak{S}_4\). Then we have:

1. \(M(W, G)\) is \(\text{Mod}(3)\)-conjugate to either \(MG_{sy}\) or \(MH_{sy}\), \(\rho(W, G)\) is \(GL(3, C)\)-conjugate to either \(G_{sy}\) or \(H_{sy}\).

2. \(M(W, G) \sim MG_{sy}\) (resp. \(MH_{sy}\)) if and only if \(\rho(W, G) \sim G_{sy}\) (resp. \(H_{sy}\)).

\(MG_{sy}\) and \(MH_{sy}\) (resp. \(G_{sy}\) and \(H_{sy}\)) in the above are certain subgroups of \(\text{Mod}(3)\) (resp. \(GL(3, C)\)), which are explained in (3.1) of Part II.

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Part I. On the automorphism group of Klein's quartic curve.

§ 1. Notations and theorem.

1.1. Let \(k\) be an algebraically closed base field of characteristic \(p \geq 0\). A curve will mean a complete nonsingular curve over \(k\). If \(C\) is a nonhyperelliptic curve of genus 3, then its canonical embedding is a quartic plane curve. Conversely, any (nonsingular) quartic plane curve is nonhyperelliptic of genus 3, and its embedding into the ambient projective plane is canonical.

Let \(C'\) and \(C\) be two quartic plane curves. We denote by \(\text{Lin}(C', C)\) the set of automorphisms of the ambient projective plane which induce isomorphisms of \(C'\) onto \(C\). Then it is known that the natural mapping of \(\text{Lin}(C', C)\) into \(\text{Iso}(C', C)\) is a bijection.

Considering a system of homogeneous coordinates, we put

\[ P^3 = \text{Proj}(k[x_1, x_2, x_3]). \]

Then we may identify the group of automorphisms of \(P^3\), \(\text{Aut}(P^3)\), with a projective linear group, \(PGL(3, k)\). In fact, if a matrix \((a_{ij})\) represents an element of \(PGL(3, k)\), its corresponding automorphism (of \(P^3\)) is defined by:
\[(x_1, x_2, x_3) \mapsto \left( \sum_{j=1}^{3} a_{1j}x_j, \sum_{j=1}^{3} a_{2j}x_j, \sum_{j=1}^{3} a_{3j}x_j \right). \]

If \( C \) is a quartic plane curve in \( \mathbb{P}^2 = \text{Proj}(k[x_1, x_2, x_3]) \) the automorphism group of \( C, \text{Aut}(C) \), is always considered as a subgroup of \( \text{PGL}(3, k) \). For a matrix \( T = (a_{ij}) \) in \( M(3, k) \), \( T^* \) denotes the homomorphism of the graded ring of \( k[x_1, x_2, x_3] \), defined by \( x_i \mapsto \sum_{j=1}^{3} a_{ij}x_j \) \((i = 1, 2, 3)\). And when \( T \) is an element of \( \text{GL}(3, k) \) and \( H \) is a subset or an element of \( \text{GL}(3, k) \), we denote \( T^{-1} \cdot H \cdot T \) by \( T^*(H) \).

We use the same notation for a quartic curve and a generator of its homogeneous ideal of definition. And we denote an element of \( \text{PGL}(3, k) \) by its representatives when there is no fear of confusion. Then, for example, if \( C \) is a quartic curve and \( H \) is a subset of \( \text{Aut}(C) \), then for any element \( T \) of \( \text{PGL}(3, k) \), \( T^*(C) \) is well-defined as a plane curve, and \( T^*(H) \) is also well-defined as a subset of \( \text{Aut}(T^*(C)) \).

1.2. Notations. We fix a primitive 7-th root of unity \( \zeta \) in \( k \) (if exists), and we denote: (cf. [11])

\[
\begin{align*}
\beta_1 &:= \zeta^2 + \zeta^4, \quad \beta_2 := \zeta^3 + \zeta^4, \quad \beta_3 := \zeta^4 + \zeta, \\
\gamma_1 &:= \zeta^2 - \zeta^4, \quad \gamma_2 := \zeta^3 - \zeta, \quad \gamma_3 := \zeta^4 - \zeta, \\
\theta_1 &:= \zeta^3 + \zeta^4 + \zeta^5, \quad \theta_2 := \zeta^4 + \zeta^5 + \zeta^6 \quad \text{and} \\
\alpha_1 &:= \beta_1 + \beta_2, \quad \alpha_2 := \beta_1 + \beta_3, \quad \alpha_3 := \beta_2 + \beta_3.
\end{align*}
\]

It is immediate to see:

1. \( \beta_1^2 = \beta_2 + 2, \ \beta_2^2 = \beta_3 + 2, \ \beta_3^2 = \beta_1 + 2, \ \beta_1 \beta_2 = \beta_1 + \beta_2, \ \beta_2 \beta_3 = \beta_3 + \beta_1, \ \beta_1 \beta_3 = \beta_3 + \beta_2, \)
2. \( \beta_1, \ \beta_2 \) and \( \beta_3 \) are the distinct three roots of the equation \( \beta^3 + \beta^2 - 2\beta + 1 = 0, \)
3. \( \theta_1 \) and \( \theta_2 \) are the distinct two roots of the equation \( (2\theta + 1)^2 + 7 = 0, \)
4. \( \beta_1 \gamma_1 = \gamma_2, \ \beta_2 \gamma_2 = \gamma_1, \ \beta_3 \gamma_3 = \gamma_1, \ \alpha_1 \gamma_1 = \gamma_2, \ \alpha_2 \gamma_2 = \gamma_1, \ \alpha_3 \gamma_3 = \gamma_2. \)

Next we define distinguished elements and a subgroup of \( \text{GL}(3, k) \) as follows: (cf. [3, p. 444])

\[
\lambda := D(\zeta^3, \zeta, \zeta^4), \quad \sigma_i := \gamma_i \cdot (\theta_1 - \theta_i)^{-1} \cdot S(\alpha_i, \beta_i, 1), \ (i = 1, 2, 3)
\]

\[
\tau := \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \text{where } D(a, b, c) = \begin{pmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & c
\end{pmatrix} \text{ and } S(a, b, c) = \begin{pmatrix}
a & b & c \\
c & a & b \\
b & c & a
\end{pmatrix}.
\]

And \( G_K := \langle \lambda, \tau, \sigma \rangle, \) where \( \sigma := \sigma_1. \)

1.2.1. LEMMA. The followings hold in \( \text{GL}(3, k) \):

\[
(1) \beta_1^2 = \beta_2 + 2, \ \beta_2^2 = \beta_3 + 2, \ \beta_3^2 = \beta_1 + 2, \ \beta_1 \beta_2 = \beta_1 + \beta_2, \ \beta_2 \beta_3 = \beta_3 + \beta_1, \ \beta_1 \beta_3 = \beta_3 + \beta_2,
\]

\[
(2) \beta_1, \ \beta_2 \) and \( \beta_3 \) are the distinct three roots of the equation \( \beta^3 + \beta^2 - 2\beta + 1 = 0, \)

\[
(3) \theta_1 \) and \( \theta_2 \) are the distinct two roots of the equation \( \gamma_2^3 + \gamma_2^2 + 7 = 0, \)

\[
(4) \beta_1 \gamma_1 = \gamma_2, \ \beta_2 \gamma_2 = \gamma_1, \ \beta_3 \gamma_3 = \gamma_1, \ \alpha_1 \gamma_1 = \gamma_2, \ \alpha_2 \gamma_2 = \gamma_1, \ \alpha_3 \gamma_3 = \gamma_2. \)
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(1) the order of $\lambda$ (resp. $\tau$, $\sigma$) is 7 (resp. 3, 2).
(2) $\sigma_1=\tau\sigma_2$, $\sigma_2=\tau\sigma_3$, $\sigma_3=\tau\sigma_1$, $\sigma_1\tau=\sigma_3$, $\sigma_2\tau=\sigma_1$.
(3) $\tau\lambda^{-1}=\lambda^3$.
(4) "defining relation" $\sigma_1\lambda^{-1}\sigma_1=\lambda^3\sigma_1\lambda^3$ (i=1, 2, 3).

Proof. These are followed from above by direct calculation.

1.3. Lemma. Assume that char($k)\neq 7$. There is an isomorphism of $PSL(2, 7)$ onto $G_K$ sending $\left[ \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right]$ (resp. $\left[ \begin{array}{cc} 2 & 0 \\ 0 & 1 \end{array} \right]$, $\left[ \begin{array}{cc} 0 & 3 \\ 1 & 0 \end{array} \right]$) to $\lambda$ (resp. $\tau$, $\sigma$). Hence the natural homomorphism of $G_K$ into $PGL(3, k)$ is injective.

Proof. We have known that the followings are defining relations for $PSL(2, 7)$:

$\lambda^7=\tau^3=\sigma^2=1$, $\tau\lambda^{-1}=\lambda^3\sigma_1\lambda^3$ and $(\lambda\tau)^3=1$.

If we take (in $PSL(2, 7)$)

$\left[ \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right]$, $\left[ \begin{array}{cc} 2 & 0 \\ 0 & 1 \end{array} \right]$ and $\left[ \begin{array}{cc} 0 & 3 \\ 1 & 0 \end{array} \right]$ in lieu of $x$, $y$ and $z$, then these satisfy the above relations. From (1.2.1) $\lambda$, $\tau^{-1}$ and $\tau^{-1}\sigma\tau$ also satisfy the relations. Therefore there is a surjective homomorphism as in the statement of the Lemma. Since $PSL(2, 7)$ is a simple group (of order 168), this is an isomorphism. Then the latter part is obvious. Q.E.D.

1.4. A couple $(C, G)$ of a curve $C$ and its automorphism group $G$ shall be called an $AM$ curve. An isomorphism of $AM$ curves of $(C', G')$ onto $(C, G)$ is an isomorphism of curves $T: C'\rightarrow C$ such that $G'=T^{-1}GT$. In this case we denote $(C', G')$ by $T^*(C, G)$ or $(T^*(C), T^*(G))$, and also write $(C', G') \equiv (C, G)$.

The purpose of this part is to prove the following theorem:

1.4.1. Theorem. When char($k)\neq 3$ (resp. char($k)=3$), $(K, Aut(K))$ is isomorphic (as $AM$ curves) to $(K, G_K)$ (resp. $(K_3, PSU(3, 3^3))$). Moreover when char($k)\neq 2$, $(K, Aut(K))$ is isomorphic to $(K, PSU(3, 3))$.

In the above, $K$ denotes the plane curve defined by $x_1x_2x_3+x_4x_5x_6+x_7x_8$, in case char($k)\neq 7$. $K_3$ denotes the curve $x_1^3+x_2^3+x_3^3$ and $K_2$ denotes the curve $x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2+x_7^2+x_8^2$. And $PSU(3, 3^3)$ denotes the injective image in $PGL(3, k)$ (in case char($k)=3$) of

$SU(3, 3^3)\{ A \in SL(3, 3^3)| A \cdot A^{(3)}=I \}$. 
where \( A^{(i)} := (a_{ij}) \) if \( A = (a_{ij}) \). It is known as a simple group of order \( 2^3 \cdot 3^3 \cdot 7 = 6048 \). \( PSL(3, 2) \) denotes the injective image in \( PGL(3, k) \) (in case \( \text{char}(k) = 2 \)) of a finite general linear group \( GL(3, 2) \). It is known as a simple group of order \( 2^3 \cdot 3 \cdot 7 = 168 \).

A part of proof. First we note that in case where \( \text{char}(k) = 7 \), \( K \) is a singular plane curve, so we omit this case. Now it follows that \( \lambda^*(K) = K \) and \( \sigma^*(K) = K \) in \( k[x_1, x_2, x_3] \) by direct calculation using (1.2). So \( G_K \) is contained in \( \text{Aut}(K) \) (in \( PGL(3, k) \)). On the other hand, when \( \text{char}(k) \neq 2 \) or 3, it follows from [7] that \( \# \text{Aut}(K) \leq 84(g-1) \) with \( g = 3 \). Thus we get that \( \text{Aut}(K) = G_K \) in these cases.

The excluded cases are settled in §2, (2.2.1) and §3, (3.1.1).

§ 2. The case \( \text{char}(k) = 2 \).

Throughout this section we assume that \( \text{char}(k) = 2 \). First we write down rather general notations for the use in Part II.

2.1. Notations. We define distinguished subgroups of \( GL(3, 2) \):

\[
G_8 := \langle R_+, R_- \rangle, \quad G_{24}(+) := \langle S_+, R_+ R_- \rangle, \quad G_{24}(-) := \langle S_-, R_+ R_- \rangle
\]

where

\[
R_+ := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R_- := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_+ := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad S_- := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.
\]

Here we have known that \( G_8 \) is a 2-Sylow subgroup of \( GL(3, 2) \) and that \( G_{24}(+) \) and \( G_{24}(-) \) are isomorphic to the symmetric group of degree 4, \( \mathfrak{S}_4 \).

Also we define distinguished families of \( AM \) curves as follows:

\[
F_8 := \text{the set of } AM \text{ curves } (C(a, b), G_8) \text{ (with parameters } a \text{ and } b) \]
\[
F_{24}(+) := \text{the set of } AM \text{ curves } (C(a, a), G_{24}(+)) \]
\[
F_{24}(-) := \text{the set of } AM \text{ curves } (C(1, b), G_{24}(-))
\]

where

\[
C(a, b) := x_1^4 + ax_1^3 + bx_2^3 + ax_1^2x_2^2 + x_1^2x_2^2 + x_3x_1^2x_2(x_1 + x_2 + x_3).
\]

When \( G \) is a subgroup of \( GL(3, k) \) (in any characteristic) we denote by \( F(G) \) the set of (nonsingular) quartic \( AM \) curves \( (C, G) \). Forgetting automorphism groups, we also use the above each family as the set of corresponding curves.

Now we prove a lemma which characterize the curve \( K_8 \).
2.1.1. Lemma. We have: $F_{2i}(+) = F(G_{2i}(+))$ and $F_{2i}(-) = F(G_{2i}(-))$. Hence $F(PSL(3, 2)) = \{K_i\}$.

Proof. Comparing coefficients we see easily that $F(\langle R_1, R_\alpha \rangle)$ is the set of curves $C(a, b, c_2, c_3)$, where $C(a, b, c_2, c_3) := x_1^4 + ax_1^3 + bx_1^2 + (x_1^3 x_2 + x_1^2 x_3 + c_2 x_1 x_3^2 + c_3 x_2 x_3^2) + x_1 x_2 x_3 (x_1 + x_2 + x_3) + (1 + c_2) x_1^2 x_3 + (1 + c_3) x_2 x_3^2$ with $a, b, c_2$ and $c_3$ in $k$. Again comparing coefficients as for $S_+$ (resp. $S_-$), we get that $F(\langle S_+, R_\alpha \rangle) = \{C(a, b, c_2, c_3) \mid a, b, c_2, c_3 \in k\}$ and that $F(\langle S_-, R_\alpha \rangle) = \{C(a, b, c_2, c_3) \mid c_2, c_3 \in k\}$. Since $\langle S_+, R_\alpha \rangle$ is equal to $PSL(3, 2)$, it follows from these facts that $F(PSL(3, 2)) = \{K_i\}$. Q.E.D.

2.2. We shall prove (2.2.1) using (2.2.2).

2.2.1. Proposition. $(K, Aut(K)) \cong (K_i, PSL(3, 2))$.

2.2.2. Lemma. Let $C$ be a curve in $F_s$, and let $T$ be an element of $GL(3, k)$. If $T^*(C)$ is again a curve in $F_s$, then $T$ is contained in $PSL(3, 2)$ (in $PGL(3, k)$).

Proof of (2.2.2). Let $C = C(a, b)$ and $T = (a_{ij})$ be as above. We denote $T^{(c)} := (a_{ij})$, $\Delta := (\Delta_{ij})$ where $\Delta_{ij}$ are the cofactors of the matrix $(a_{ij})$, and put $\Delta \cdot T^{(c)} = (b_{ij})$. Then we have (in $k[x_1, x_2, x_3]$):

$$T^*(x_1 x_2 x_3 (x_1 + x_2 + x_3)) = T^*(x_1 x_2 x_3 (x_1 + x_2 + x_3)) = (a_{11}^2 x_1^2 + a_{12} x_1 x_2 + a_{13} x_1 x_3)(a_{21} x_1 + a_{22} x_2 + a_{23} x_3)
+ (a_{11}^2 x_1^2 + a_{12} x_1 x_2 + a_{13} x_1 x_3)(a_{21} x_1 + a_{22} x_2 + a_{23} x_3)
+ (a_{11}^2 x_1^2 + a_{12} x_1 x_2 + a_{13} x_1 x_3)(a_{21} x_1 + a_{22} x_2 + a_{23} x_3)
+ (a_{11}^2 x_1^2 + a_{12} x_1 x_2 + a_{13} x_1 x_3)(a_{21} x_1 + a_{22} x_2 + a_{23} x_3).
$$

Thus we have:

1. (the coefficient of $x_1^2 x_2 x_3$ in $T^*(C(a, b))$) = (the coefficient of $x_1^2 x_2 x_3$ in $T^*(x_1 x_2 x_3 (x_1 + x_2 + x_3))$) = $a_{11}^2 \Delta_{11} + a_{12}^2 \Delta_{12} + a_{13}^2 \Delta_{13} = b_{11}$.

Similarly we have:

2. (the coefficient of $x_2^2 x_3 x_1$ (resp. $x_3^2 x_2 x_1$) in $T^*(C(a, b))$) = $b_{12}$ (resp. $b_{13}$).

3. (the coefficient of $x_1^2 x_3 x_1$ (resp. $x_2^2 x_3 x_2$) in $T^*(C(a, b))$) = $b_{21}$ (resp. $b_{22}$, $b_{23}$, $b_{24}$, $b_{25}$).

Since $T^*(C(a, b))$ is a curve in $F_s$, we have that $\Delta \cdot T^{(c)} = (b_{ij}) = I$ in $PGL(3, k)$. On the other hand we have $\Delta \cdot T = I$ in $PGL(3, k)$. It follows that $T = T^{(c)}$ in $PGL(3, k)$. This means that $T$ is contained in $PSL(3, 2)$. Q.E.D.
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Proof of (2.2.1). It follows from (2.1.1) and (2.2.2) that \( \text{Aut}(K_e) = PSL(3, 2) \). On the other hand it is easy to see that \( S(\beta, \alpha, 1)^*K = K_e \) (as curves) by (1.2). Thus we conclude that \( (K, \text{Aut}(K)) \) is isomorphic (as AM curves) to \( (K_e, PSL(3, 2)) \). Q. E. D.

Also from (2.1.1), (2.2.1) and (2.2.2) we get:

2.2.3. REMARK. \( G_{24}(+) \) and \( G_{24}(-) \) are not \( PGL(3, k) \)-conjugate to each other.

§ 3. The case \( \text{char}(k)=3 \).

In this section we assume that \( \text{char}(k)=3 \).

3.1. We shall prove (3.1.1) using (3.1.2).

3.1.1. PROPOSITION. \( (K, \text{Aut}(K)) \cong (K, PSU(3, 3^3)) \).

3.1.2. LEMMA. Let \( T \) be an element of \( GL(3, k) \) such that \( T^*(K_i) \) is in \( F_{24} \). Then \( T \) is contained in \( PSU(3, 3^3) \) (in \( PGL(3, k) \)), and \( T^*(K_i) = K_i \).

In the above, \( F_{24} \) denotes (in general when \( \text{char}(k) \neq 2 \)), the set of AM curves \( (C(a), G_{24}) \) where \( C(a) \) is a plane curve defined by: \( x_1^4+x_2^4+x_3^4 + a(x_1^4 x_2^4 + x_2^4 x_3^4 + x_3^4 x_1^4) \), \( a \in k \), and \( G_{24} \) is a subgroup \( \langle R, S \rangle \) of \( GL(3, k) \), with

\[
R = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

Proof of (3.1.2). Let \( T = (a_{ij}) \) and \( T^*(K_i) = (b_{ij}) \). First we note that:

\[
T^*(K_i) = (a_{ij} x_1 + a_{12} x_2 + a_{13} x_3)^4 + (a_{21} x_1 + a_{22} x_2 + a_{23} x_3)^4 + (a_{31} x_1 + a_{32} x_2 + a_{33} x_3)^4
\]

\[
= b_{11} x_1^4 + b_{12} x_1^4 + b_{13} x_1^4 + b_{21} x_2^4 + b_{22} x_2^4 + b_{23} x_2^4 + b_{31} x_3^4 + b_{32} x_3^4 + b_{33} x_3^4
\]

Hence it follows by the assumption that \( T^*(K_i) = K_i \) and that \( T^*(T^3) = I \) (in \( PGL(3, k) \)). Then we have also that \( T^*(T^3) = K_i \), so that \( T^*(T^3) = I \) i.e. \( T^*(T^3) = I \). Hence we get that \( T = T^3 \) (in \( PGL(3, k) \)). Put \( c^{-4} T = T^3 \) in \( GL(3, k) \) with some \( c \) in \( k \). Then we have that \( cT \) is in \( GU(3, 3^3) \) and so that \( (\det(cT))^3 \cdot cT = I \) in \( SU(3, 3^3) \). Q. E. D.

Proof of (3.1.1). It also follows from the above proof that \( PSU(3, 3^3) \) is
contained in $\text{Aut}(K_i)$. So we have that $\text{Aut}(K_i) = \text{PSU}(3, 3^e)$. On the other hand it is easy to see that $S(\beta, \alpha, 1)^*(K) = K_i$ by (1.2). Thus we conclude that $(K, \text{Aut}(K))$ is isomorphic to $(K_i, \text{PSU}(3, 3^e))$. Q. E. D.

3.2. Remark. In the similar line (as in (3.1)) we also have that $\text{Aut}(X_{q+i})$ is isomorphic to $\text{PU}(3, q^2)$, if $\text{char}(k) \neq p$ is positive and $q = p^n > 3$ with $n \geq 1$. In the above, $X_{q+i}$ denotes the (nonsingular) plane curve (of genus 2) defined by: $x^{q+1} + x^{q+1} + x^{q+1}$. Hence the order of $\text{Aut}(X_{q+i})$ is $(q^2 - 1)^2$. Moreover if $(3, q+1) = 1$, then $\text{PU}(3, q^2) = \text{PSU}(3, q^2)$ is a simple group. Here we note that this curve is isomorphic to the curve defined by: $y^q + y = x^{q+1}$, (e. g. [8, p. 528]).

Part II. On curves of genus three which have automorphism groups isomorphic to $\mathfrak{S}_4$.

§ 1. Normal forms.

The purpose of this section is to prove the following theorem:

1.1. Theorem. Let $(C, G)$ be an AM curve of genus three. Assume that $G$ is isomorphic to $\mathfrak{S}_4$. Then there is an isomorphism $T$ (of AM curves) such that:

(i) $T^*(C, G)$ is in $F_{st}, hF_{st}$ or $hF'_{st}$, when $\text{char}(k) \neq 2$, or
(ii) $T^*(C, G)$ is in $F_{24}(+) or F_{24}(-)$, when $\text{char}(k) = 2$.

In the above we denote:

$F_{st}$ = the set of AM curves $(C(a), G_{st})$ (with a parameter $a$), (3.1 of Part I),
$hF_{st} = \{\text{the AM curve } (C^*, hG_{st})\}$,
$hF'_{st} = \{\text{the AM curve } (C^*, hH_{st})\}$,

where $C^*$ denotes the hyperelliptic curve (in case where $\text{char}(k) \neq 2$ or 3) defined by: $y^q = x^8 + 14x^4 + 1$, and $G_{st} = \langle A_4, J, T_4 \rangle$, $H_{st} = \langle A_4, T_3 \rangle$. In the above we denote by $J$ (resp. $A_4, T_3$) the automorphism of $C^*$ defined by $(x, y) \rightarrow (x, -y)$ (resp. $(ix, y), (-i(x-1) \cdot (x+1)^{-1}, -4y(x+1)^{-1})$, ($i$ denotes $\sqrt{-1}$).

1.2. The case: $\text{char}(k) \neq 2$ and $C$ is nonhyperelliptic. Then we may assume that $(C, G)$ is a quartic plane AM curve. Since it is obvious that $F(G_{st}) = F_{st}$ (cf. (2.1 of Part I)), it suffices to show:

1.2.1. Lemma. Assume that char$(k) \neq 2$. Let $H$ be a subgroup of $PGL(3, k)$
which is isomorphic to $\mathcal{S}_4$. Then $H$ is $PGL(3, k)$-conjugate to $G_4$.

**Proof.** We denote by $P\cdot PGL$ (resp. $D\cdot PGL$) the set of elements of $PGL(3, k)$ which are represented by $(a_{ij})$, where $a_{11}=a_{22}=a_{13}=a_{23}=0$ (resp. $a_{ij}=0$ if $i \neq j$). Also we denote $\langle S^2, RS^3R^{-1} \rangle$ by $G_4$.

Let $V=\langle A_1, A_2 \rangle$ be the (unique) normal subgroup of $H$ of order $4$. We may assume that $A_1=S^2$ by Jordan's canonical form. Then $A_2$ is contained in $P\cdot PGL$, which is equal to the centralizer of $S'$ in $PGL(3, k)$, $C_{PGL}(S')$. Since $A_2=I$ (in $PGL(3, k)$), there is an element $T$ in $P\cdot PGL$ such that $T^*(A_2)$ is in $D\cdot PGL$. Thus we get that $T^*(V)=\langle T^*(A_1), T^*(A_2) \rangle=G_4$. So we may assume that $V$ is equal to $G_4$.

Next it is easy to show that $C_{PGL}(G_4)=D\cdot PGL$ and that the normalizer of $G_4$ in $PGL(3, k)$, $N_{PGL}(G_4)$ equals $\langle R, S' \rangle \cdot C_{PGL}(G_4)$, where $S'=S^2 \cdot RSR$. Therefore $H$ contains an element of the form $RD$, where $D=D(a, \beta)$ (cf. (1.2 of Part I)). Let $v$ be a solution of the equation $\alpha \beta v^3=1$. Then we have that $D(\beta v^2, v, 1)^*(RD)=R$ (in $PGL(3, k)$). Thus we may assume that $R$ belongs to $H$.

Since $H$ is isomorphic to $\mathcal{S}_4$, we have that $N_H(\langle R \rangle)=\langle R, S'D' \rangle$ for some $D'=D(\gamma, \delta, 1)$. It follows from $\langle S'D'^3=I \rangle$ that $\gamma^3=1$. And it follows from $S'D' \cdot R(S'D')^{-1}=R^{-1}$ that $\gamma^3=\delta$. Then we have that $D^*(S'D')=S'$. Since this $D'$ is in $C_{PGL}(S')$, we get that $D^*(H)=\langle D^*(S') \rangle = G_4$. This completes the proof of (1.2.1), and hence the theorem (1.1) in case where $\text{char}(k)\neq 2$ and $C$ is nonhyperelliptic.

1.3. The case: $C$ is hyperelliptic.

First we show:

1.3.1. **Lemma.** Assume that $\text{char}(k)\neq 2$.

(1) Let $H$ be an abelian subgroup of $PGL(2, k)$ of type $(2, 2)$. Then $H$ is $PGL(2, k)$-conjugate to $H_4$, where $H_4$ denotes $\langle A^3, B \rangle$.

(2) $N_{PGL(2, k)}(H_4)$ is equal to $\langle \Delta, T_3 \rangle$ and is isomorphic to $\mathcal{S}_4$.

In the above we denote $\begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}$ (resp. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$) by $\Delta$ (resp. $B$, $T_3$).

Also we shall denote $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ by $D(\alpha, \beta)$.

**Proof.** (1) Let $H=\langle A_1, A_2 \rangle$. We may assume that $A_1=A^2$ by the Jordan's canonical form. Then $A_2$ is of the form $D(\beta, 1)B$. Put $T=D(\beta, 1)B$ with $\beta^3=\alpha$. Then we have that $T^{-1}HT=\langle T^{-1}A_1T, T^{-1}A_2T \rangle=\langle A^3, B \rangle=H_4$.

(2) It is easy to show that $C_{PGL(2, k)}(H_4)=H_4$. Since we have that $B' A^3 B'^{-1}$
Next we shall show the theorem (1.1) in case where $C$ is hyperelliptic. In this case we have a natural exact sequence $\langle J \rangle \to \text{Aut}(C) \to \text{PGL}(2, k)$. Since $G$ is isomorphic to $\mathfrak{S}_4$, we have that the image $G$ of $G$ in $\text{PGL}(2, k)$ is also isomorphic to $\mathfrak{S}_4$. Thus $\text{char}(k)$ must be different from 2, because there is no elements of order 4 in $\text{PGL}(2, k)$ in case $\text{char}(k)=2$. Then $C$ is determined by $f(x, z)$, where $f(x, z)$ is a homogeneous form of degree 8 which is a semi-invariant with respect to $G$. Then we may assume by (1.3.1) that $G=\text{PGL}(2, k)(H_0)$. Since $f(x, z)$ is a semi-invariant for $A$, we have that $f(x, z) = \alpha x^8 + \beta x^4 z^4 + \gamma z^8$ for some $\alpha, \beta$ and $\gamma$. Moreover since $f(x, z)$ is a semi-invariant for $B$, we have that Case 1: $\alpha + \gamma = 0$, $\beta = 0$, or Case 1: $\alpha = \gamma$. In Case 1, $f(x, z)$ cannot be a semi-invariant for $T_3$. So Case 1 does not happen. In Case 2, since $f(x, z)$ is a semi-invariant for $T_3$, we have that $14\alpha = \beta$ i.e. $f(x, z) = \alpha(x^8 + 14x^4 z^4 + z^8)$. Thus we see that $C$ is defined by $y^2 = x^8 + 14x^4 + 1$. Since $G=\langle A, T_3 \rangle$, and since $A_i$ and $T_3$ are automorphisms of $C$, we have that $G$ is contained in $\langle A_4, T_3, J \rangle$. On the other hand $T_3$ is in $G$, because there are no element of order 6 in $\mathfrak{S}_4$. Thus we obtain that $G=\langle A_4, T_3 \rangle$ or $\langle A_4, T_3 \rangle$. This completes the proof of the fact that $(C, G)$ isomorphic to $(C^*, hG_{24})$ or $(C^*, hH_{23})$, in case where $C$ is hyperelliptic.

1.4. The case: $\text{char}(k)=2$. Then we may assume that $C$ is nonhyperelliptic. And it follows from the Jordan's canonical form that we may assume that $R_+R_-$ is in $G$. Then $C$ equals to some $C(a, b, c_1, c_2)$ in $F(\langle R_+R_- \rangle)$ (cf. (2.1.1 of Part I). If $T = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} (\alpha, \beta \in k)$, then $T$ is in $\text{PGL}(R_+R_-)$ and $T^*(C) = C(a', b', c'_1, c'_2)$ in $F(\langle R_+R_- \rangle)$, where $c'_1 = c_1 + c_2(\alpha^2 + \alpha) + \alpha^4 + \alpha^3 + \beta^2 + \beta$, and $c'_2 = c_1 + \alpha^2 + \alpha$. For suitable choice of $\alpha$ and $\beta$, we get that $T^*(C)$ is a curve in $F_8$. Hence we may assume that $C$ is in $F_8$ with $R_+R_-$ in $G$. It follows from (2.2.2 of Part I) that $\text{Aut}(C)$ is contained in $\text{PSL}(3, 2)$. It is easy to see that $C_{\text{PSL}(3, 2)}(R_+R_-) = G_{24}$. So we have that $G_{24}$ is contained in $G$. Therefore the normal subgroup of $G$ of order 4 is either $\langle R_-, (R_+R_-) \rangle$ or $\langle R_+, (R_+R_-) \rangle$. Since $N_{\text{PSL}(3, 2)}(R_+, (R_+R_-)) = G_{24}(+)$, and $N_{\text{PSL}(3, 2)}(R_-, (R_+R_-)) = G_{24}(-)$, we have that $G = G_{24}(+) \text{ or } G_{24}(-)$. On the other hand, since $F(G_{24}(+)) = F_{24}(+) \text{ and } F(G_{24}(-)) = F_{24}(-)$ (2.1.1 of Part I), we get that $(C, G)$ is a member of $F_{24}(+) \text{ or } F_{24}(-)$. This completes the proof.
On Certain Curves of Genus Three with Many Automorphisms

§ 2. Isomorphism classes.

The purpose of this section is to prove the following theorem:

2.1. THEOREM. Assume that char(k)≠2. Let C(a) and C(a') be two curves in $F_{2^9}$, where $a≠3θ_1$ or $3θ_2$. Then C(a) is isomorphic to C(a') if and only if $a=a'$.

PROOF. To prove the “only if” part, we assume that $C(a)\cong C(a')$ and $a≠a'$. First it is easy to see that $C_{PGL}(G_{2^9})=\{I\}$. Since any automorphism of $G_{2^9}$ is an inner automorphism, we also have that $N_{PGL}(G_{2^9})=G_{2^9}$. Therefore by the assumption it follows that $\text{Aut}(C(a))$ contains strictly $G_{2^9}$. Then we apply a result on the classification of nonhyperelliptic AM curves of genus three [5], and it follows that $C(a)$ is isomorphic to $K$ or $K_4$.

(1) The case: $C(a)\cong K$. When char(k)=3, it follows from (3.1.2 of Part I) that $a=0$, where this is the excluded value. When char(k)≠3, we note that $\#\text{Aut}(K_4)=96$, and that $C_{\text{Aut}(K_4)}(S^*)$ is a 2-Sylow subgroup of $\text{Aut}(K_4)$ with $\langle D(i, i, -1) \rangle$ as its center. So any 2-Sylow subgroup of $\text{Aut}(K_4)$ has a cyclic subgroup of order 4 as its center. Since $C_{PGL}(S^*, R S^* R^{-1})$ is contained in $D\cdot PGL$, $\text{Aut}(C(a))$ contains an element of $D\cdot PGL$ of order 4. Then we have at any rate that $a=0$. Also we have that $a'=0$. These lead to a contradiction to the assumption on $a$ and $a'$.

(2) The case: $C(a)\cong K$. We may assume that char(k)≠3, by (1.4.1 of Part I). If we denote by $S_6$ (resp. $S_5$) $S(\zeta^3\alpha_0, \zeta^4\beta_1, 1)$ (resp. $S(\zeta\alpha_0, \zeta^4\beta_1, 1)$) (cf. (1.2 of Part I)) then by direct calculations we see that $S_6^*(K)=C(3θ_1)$ (in $F_{2^9}$) and $S_5^*(K)=C(3θ_2)$ (in $F_{2^9}$). Let $T$ be an isomorphism of $K$ onto $C(a)$. Then $T^*(G_{2^9})$ is $G_K$-conjugate to either $S_6^{-1*}(G_{2^9})$ or $S_5^{-1*}(G_{2^9})$, since $G_K=\text{Aut}(K)$ (1.4.1 of Part I) and $G_K$ is isomorphic to $PSL(2, 7)$. Hence replacing $T$ if necessary, we may assume that $TS_0$ or $T\overline{S}_0$ is contained in $N_{PGL}(G_{2^9})=G_{2^9}$, which is contained in $\text{Aut}(C(a))$. Thus we have at any rate that $a=3θ_1$ or $3θ_2$, which are the excluded values. This completes the proof of (2.1).

2.2. REMARK. We have an analogous result for the case char(k)=2, by (2.2.2 of Part I):

Assume that char(k)=2. Let $C(a, b)$ and $C(a', b')$ be two curves in $F_8$. Then $C(a, b)$ is isomorphic to $C(a', b')$ if and only if $a=a'$ and $b=b'$.
§ 3. Subgroups of Mod(3) which are isomorphic to \( \mathfrak{S}_3 \), and their representations.

In this section we work in the category of (compact) Riemann surfaces.

3.1. Notations and theorem.

3.1.1. Let \( W_0 \) be a fixed Riemann surface of genus 3. For each Riemann surface \( W \) of genus 3, we consider the pairs \( (W, \alpha) \), where \( \alpha \) are homotopy classes of orientation-preserving (or shortly o.p.) homeomorphisms of \( W_0 \) onto \( W \). Two such pairs \( (W, \alpha) \) and \( (W', \alpha') \) are said to be conformally equivalent if there is a conformal mapping of \( W \) onto \( W' \) which is an element of \( \alpha' \alpha^{-1} \). We denote by \( \langle W, \alpha \rangle \) the equivalence class of \( (W, \alpha) \). And the set of these classes is called the Teichmüller space \( T(3) \) of genus 3. \( T(3) \) becomes a metric space [9], and moreover a (simply connected) complex manifold of dimension \( 3g-3 \) with \( g=3 \) [2].

Let \( G(W_0) \) be the group of o.p. homeomorphisms of \( W_0 \). Each \( c \) in \( G(W_0) \) defines a well-defined permutation \( c^* \) of \( T(3) \) sending \( \langle W, \alpha \rangle \) to \( \langle W, \alpha \cdot c^{-1} \rangle \). In fact this \( c^* \) is a biholomorphic mapping. And so we have a group homomorphism of \( G(W_0) \) into \( \text{Aut}(T(3)) \), the group of biholomorphic mappings of \( T(3) \). We denote its image by \( \text{Mod}(3) \). For \( \langle W, \alpha \rangle \) in \( T(3) \), we have a natural group homomorphism (denoted by \( M_\alpha \) of \( \text{Aut}(W) \) into \( \text{Mod}(3) \) defined by \( \sigma \mapsto \langle \alpha^{-1} \sigma \alpha \rangle^* \). It is known that \( M_\alpha \) defines an isomorphism of \( \text{Aut}(W) \) and the isotropy subgroup of \( \text{Mod}(3) \) at \( \langle W, \alpha \rangle \) (e.g. [6, p. 16, Corollary]). For an AM Riemann surface \( (W, G) \) (defined as in (1.4 of Part I)), taking a homotopy class \( \alpha \) of \( W_0 \) onto \( W \), we define a homomorphism (denoted by \( M(W, G) \)) of \( \text{Aut}(W) \) into \( \text{Mod}(3) \) as above. Then we note that its image \( M(W, G) \) is determined up to \( \text{Mod}(3) \)-conjugacy.

3.1.2. For an AM Riemann surface \( (W, G) \) of genus 3, taking a basis \( \varphi_1, \varphi_2, \varphi_3 \) of the space of holomorphic differentials, we define a representation, \( \rho(W, \sigma) \), of \( \text{Aut}(W) \) on the space which is defined by: \( \rho(W, \sigma)(a_{ij}) = \rho(W, \sigma) \) in \( GL(3, \mathbb{C}) \), where \( \sigma^*(\varphi_i) = \sum^3_{j=1} a_{ij} \varphi_j \) (\( \sigma \in \text{Aut}(W) \)). Then we note that the image \( \rho(W, G) \) of \( G \) is determined up to \( GL(3, \mathbb{C}) \)-conjugacy.

The purpose of this section is to prove the following theorem:

3.1.3. Theorem. Let \( (W, G) \) be an AM Riemann surface of genus three. Assume that \( G \) is isomorphic to \( \mathfrak{S}_3 \). Then we have:
(1) \( M(W, G) \) is \( \text{Mod}(3) \)-conjugate to either \( MG_{24} \) or \( MH_{24} \). \( \rho(W, G) \) is \( GL(3, C) \)-conjugate to either \( G_{24} \) or \( H_{24} \).

(2) \( M(W, G) \sim MG_{24} \) (resp. \( MH_{24} \)) if and only if \( \rho(W, G) \sim G_{24} \) (resp. \( H_{24} \)).

In the above we denote by \( MG_{24} \) (resp. \( MH_{24} \)) the subgroup \( M(C, hG_{24}) \) (resp. \( M(C, hH_{24}) \)) of \( \text{Mod}(3) \). And we denote by \( G_{24} \) (resp. \( H_{24} \)) the subgroup \( \langle R, S \rangle \) (resp. \( \langle R, -S \rangle \)) of \( GL(3, C) \) (cf. (3.1 of Part I)).

3.2. Our proof is based on the following several lemmas:

3.2.1. Lemma. Let \((C(a), G_{24})\) is an \( AM \) Riemann surface in \( F_{24} \). Then \( \rho(C(a), G_{24}) \) is \( GL(3, C) \)-conjugate to \( G_{24} \).

Proof. Let \( F(x_1, x_2, x_3) \) be the homogeneous polynomial defining \( C(a) \). And we denote by \( x \) and \( y \) the functions on \( C(a) \), \( x=x/x_3 \) and \( y=x/x_3 \). Since \( C(a) \) is a nonsingular plane curve which meets the line defined by \( x_3=0 \) transversally, the differentials \( xF_2^1dx \), \( yF_2^1dx \) and \( F_2^1dx \) form a basis of the space of holomorphic differentials, where \( F_2=F_2(x, y, z)=\left(\frac{\partial}{\partial x}\right)(x, y, 1) \). If \( \rho(C(a), G_{24}) \) is the representation with respect to this basis, then we have that \( \rho(C(a), G_{24})=S \), since \( S^*(xF_2^1dx)=-xF_2^1dx \), \( S^*(xF_2^1dx)=xF_2^1dx \) and \( S^*(yF_2^1dx)=xF_2^1dx \). On the other hand we have that \( R^*(xF_2^1dx)=(4x^3+2a((yx^{-1})x^{-1}+x^{-1}))^{-1}d(yx^{-1})=(4+2a(x^3+y^3))^{-1}x(xdy+xdx)=xF_2^1dx \), since \( F_1(x, y)dx+F_2(x, y)dy=0 \).

Hence we also have that \( R^*(yF_2^1dx)=x^{-1}R^*(xF_2^1dx)=xF_2^1dx \). Thus we get that \( \rho(C(a), R)=R \). Therefore we conclude that \( \rho(C(a), G_{24})=G_{24} \). Q. E. D.

3.2.2. Lemma. Let \( C^* \) be the hyperelliptic surface in (1.1). Then \( \rho(C^*, hG_{24}) \) (resp. \( \rho(C^*, hH_{24}) \)) is \( GL(3, C) \)-conjugate to \( G_{24} \) (resp. \( H_{24} \)).

Proof. Let \( \rho(C^*, ) \) be the representation of \( \text{Aut}(C^*) \) with respect to the basis: \( i(x^3-1)y^{-1}dx, (x^3+1)y^{-1}dx \) and \( 2ixy^{-1}dx \). First it is obvious that \( \rho(C^*, J)=J \). Next it follows easily that:

\[
\begin{align*}
(A_4J)^*(i(x^3-1)y^{-1}dx) &= i(-x^3-1)(-y)^{-1}dxx=(x^2+1)y^{-1}dx, \\
(A_4J)^*((x^3+1)y^{-1}dx) &= i(x^2-1)y^{-1}dx, \quad \text{and} \\
(A_4J)^*(2ixy^{-1}dx) &= 2ixy^{-1}dx.
\end{align*}
\]

Hence we obtain that \( \rho(C^*, A_4J)=S \) and \( \rho(C^*, A_4)=S \). We also have that:

\[
\begin{align*}
T_2^*(y^{-1}dx) &= i(x+1)^2(2y)^{-1}dx, \\
T_2^*(x^{-1}y^{-1}dx) &= (x-1)(2y)^{-1}dx.
\end{align*}
\]

Hence we obtain that:

\[ T_s^g(x^2 - 1)y^{-1}dx = (x^2 + 1)y^{-1}dx, \quad T_s((x^2 + 1)y^{-1}dx) = 2ixy^{-1}dx \quad \text{and} \]
\[ T_s^g(2ixy^{-1}dx) = i(x^2 - 1)y^{-1}dx. \]

Therefore it follows that \( p(C*, T_s) = R \). Combining these results, we have that \( p(C, hG_{24}) = G_{24} \) and \( p(C, hH_{24}) = H_{24} \). Q.E.D.

3.2.3. REMARK. \( G_{24} \) and \( H_{24} \) are not \( GL(3, \mathbb{C}) \)-conjugate are each other, since \( <S> \) and \( <S> \) are not conjugate.

3.3. Now we prove the following proposition:

3.3.1. PROPOSITION. Let \( C(a) \) and \( C(a') \) be two Riemann surfaces in \( F_{24} \). Then there exists an orientation-preserving homeomorphism \( f \) of \( C(a) \) onto \( C(a') \) such that \( f \cdot A = A \cdot f \) for each automorphism \( A \) in \( G_{24} \).

PROOF. We shall prove this proposition in several steps.

Step 1. We denote by \( C' \) a Zariski-open subset \( \{ a \mid C(a) \in F_{24} \} \) of \( C \). We fix an element \( a_0 \) of \( C' \). Let \( L \) be a topological embedding of \( R \) to \( C' \) such that \( L(0) = a_0 \). For \( \varepsilon > 0 \), we denote by \( L_\varepsilon \) the restriction of \( L \) to the open interval \( (-\varepsilon, \varepsilon) \). And we also denote by \( L_\varepsilon \) its image in \( C' \).

Then it suffices to show:

CLAIM. There exists an \( \varepsilon > 0 \) such that for any \( a \) in \( L_\varepsilon \), there is an o.p. homeomorphism \( f_a \) of \( C(a_0) \) to \( C(a) \) with the property that \( f_a \cdot A = A \cdot f_a \) for each \( A \) in \( G_{24} \).

If we prove this Claim, then we obtain a desired mapping after composing of finitely many such mappings as in the Claim.

In the following we shall prove this Claim.

Step 2. Let \( a_0 \) and \( L \) be as above. If \( n_1(a) \) and \( n_2(a) \) are the two solutions (in \( C \)) of the equation: \( n^2 + 2an + (a + 2) = 0 \), then we denote \( N_k(a) = 1 + 2(n_k(a) + 1)^2 \cdot n_k(a)^{-1} \) (\( i = 1, 2 \)). If \( \varepsilon \) is sufficiently small, then we may assume that the mapping \( N_1 \) of \( L_\varepsilon \) to \( C \) is continuous, since \( N_1(a) \) and \( N_2(a) \) are distinct (and different from 0) for each \( a \) in \( C' \).

Next we choose a quasi-conformal mapping \( \phi \) of \( P^1 \) onto \( P^1 \) such that \( \phi(0) = 0, \phi(\infty) = \infty, \phi(N_1(a_0)) = 1 \) and \( \phi(N_2(a_0)) = i \). We denote the continuous
Let $C$ be the complex subspace of $P^2 \times L_\varepsilon$ defined by the locus of the equation:

$$x_1^4 + x_2^4 + x_3^4 + a(x_1^4 x_2^4 + x_2^4 x_3^4 + x_3^4 x_1^4) = 0.$$ 

Then we have the following Claim:

**Claim.** (1) If we define the continuous mapping $\pi$ of $C$ onto $P^1 \times L_\varepsilon$ by sending $(x_1, x_2, x_3, a)$ to \((\phi(1+(x_1^4+x_2^4)(x_3^4+x_2^4)(x_2^4+x_1^4)(x_1^4 x_2^4 x_3^4)^{1/2}), a)\), then it is the quotient mapping of $C$ onto $C/G_{2i}$. 

(2) The o.p. continuous mapping $\pi_a : \pi^{-1}(a) \to P^1$ (the fiber of $\pi$ over $a$) is the natural mapping of $C(a)$ onto $C(a)/G_{2i}$. 

(3) The branch points of $\pi_a$ are $0, \infty, N_1(a)$ and $N_2(a)$. 

**Proof.** We have (1) and (2) from the fact that the holomorphic mapping of $C(a)$ to $P^1$ defined by $(x_1, x_2, x_3) \to 1+(x_1^4+x_2^4)(x_3^4+x_2^4)(x_2^4+x_1^4)(x_1^4 x_2^4 x_3^4)^{-1/2}$ is the quotient mapping $C(a) \to C(a)/G_{2i}$. 

Since $G_{2i}$ is isomorphic to $\mathbb{Z}_2$, it is easy to see that the branch points are the images of the following 4 points of $C(a)$; $(1, 0, \omega^n)$: a fixed point of $R$ (in $C(a)$), where $\omega$ is a solution of the equation $\omega^4+\omega+1=0, (\ast, 1, 0)$: a fixed point of $S^4$, $(1, 1, \sqrt{N_1(a)})$: a fixed point of $S^4RSR$ ($i=1, 2$). These images are in fact $0, \infty, N_1(a)$ and $N_2(a)$. Q.E.D. 

**Step 3.** We define a mapping $g$ of $P^1 \times L_\varepsilon$ into $P^1 \times L_\varepsilon$ by $(P, a) \to (\text{Re}(P)N_1(a)+\text{Im}(P)N_2(a), a)$ (if $P \neq \infty$), and $(\infty, a) \to (\infty, a)$. If $\varepsilon$ is sufficiently small, then it follows easily that:

1. $g$ is a homeomorphism such that $g(0, a) = (0, a), g(\infty, a) = (\infty, a)$ and $g(N_1(a), a) = (N_1(a), a)$ ($i=1, 2$).
2. The fiber of $g$ over $a$ (denote it by $g_a$) is an o.p. homeomorphism.

**Step 4.** $B(a)$ denotes the set $\{(Q, a) \in P^1 \times L_\varepsilon \mid Q$ is a branch point of $\pi_a : C(a) \to P^1\}$, and $B$ denotes the union $\bigcup_{a \in L_\varepsilon} B(a)$. Since the action of $G_{2i}$ on $C \setminus \pi^{-1}B$ is fixed-point free, the restriction of $\pi$ to $C \setminus \pi^{-1}B$ into $P^1 \times L_\varepsilon \setminus B$ is surjective and locally homeomorphic.

For a point $P$ of $C(a_0) \setminus \pi_a^{-1}B(a_0)$ and $a$ in $L_\varepsilon$, let $L(P, a)$ be the lifting with initial point $P$ (considered as a point of $C$) of the $R$-curve from $[0, t_a]$ to $P^1 \times L_\varepsilon$ (where $L(t_a)=a$) defined by $t \to g(\pi_a(P), L(t))$. Then we have a homeomorphism (denoted by $f$) of $(C(a_0) \setminus \pi_a^{-1}B(a_0)) \times L_\varepsilon$ onto $C \setminus \pi^{-1}B$, sending $(P, a)$ to the end point of $L(P, a)$. This mapping has the property that $f(AP, a) = Af(P, a)$ for
any automorphism \( A \) in \( G_{24} \), since \( Af(P, a) \) is the end point of the \( R \)-curve \( AL(P, a) \) which is equal to \( L(AP, a) \).

It is obvious that \( f \) can be uniquely extended to a homeomorphism (again denoted by \( f \)) of \( C(a_0) \times L \), onto \( C \), and that \( f \) has the property that \( f(AP, a) = Af(P, a) \), because \( C \rightarrow L \) is a proper mapping.

Step 5. The fiber (denoted by \( f_a \)) of \( f \) over \( a \in L \) is the desired homeomorphism of \( C(a_0) \) onto \( C(a) \) with the property that \( f_a A = Af_a \) for each \( A \) in \( G_{24} \). The fact that \( f_a \) is orientation-preserving is followed from (2) of Claim in Step 2 and from (2) of Step 3.

Q. E. D. of (3.3.1).

3.3.2. Corollary. Let \( (C(a), G_{24}) \) and \( (C(a'), G_{24}) \) be two AM Riemann surfaces in \( F_{24} \). Then \( M(C(a), G_{24}) \) and \( M(C(a'), G_{24}) \) are Mod(3)-conjugate to each other.

Proof. Let \( f \) be as in (3.3.1). If we take a homotopy class \( \alpha \) of \( W_0 \) onto \( C(a) \), then we have that \( M_{f, \alpha}(A) = ((f \cdot \alpha)^{-1} A(f \cdot \alpha))^* = (\alpha^{-1} \cdot f^{-1} Af \cdot \alpha)^* = M_{\alpha}(f^{-1} Af) = M_{\alpha}(A) \). Thus we have that \( M(C(a), G_{24}) \sim M(C(a'), G_{24}) \). Q. E. D.

3.4. Proof of the theorem: Let \( (W, G) \) be as in (3.1.3).

First we note by (3.2.1), (3.2.2) and (1.1) that \( \rho(W, G) \) is \( GL(3, C) \)-conjugate to either \( G_{24} \) or \( H_{24} \), and that \( \rho(W, G) \sim G_{24} \) (resp. \( H_{24} \)) if and only if \( (W, G) \) is an element of \( F_{24} \) or \( hF_{24} \) (resp. of \( hF_{24} \)), up to isomorphisms of AM Riemann surfaces.

For the rest of this section we shall prove the similar results as above concerning the subgroups of \( \text{Mod}(3) \). In general, when \( H \) is a finite subgroup of \( \text{Mod}(3) \), we denote by \( T(3)^H \) the fixed set point \( \langle W', \alpha \rangle \mid c^* \langle W', \alpha \rangle = \langle W', \alpha \rangle \) for all \( c^* \) in \( H \). If \( \langle W', \alpha \rangle \) is an element of \( T(3)^H \), we consider the AM Riemann surface \( (W', G') \) where \( G' = M_{\alpha}(H) \), and we denote by \( d(H) \) the number: \( 3 \cdot (\text{genus of } W'/G') - 3 + \#(\text{branch points for } W' \rightarrow W'/G') \). Then it follows from [4] that \( T(3)^H \) is a simply connected submanifold of \( T(3) \) of dimension \( d(H) \). Since the genus of \( C^*/hG_{24} \) (resp. \( C^*/hH_{24} \)) is 0 (resp. 0) and \( \# \) (branch points for \( C^* \rightarrow C^*/hG_{24} \) (resp. \( C^*/hH_{24} \))) is 4 (resp. 3), we have by definition that \( d(MG_{24}) = 1 \) (resp. \( d(MH_{24}) = 0 \)). Thus in particular it follows that \( MG_{24} \) is not \( \text{Mod}(3) \)-conjugate to \( MH_{24} \). Since \( \text{Mod}(3) \) acts on \( T(3) \) properly discontinuously, it follows from the classification (1.1) and (2.1) that \( T(3)^{MG_{24}} \) contains an element \( \langle W, \alpha \rangle \) such that \( \langle W, M_{\alpha}^{-1}(MG_{24}) \rangle \) is an AM Riemann surface in \( F_{24} \) up to isomorphisms. Hence by (3.3.2) we have that \( M(C(a), G_{24}) \)
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is conjugate to $MG_{24}$ for any $AM$ Riemann surface $(C(a), G_{24})$ of $F_{24}$. Thus we obtain that $M(W, G)$ is $\text{Mod}(3)$-conjugate to either $MG_{24}$ or $MH_{24}$, and that $M(W, G)$ is $MG_{24}$ (resp. $MH_{24}$) if and only if $(W, G)$ is an element of $F_{24}$ or $hF_{24}$ (resp. of $hF'_{24}$), up to isomorphisms of $AM$ Riemann surfaces.

The above two results completes the proof of (3.1.1).

References


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