THE APPROXIMATE SECTION EXTENSION PROPERTY
AND HEREDITARY SHAPE EQUIVALENCES(1)

By

Tatsuhiko YAGASAKI

Abstract. In this paper, the concept of the approximate section extension property (ASEP) is introduced. It is shown that hereditary shape equivalences are exactly the maps with the hereditary ASEP and every UV^n-1-map with n-dimensional range has the ASEP.

0. Introduction.

In this paper we will introduce the concept of the approximate section extension property (ASEP), which is a shape version of the section extension property, [Do]. The ASEP is defined in Section 1 to maps between metric spaces. However, using resolutions of maps, this can be extended to a general case (see Section 3). Main results of the paper are contained in Section 2. We prove the followings:

1) Hereditary shape equivalences (HSE’s) are exactly the maps with the hereditary ASEP. In particular, any pull back of a HSE is also HSE.

2) Every UV^n-1-map with an n-dimensional range and every UV^n-map between ANR’s has the ASEP. If the range is a manifold, then an appropriate converse holds.

One can regard these results as a shape version of some results in the fiber homotopy theory, [Do].

Here, we list some notations to be used throughout the paper. In Sections 1 and 2, spaces are assumed to be metrizable. If A is a subset of a space X, \( \bar{A} \) is the closure of \( A \) and \( \text{inc}(A,X) \) denotes the inclusion map \( A \subseteq X \). \( \text{Cov}X \) is the set of all normal coverings of \( X \). For \( \mathcal{U} \in \text{Cov}Y \), \( \text{st} \mathcal{U} \) is the star of \( \mathcal{U} \). We say two maps \( f, g : X \to Y \) are \( \mathcal{U} \)-near and write \( (f,g) \leq \mathcal{U} \) when each \( x \in X \) admits a \( V \in \mathcal{U} \) such that \( f(x), g(x) \in V \). An ANR is one for metric spaces. A polyhedron is the body \( |K| \) of a simplicial complex \( K \) with the CW-topology.

We refer to [DS] for the definitions of basic terms in Shape theory, and to [A1] for relation theoretic terms. One should refer to [Do] in Sections 1, 2 and to [Ma] in Section 3.

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1. Approximate section extension property.

Let \( f : X \to Y \) be a map between metric spaces. We can find closed embeddings \( i : X \to M \), \( j : Y \to N \) into ANR's and a map \( F : M \to N \) with \( Fi = jf \) ([H]). Consider a pair \((U, V)\) of open neighborhoods (nbd's) \( U \) of \( i(X) \) in \( M \) and \( V \) of \( j(Y) \) in \( N \) with \( F(U) \subset V \). We call such a pair an admissible pair for \( f \) w.r.t. \( F \).

**Proposition 1.1:** Under the above notations, the following conditions (1)–(3) are equivalent. Furthermore if the map \( f \) satisfies the condition (1) (eq., (2), (3)), for some \( i, j \) and \( F \) as above, then \( f \) satisfies the condition (1) for any such \( i, j \) and \( F \).

(1) For each admissible pair \((U, V)\) \( U \in \text{Cov} \) \( U \) and \( V \in \text{Cov} \) \( V \), there exist an admissible pair \((U_1, V_1)\) such that for each closed set \( A \) of \( V_1 \) and each map \( s : A \to U_1 \) with \((F, \text{inc}(A, V_1)) \leq V \), there exist an open nbd \( W \) of \( j(Y) \) in \( V_1 \) and a map \( S : W \to U \) with \((F, \text{inc}(W, V)) \leq V \) and \((S|_{A,w}, (S|_{A,w})) \leq U \).

(2) For each admissible pair \((U, V)\) \( U \in \text{Cov} \) \( U \) and \( V \in \text{Cov} \) \( V \), there exist \((U_1, V_1)\) such that for each closed set \( A \) of \( Y \) and each map \( s : A \to U_1 \) with \((F, \text{inc}(A, V_1)) \leq V \), there exists a map \( S : Y \to U \) with \((F, \text{inc}(Y, V)) \leq V \) and \((S|_{A}, (S|_{A})) \leq U \).

(3) For each open nbd \( U \) of \( i(X) \) in \( M \) and \( V \in \text{Cov} \) \( N \), there exist an open nbd \( U_1 \) of \( i(X) \) in \( U \) and \( V_1 \in \text{Cov} \) \( V \) such that for each closed set \( A \) of \( Y \) and each map \( s : A \to U_1 \) with \((F, \text{inc}(A, V_1)) \leq V \), there exists a map \( S : Y \to U \) with \((F, \text{inc}(Y, N)) \leq V \) and \((S|_{A}, (S|_{A})) = s \).

**Proof:** Note that \( U \) and \( U_1 \) are ANR's. Therefore (2) \( \to \) (1) follows from the nbd extension property of ANR's and (2) \( \to \) (3) follows from the homotopy extension theorem ([H]). (1) \( \to \) (2) and (3) \( \to \) (2) are obvious. For the latter statement, see Proposition 3.2, which implies the same conclusion under a more general setting.

**Definition 1.2:** We say the map \( f \) has the approximate section extension property (ASEP) provided \( f \) satisfies the conditions in Proposition 1.1.

If \( f \) is a proper map (i.e., the inverses of compact sets are compact), then we can reduce the above conditions to a simpler one. For later use, we shall work in the setting of relation.

Suppose \( M \) and \( N \) are ANR's, \( Y \) is a closed subset of \( N \), \( R : Y \to M \) is an (upper semi-) continuous relation with compact point images (i.e., for each \( y \in Y \), \( R(y) \) is compact) ([A1]) and \( p : N \times M \to N \) is the projection. Note that \( p(R) \subset Y \).
Proposition 1.3: Under the above notations, the projection $p: R \to Y$ has the ASEP iff

(1) each nbd $U$ of $R$ in $Y \times M$ contains a nbd $V$ of $R$ such that

(2) for each closed set $A$ of $Y$ and each map $s: A \to V$ with $ps = \text{inc}(A, Y)$, there exists a map $S: Y \to U$ with $pS = 1_Y$ and $S |_A = s$.

Let $f: X \to Y$ be a proper map and $M$ an ANR containing $X$ as a closed subset. Consider the continuous relation $f^{-1} = \bigcup \{ y \times f^{-1}(y) : y \in Y \} : Y \to M$. Since the projection $p: f^{-1} \to Y$ corresponds to $f$ by the identification $f^{-1} \to X: (f(x), x) \mapsto x$, we get the following.

Corollary 1.4: Under the above notation, $f$ has the ASEP iff (1) holds with $R$ replaced by $f^{-1}$.

Remark 1.5: The map $f$ is said to be approximately invertible ([A2]) if each nbd $U$ of $f^{-1}$ in $Y \times M$ admits a map $S: Y \to U$ such that $pS = 1_Y$.

Proof of 1.3: From Proposition 1.1, $p$ has the ASEP iff

(1): for each nbd $U'$ of $R$ in $N \times M$ and $\cup \in \text{Cov} N$, there exist a nbd $V'$ of $R$ in $U'$ and $\cup \in \text{Cov} N$ such that

(2): if $A$ is a closed set of $Y$ and $s': A \to V'$ is a map with $(ps', \text{inc}(A, N)) \leq \cup$, then there exists $S': Y \to U'$ with $(pS', \text{inc}(Y, N)) \leq \cup$ and $S' |_A = s'$.

(2) $\to$ (1): Given $U'$ and $\cup$ as in (1). By [A1], Lemma A-8, there exist an open nbd $U_i$ of $R$ in $U''$ and $\cup_i = \{ U_i \}_{i \in \mathbb{A}} \in \text{Cov} N$ such that $\bigcup \bigcup_i \subset U''$ for each $\mathbb{A}$ and $\cup_i < \cup$. Use (2) to $U = U_i \cap (Y \times M)$ and we have an open nbd $V$ of $R$ in $U$ which satisfies (2). By [A1], Lemma A-8, there exist a nbd $V'$ of $R$ in $U_i$ and a $\cup = \{ V_i \}_{i \in \mathbb{A}} \in \text{Cov} N$ such that $\bigcup \bigcup_i \subset V$ for each $\mathbb{A}$ and $\cup_i - \text{near maps to } N$ are $\cup_i$-homotopic. Then $V'$ and $\cup$ satisfy (2). In fact, take $s'$ as in (2). Since $(ps', \text{inc}(A, N)) \leq \cup$, we can define $s: A \to V$ by $s(y) = (y, ps'(y))$ ($y \in A$), where $\pi: N \times M \to M$ is the projection. By (2), the map $s$ extends to a map $S: Y \to U$ with $pS = 1_Y$, and since $ps'$ is $\cup_i$-homotopic to $\text{inc}(A, N)$, using the homotopy extension theorem, $ps'$ extends to a map $q: Y \to N$ which is $\cup_i$-homotopic to $\text{inc}(Y, N)$. Then the desired map $S': Y \to U'$ is defined by $S'(y) = (q(y), \pi S(y))$ ($y \in Y$).

(1) $\to$ (2): The proof is similar and omitted.

We use 1.4 to obtain the Uniformization Theorem for the ASEP. Compare this with [Do], Theorem 2.7.

Theorem 1.6: Suppose $f: X \to Y$ is a proper map. If each $y \in Y$ admits a
(not necessarily open) nbd $V$ in $Y$ such that $f|_{f^{-1}(V)}: f^{-1}(V) \to V$ has the ASEP, then $f$ itself has the ASEP.

**Proof:** Consider the following property $\mathcal{P}(A)$ for each subset $A$ of $Y$.

$\mathcal{P}(A)$: $f_A$ has the ASEP, where $f_A=f|_{f^{-1}(A)}: f^{-1}(A) \to A$. In order to show $\mathcal{P}(Y)$ holds, by [Mi], Theorem 5.5, it suffices to show $\mathcal{P}$ is an $F$-hereditary property, that is, satisfies the following conditions:

1. If $f_A$ has the ASEP and $B$ is a closed subset of $A$, then $f_B$ has the ASEP.
2. Suppose $A=A_1 \cup A_2 \subseteq Y$ and $A_1$ and $A_2$ are closed in $Y$. If $f_i=f_{A_i}$ has the ASEP $(i=1,2)$, then $f_A$ has the ASEP.
3. Suppose $A= \bigcup \{A_i: x \in M\} \subseteq Y$ and $\{A_i\}_{i \in \mathbb{A}}$ is discrete in $Y$. If each $f_{A_i}$ has the ASEP, then $f_A$ has the ASEP.

(F1) and (F3) are easily verified, we will prove (F2). We use the same notation as in Corollary 1.4. Note that $f_1^{-1}=f^{-1} \cap (A_i \times M)$ $(i=1,2)$. To see the ASEP of $f_A$, let $U$ be any open nbd of $f_1^{-1}$ in $A \times M$. Since $f_1$ has the ASEP, the nbd $U \cap (A_i \times M)$ of $f_1^{-1}$ contains an open nbd $V_i$ of $f_1^{-1}$ in $A_i \times M$ which satisfies the condition (*) in Proposition 1.3 w.r.t. $f_1^{-1}$. In turn, by the ASEP of $f_A$, the open nbd $U_2=\bigcup(V_1 \cap (U-(A_i \times M))) \cap (A_i \times M)$ of $f_1^{-1}$ in $A_i \times M$ contains an open nbd $V_2$ of $f_1^{-1}$ in $A_i \times M$ which satisfies (*) w.r.t. $f_1^{-1}$. Then the nbd $V=(V_1-(A_i \times M)) \cup V_2$ of $f_1^{-1}$ in $U$ satisfies (*) w.r.t. $f_1^{-1}$ and $U$.

2. The ASEP, HSE's and UV*-maps.

We begin with a reference to [A1]. Let $f:X \to Y$ be a proper map between metric spaces, and $M$ an ANR containing $X$ as a closed subset.

**Theorem ([A1], Theorem 4.5):** The map $f$ is a HSE iff the relation $f^{-1}: Y \to M$ is slice trivial, that is, satisfies the following:

For each nbd $U$ of $f^{-1}$ in $Y \times M$ there exist a nbd $V$ of $f^{-1}$ and maps $\phi: V \times [0,1] \to U$ and $S: Y \to U$ such that $\phi=\text{inc}(V, U)$, $\phi(y, x)=S(y)$, $p \phi(y, x, t)=y$, $p S=1_Y$ $((y, x) \in V, t \in [0,1])$, where $p: Y \times M \to Y$ is the projection.

We will use the above theorem to get another characterization of HSE's in term of the ASEP, which corresponds to [Do], Proposition 3.1.

Let $\alpha: B \to Y$ be a proper map. Define $E$ and $f_\alpha: E \to B$, $\beta: E \to X$ by $E=\{(b, x) \in B \times X: \alpha(b)=f(x)\}$, $f_\alpha(b, x)=b$, $\beta(b, x)=x$, $((b, x) \in E)$. The map $f_\alpha$ is called the map induced from $f$ by $\alpha$. Since $E=\bigcup \{b \times f^{-1}(y) : y \in Y\}$, if we regard $E$ as a relation from $B$ to $X$, we have:

1. $E=f_\alpha^{-1}$, therefore $E$ is continuous and has compact point images.
2. $\alpha \times 1_M: B \times M \to Y \times M$ is a closed map and $E=(\alpha \times 1_M)^{-1}(f^{-1})$. 

**Lemma 2.1:** Under the above notation, the following conditions are equivalent.

a) $f_*$ has the ASEP.

b) Each nbd $U$ of $f^{-1}$ in $Y \times M$ contains a nbd $V$ of $f^{-1}$ such that if $A$ is a closed set of $B$ and $\alpha_0: A \rightarrow V$ is a map with $p\alpha_0 = \alpha|_A$, then there exists a map $\alpha': B \rightarrow U$ with $p\alpha' = \alpha$ and $\alpha'|_A = \alpha_0$.

**Proof:** By Proposition 1.3, a) is equivalent to 1.3 (♯), with $R$ and $Y$ replaced by $E$ and $B$. By the observation 2), $\alpha \times 1_M$ gives the correspondence between a nbd base of $f^{-1}$ in $Y \times M$ and a nbd base of $E$ in $B \times M$, so that the maps $\alpha_0$ and $\alpha'$ as in b) correspond to the maps $s$ and $S$ as in (♯) and (*). From this follows 2.1. Compare this with [Do], Proposition 3.1.

Let $\tilde{f}$ be the map induced from $f$ by $fr$, i.e., $\tilde{f} = f_{|r}$, where $r: X \times [0,1] \rightarrow X$ is the projection.

**Theorem 2.2:** Let $f: X \rightarrow Y$ be a proper map. The following conditions are equivalent.

a) For each proper map $\alpha: B \rightarrow Y$, $f_*$ has the ASEP.

b) $f$ is approximately invertible (see Remark 1.5) and $\tilde{f}$ has the ASEP.

c) $f$ is a HSE.

d) Each nbd $U$ of $f^{-1}$ in $Y \times M$ contains a nbd $V$ of $f^{-1}$ such that for any proper map $\alpha: B \rightarrow Y$, Lemma 2.1, b) holds.

Since $(f_*)_{\beta} = f_{\alpha \beta}$ for any maps $C \xrightarrow{\beta} B \xrightarrow{\alpha} Y$, we have the following.

**Corollary 2.3:** If $f$ is a HSE, then $f_*$ is a HSE for any proper map $\alpha: B \rightarrow Y$.

**Remark 2.4:** By [A1], Lemma 4.6, we see the relation $E = f^{-1}\alpha: B \rightarrow M$ is slice trivial if $f^{-1}: Y \rightarrow M$ is slice trivial. This also implies 2.3.

**Proof of 2.2:** (a) $\rightarrow$ (b) follows from 1.4. (b) $\rightarrow$ (c): It suffices to show that each open nbd $U$ of $f^{-1}$ in $Y \times M$ admits a slice contraction $\phi: f^{-1} \times [0,1] \rightarrow U$ of $f^{-1}$ in $U$. (See [A1], Lemma 4.3.) Applying Lemma 2.1 to $\tilde{f}$, we can find an open nbd $V$ of $f^{-1}$ in $U$ such that each map $g: X \times [0,1] \rightarrow V$ with $pg = fr|_{X \times [0,1]}$ has an extension $G: X \times [0,1] \rightarrow U$ with $pG = fr$. Since $f$ is approximately invertible, there exists a map $S: Y \rightarrow V$ with $PS = 1_Y$. Then the map $g: X \times [0,1] \rightarrow V$ defined by $g(x, 0) = (f(x), x) \in f^{-1}$ and $g(x, 1) = S(f(x)) \in V$ $(x \in X)$ admits an extension $G$ as above. Define $\phi: f^{-1} \times [0,1] \rightarrow U$ by $\phi(y, x, t) = G(x, t)$ ($(y, x, t) \in f^{-1} \times [0,1]$). (c) $\rightarrow$ (d): See the proof of [A1], Proposition 2.2. (d) $\rightarrow$ (a) follows from Lemma 2.1.
Next we study UV*-maps with the ASEP. We recall some definitions of basic terms. For \( n \geq 0 \), \( S^n \) denotes the standard \( n \)-sphere and \( B^n \) denotes the \( n \)-ball. Suppose \( X \) be a metric space and \( i: X \to M \) is a closed embedding into an ANR \( M \). We say \( X \) is UV\(^n\) (or ACM) provided each nbhd \( U \) of \( i(X) \) in \( M \) contains a nbhd \( V \) of \( i(X) \) such that every \( \alpha: S^k \to V \) \((0 \leq k \leq n)\) is null homotopic in \( U \). The definition does not depend on the choice of such an embedding \( i \). \( X \) is UV\(\sim\) if \( X \) is UV\(^n\) for each \( n \geq 0 \). A UV\(\sim\)-map \((0 < \omega)\) is an onto map each point inverse of which is UV\(^n\). For the details, see [Dy], [K] and [L]. We will prove the following theorem.

**Theorem 2.5:** Suppose \( f: X \to Y \) is a closed onto map.
(1) If \( f \) is a UV\(^n\)-map and \( \dim Y \leq n < \infty \), then \( f \) has the ASEP.
(2) If \( f \) is a UV*-map and \( X \) and \( Y \) are ANR's, then \( f \) has the ASEP.

The proof is based on the following lifting lemmas.

**Lemma 1 ([Dy], Lemma 8.3):** Under the same notation as in Proposition 1.1, suppose the map \( f \) is a closed UV\(^n\)-map \((0 \leq n < \infty)\). Then for each admissible pair \((U, V)\) for \( f \) w.r.t. \( F \) and each \( \mathcal{V} \in \mathcal{V}(V) \), there exist \((U_1, V_1) \supseteq (U, V)\) and \( \mathcal{V}_1 \in \mathcal{V}(V_1) \) such that

\[
(*) \text{ if } (P, Q) \text{ is a polyhedral pair with } \dim P \leq n \text{ and } q: P \to V, \ h: Q \to U, \text{ are maps with } (g|_Q, Fh) \leq \mathcal{V}_1, \text{ then there exists a map } g': P \to U \text{ with } g'|_Q = h \text{ and } (Fg', g) \leq \mathcal{V}.'
\]

**Lemma 2 ([K], Theorem 1, Part II):** Let \( f: X \to Y \) be an LC\(\omega\)-dense map. Then for each locally finite open covering \( \mathcal{U} \) of \( Y \), polyhedral pair \((P, Q)\) and maps \( g: P \to Y, \ h: Q \to X \) with \( g|_Q \approx fh \) \((\mathcal{U}\)-homotopic), there exists a map \( g': P \to X \) with \( g'|_Q = h \) and \( fg' \approx g \) \((\mathcal{U}\)-homotopic).

**Remark 2.6:** 1) Among closed onto maps with ANR domains, LC\(\omega\)-(dense) maps coincide with UV*-maps.

2) Theorem 2.5, (2) holds even if \( X \) is an approximate polyhedron (AP) ([Ma]) and \( f \) is an LC\(\omega\)-dense map.

However, it will be shown that Taylor’s map \( T: X \to Q \) does not have the ASEP (Example 3.7). Hence we can’t omit the assumption on \( X \) if \( \dim Y = \infty \).

**Proof of 2.5:** (1) Under the same notation as in Proposition 1.1, we will show that \( f \) satisfies the condition (2) in 1.1. Given any admissible pair \((U, V)\), \( \mathcal{U} \subseteq \mathcal{V}(U) \) and \( \mathcal{V}_1 \subseteq \mathcal{V}(V) \). Let \( \mathcal{V}_1 \) be a star refinement of \( \mathcal{V} \). By Lemma 1, we get \((U_1, V_1) \supseteq (U, V)\) and \( \mathcal{V}_1 \subseteq \mathcal{V}(V) \), which satisfy (*)}. We must verify that
(U, V), \subset V_1 satisfy the required condition in Proposition 1.1, (2) w. r. t. (U, V), U and C. Let A be a closed set of Y, W an open nbd of A in V, and s : W → U, a map with \((F_s, \text{inc}(W, V)) \subseteq C V_1\). Take an open nbd W, of A with \(W_i \subseteq W\) and a common refinement \(\mathcal{Y}_i \in \text{Cov}\) of coverings \(\mathcal{V}_i \in \mathcal{V}_1\), \(\{V_i - A, W_i\}\) and \(s^{-1}(\mathcal{U} |_{V_i}) \cup \{V_i - A\}\). Since \(\dim Y \leq n\) and \(V_i\) is an ANR, there exist a polyhedron P and maps \(Y \rightarrow P \rightarrow V_i\) such that \(\dim P \leq n\) and \((r_i, \text{inc}(Y, V_i)) \subseteq \mathcal{V}_1\) ([H], Theorem 6.1). Choose a triangulation \(K\) of \(P\) such that \(|\sigma| : \sigma \in K\) refines \((r_i^{-1}(W), P \rightarrow r_i^{-1}(W_1))\), and put \(Q = \cup |\sigma| : \sigma \in K\), \(|\sigma| \cap r_i^{-1}(W_1) \neq \emptyset\). Then \(Q\) is a subpolyhedron of \(P\) and \(r(A) \subset Q\), \(r(Q) \subset W\). Since \((F_i \sigma |_Q, |\sigma|) \subseteq C V_1\), by (*), we obtain a map \(r' : P \rightarrow U\) with \((F_i r', r) \subseteq C V\) and \(r'_|Q = sr|_Q\). Put \(S = r' : Y \rightarrow U\), then it is easy to see that \((FS, \text{inc}(U, V)) \subseteq C V\) and \((S, s, s|_A) \subseteq U\). This completes the proof of (1).

(2) is verified by the same argument as in (1), using Lemma 2 instead of Lemma 1.

The next theorem is a partial converse of the above theorem.

**Theorem 2.7:** Let \(f : X \rightarrow Y\) be a proper (onto) map with the ASEP and \(y \in Y\). If each nbd \(V\) of \(y\) in \(Y\) contains an \(n\)-sphere contractible in \(V\), then \(f^{-1}(y)\) is UV*.

**Proof.** Take a closed embedding \(X \hookrightarrow M\) into an ANR \(M\). To show \(f^{-1}(y)\) is UV* (in \(M\)), let \(U_1\) be any open nbd of \(f^{-1}(y)\) in \(M\). We must find a nbd \(V_1\) of \(f^{-1}(y)\) in \(U_1\) as in the definition of UV*-property. Take an open nbd \(V_1\), \(V_2\) of \(y\) in \(Y\) with \(f^{-1}(V_1) \subset U_1\), \(V_2 \subset V_1\) and let \(\bar{U} = V_1 \times U_1 \cup (Y - V_2) \times M\). Then by Corollary 1.4, we get an open nbd \(\bar{V}\) of \(f^{-1}\) in \(\bar{U}\) which satisfies Proposition 1.3, (*). Take a nbd \(V_2\) of \(y\) in \(V_2\) and a nbd \(U_2\) of \(f^{-1}(y)\) in \(U_1\) such that \(V_2 \times U_2 \subset \bar{V}\). By the assumption, \(V_2\) contains an \(n\)-sphere \(S^n\) contractible in \(V_2\). Now for \(0 \leq i \leq n\), consider \(S^i \subset S^n\) and given any map \(a : S^i \rightarrow U_2\). Then the map \(s : S^i \rightarrow \bar{V}\) defined by \(s(x) = (x, a(x))\) \((x \in S^i)\) can be extended to a map \(S : Y \rightarrow \bar{U}\) with \(pS = 1_Y\). Since \(S^i \simeq 0\) in \(V_2\), we have a map \(h : B^{i+1} \rightarrow V_2\) such that \(h|_{S^i} = \text{inc}(S^i, Y)\). Then \(\piSh : B^{i+1} \rightarrow U_1\) is an extension of \(a\), where \(\pi : Y \times M \rightarrow M\) is the projection. This completes the proof.

**Corollary 2.8:** Let \(f : X \rightarrow Y\) be a proper map.

1. Suppose \(\dim Y \leq n\) \((0 \leq n < \infty)\) and each non-empty open set \(U\) of \(Y\) contains an \((n-1)\)-sphere contractible in \(U\). Then the map \(f\) has the ASEP iff \(f\) is a UV*-map.

2. Suppose \(X\) and \(Y\) are ANR's and each non-empty open set \(U\) of \(Y\) contains an \(n\)-sphere for each \(n \geq 0\). Then \(f\) has the ASEP iff \(f\) is a UV*-map.

**Remark 2.9:** If \(\dim Y = n\) \((0 \leq n < \infty)\) and \(Y\) contains a dense open set which
is an n-manifold, then \( Y \) satisfies the condition in (1). If \( Y \) is a \( Q \) (or \( \mathcal{L}_2 \))-manifold, then \( Y \) satisfies the condition in (2).

3. Generalization.

In this section, we extend the definition of the ASEP to the general case. The manner of the generalization has been established in [Ma], where the concept of shape fibrations is generalized using \( (AP-) \) resolutions of maps. We shall show the same method works for the ASEP as well as shape fibrations.

We use the same notations as in [Ma]. In addition, \( A(p) \) denotes the set of all admissible pairs of a map \( p \) of systems. In this section, spaces are not assumed to be metrizable. Let \( B \) be a space and \( A \subset V \) subsets of \( B \). By definition, \( V \) is a halo of \( A \) in \( B \) if there is a map \( \tau : B \to [0,1] \) with \( A \subset \tau^{-1}(1) \) are \( B - V \subset \tau^{-1}(0) \).

**Definition 3.1:** Let \( p : E \to B \) be a map of systems, where \( E = (E_i, q_i, \Lambda) \), \( B = (B_i, r_{pi}, M) \) and \( p = (p_i, \phi) \). We say \( p \) has the ASEP provided the following holds:

For each \( (\lambda, \mu) \in A(p) \) and each \( U_1 \in \text{Cov} E_i, \cup_1 \in \text{Cov} B_i \), there exist \( (\lambda_i, \mu_i) \geq (\lambda, \mu) \) in \( A(p) \) and \( u_1 \in \text{Cov} B_i \) which satisfies the following:

(*) Suppose \( \mu_1 \geq \mu_i, V \) is a halo of a subset \( A \) in \( B_i \) and \( s : V \to E_i \), is a map with \( (p_i, \mu_i, r_{pi}|_V) \). Then there exist \( \mu_0 \geq \mu_2 \) and a map \( S : B_{pi} \to E_i \) with \( (p_i, S, r_{pi}) \) \( \leq \) \( \cup_1 \) and \( (S|_{p_i}^{-1}(A), q_i, s_{pi}|_V \cup_i^{-1}(A)) \) \( \leq \) \( U_1 \).

**Proposition 3.2:** Let \( (q, r, p) \) and \( (q', r', p') \) be two \( AP \)-resolutions of a map \( p : E \to B \). If \( p \) has the ASEP, then so does \( p' \).

Note that under the notation of Proposition 1.1, all admissible pairs form an ANR (hence \( AP \))-resolution of the map \( f \). Therefore by Proposition 1.1, the following definition extends Definition 1.2.

**Definition 3.3:** Let \( p : E \to B \) be a map. We say the map \( p \) has the ASEP provided some (eq., any) \( AP \)-resolution of \( p \) has the ASEP.

We can extend the concept of approximate invertibility ([A2]) by the same method.

**Proposition and Definition 3.4:** Suppose \( p : E \to B \) be a map and \( (q, r, \zeta) \) is an \( AP \)-resolution of \( p \), where \( q = (q_1) : E \to E = (E_i, \Lambda) \), \( r = (r_i) : B \to B = (B_i, r_{pi}, M) \) and \( \zeta = (\zeta_i) : E \to B \). Then the following conditions are equivalent and depend only on the map \( p \).

1. For each \( (\lambda, \mu) \in A(p) \) and each \( U_1 \in \text{Cov} B_i \), there exist \( \mu_1 \geq \mu_i \) in \( M \) and a
map \( S : B_n \to E_i \) with \( (p_S, r_{r_n}) \leq \mathcal{U}_n \).

(2) For each \( (\lambda, \mu) \in A(p) \) and each \( \mathcal{V}_n \in \text{Cov} B_n \), there exists \( S : B \to E_i \) with \( (p_S, r_{r_n}) \leq \mathcal{V}_n \).

We say the map \( p \) has approximate sections when the above conditions are satisfied.

**Remark 3.5:** T. Watanabe also introduced ([W], Section 24) the concept of weak approximative dominations, which is essentially the same as ours.

**Remark 3.6:**
1) Every ANR-resolution of a space \( X \) induces an HPol-expansion of \( X \) in the category \( \text{pro-}{\text{HTop}} \). Therefore by the definition, if a map \( p : E \to B \) has approximate sections then \( p \) induces a weak domination in Shape category.

2) Let \( f : X \to Y \) be a map between metric spaces. If the map \( f \) has approximate sections, then \( f \) has a dense image.

The proof of 3.2 is similar to those of [Ma], Theorem 4, or [MR], Theorem 1. For the sake of completeness, we give a full detail of the proof. As for notations, let \( q = (q_i) : E \to E = (E_i, q_{u_i}, A), r = (r_p) : B \to B = (B_n, r_{r_n}, M), \) \( p = (p_p) : E \to B, q' = (q'_i) : E \to F = (F_i, q'_{u_i}, K), r' = (r'_p) : B \to C = (C, r'_{r_n}, N), p' = (p'_p) : F \to C \). We need the following lemma. See [Ma], Definition 3 and [MR], Theorem 1.

**Lemma 3.7:** Under the above notation:

1) For each \( (\kappa, \nu) \in A(p'), \mathcal{W} \in \text{Cov} F, \) and \( \mathcal{Z} \in \text{Cov} C, \) there exist \( (\lambda, \mu) \in A(p) \) and maps \( f : E_i \to F_n, \) \( g : B_n \to C_n \) with \( (f_{q_i}, q'_i) \leq \mathcal{W}, (g_{r_i}, r'_p) \leq \mathcal{Z} \) and \( (p'_p, f, q_{p_i}) \leq \mathcal{Z} \).

2) For each \( (\kappa, \nu) \in A(p'), \mathcal{W} \in \text{Cov} F, \) and \( \mathcal{Z} \in \text{Cov} C, \) there exist \( \mathcal{W}' \in \text{Cov} F, \) and \( \mathcal{Z}' \in \text{Cov} C, \) which refine \( \mathcal{W} \) and \( \mathcal{Z} \) resp. and satisfy the following:

For each \( (\lambda, \mu) \in A(p) \) and maps \( f : E_i \to F_n, \) \( g : B_n \to C_n, \) with \( (f_{q_i}, q'_i) \leq \mathcal{W}', (g_{r_i}, r'_p) \leq \mathcal{Z}' \) and \( (p'_p, f, q_{p_i}) \leq \mathcal{Z}' \) and for each \( (\lambda, \mu) \geq (\lambda, \mu) \) in \( A(p) \) and \( \mathcal{U}_i \in \text{Cov} E_i, \) \( \mathcal{V}_i \in \text{Cov} B_{r_i}, \) there exist \( (\kappa, \nu) \geq (\kappa, \nu) \) in \( A(p') \) and two maps \( f_i : F_{i_1} \to E_{i_1}, \) \( g_i : C_{i_1} \to B_{r_i} \) with \( (f_{i q'_{i}}, q'_{i}) \leq \mathcal{W}_i, (g_{i r'_{i}}, r_{r_i}) \leq \mathcal{V}_i, (p_{r_{i_1}}, f_i, q_{i p_i}) \leq \mathcal{V}_i \) and \( (f_{q_{i_1}}, f_i, q'_{i_1}) \leq \mathcal{W}, (g_{r_{i_1}}, g_{i r_i}) \leq \mathcal{Z}_i \).

**Proof of 3.2:** The proof consists of the construction of the following diagram by Lemma 3.7 and the ASEP of \( p \).
To show \( p' \) has the ASEP, given \((\epsilon,\nu)\in A(p')\), \(W'\in\text{Cov} F\), and \(Z\in\text{Cov} C.\) Take \(W'\in\text{Cov} F\), and \(Z\in\text{Cov} C\) with \(\text{st} W' < W\), \(\text{st} Z < Z\). Apply 3.7, (2) to \((\epsilon,\nu), W', Z\) and we obtain

(a): \(W'\in\text{Cov} F\), and \(Z\in\text{Cov} C\) as in 3.7, (2).

By 3.7, (1), there exist \((\lambda, \mu)\in A(p)\) and maps \(f : E_1 \to F\) and \(g : B_{p} \to C\) with \((f_1, q') \leq W', (\mu, f, g_{p, q}) \leq Z\). Since \(p\) has the ASEP, there exist

(b): \((\lambda, \mu)\geq (\lambda, \mu)\) and \(\nu\in\text{Cov} B_{1}\) which satisfy 3.1, (*) w.r.t. \((\lambda, \mu), f^{-1}(W')\) \(\in\text{Cov} E_1\) and \(g^{-1}(Z')\in\text{Cov} B_1\).

Take \(\nu'\in\text{Cov} B_{1}\) with \(\text{st} \nu' < \nu, \lambda(\nu, q)\nu'^{-1}(Z')\). By 3.7, (2) (in this case, we apply the lemma only to \(q'\) and \(q''\)), there exists

(c): \(\nu'\in\text{Cov} B_{1}\) as in 3.7, (2) w.r.t. \(\mu\) and \(\nu'\).

We apply (a) to the data \((\lambda, \mu), f, g, (\lambda, \mu)\) and \(\nu'\in\text{Cov} B_{1}\) to get \((\epsilon, \nu)\geq (\epsilon, \nu)\) in \(A(p)\) and maps \(f : F_1 \to F_1\) and \(g : C_1 \to C_1\) with \((g_1, q'), r_{p} \leq \nu'\), \((\mu, q, f_1)\), \(g_1, q_{p, q} \leq W', (\mu, g_1, q_{p, q}) \leq Z'\).

Claim: \((\epsilon, \nu)\) and \(g^{-1}(\nu')\in\text{Cov} C_{1}\) satisfy 3.1, (*) w.r.t. \((\epsilon, \nu), W', Z\).

To see this, suppose \(\nu' \geq \nu\) in \(N, V\) is an open halo of a subset \(D\) in \(C_{2}\) and \(r' : V \to F_{1}\) is a map with \((r_{p, q}, q', r_{p, q}) \leq q_{1}^{-1}(\nu')\). Take a \(\text{haloing function} \: \tau : C_{2} \to [0, 1] \) with \(D \subset \tau^{-1}(1)\) and \(C_{2} - \tau^{-1}(0)\). Let \(V_{i} = \tau^{-1}(1/2, 1)\). Then \(V\) is a halo of \(V_{i}, D \subset V_{1}\), and \(r_{p, q}^{-1}(Z'), [C_{2} - D, V_{1}], (\mu, r_{p, q}^{-1}(W'))\in\text{Cov} C_{2}\). Take \(\nu'\in\text{Cov} C_{2}\) which refines the above three coverings. Apply 3.7, (2) to \(\nu'\) and \(Z\), we obtain

(d): \(Z'\in\text{Cov} C_{2}\) as in 3.7, (2).

We apply (c) to the data \(\nu, g_{1}, \nu_{2}, Z'\) and obtain \(\mu_{2} > \mu_{1}\) and a map \(g_{2} : B_{p_{2}} \to C_{2}\) with \((g_{p_{2}, q_{2}}, r_{p_{2}}, r_{p_{2}}) \leq Z'\) and \((\mu_{2}, q_{p_{2}, q}, r_{p_{2}}) \leq W'\). Let \(U = g_{1}^{-1}(V), A = g_{1}^{-1}(V)\) and \(s = f_{1} s_{1} : U \to E_{1}\). Then \(U\) is a halo of \(A\) in \(B_{p_{2}}\) and \((p_{1}, s_{1}, r_{p_{1}, q_{1}}) \leq W'. \)
Therefore (b) gives $\mu_s \geq \mu_s$ and a map $S : B_{g_3} \to E_4$ with $(p_\mu S_\mu, r_{g_3}) \leq g^{-1}(Z')$ and $(S|_{g_3} r_{g_3}) q_{g_3} s_{g_3} : (s_{g_3} r_{g_3})^{-1}(A) \leq f^{-1}(W')$. Apply (d) to $\mu_s$, $g_3$, and we get $\nu_s \geq \nu_s$ and a map $g_3 : C_{g_3} \to B_{g_3}$ with $(g_3 r_{g_3} g_3, r_{g_3}) \leq Z'$. Let $S' = f S g_3 : C_{g_3} \to E_4$. Then adjacent maps of the followings are $Z'$-near:

$$p_{s'} S', p_{s'} g_3 S g_3, r_{g_3} g_3, g_3 r_{g_3} g_3 r_{g_3} g_3, r_{g_3} g_3 r_{g_3} g_3, r_{g_3}.$$

This implies $(p_{s'} S', r_{g_3}) \leq Z'$. It remains only to show $(q_{s'}, s' r_{g_3}) | c_{g_3}^{-1}(D), S' r_{g_3}^{-1}(D) \leq W'$. First note that $g_3 (r_{g_3}^{-1}(D)) \subseteq r_{g_3}^{-1}(A)$ and that, since $(g_3 r_{g_3} g_3, r_{g_3}) \leq (q_{s'}, s')^{-1}(W') \cup (C_{g_3} - D)$, maps $g_3 r_{g_3} g_3, r_{g_3} : r_{g_3}^{-1}(D) \to V$ are $(q_{s'}, s')^{-1}(W')$-near. Thus adjacent maps of the followings are $W'$-near on $r_{g_3}^{-1}(D)$:

$$q_{s'}, s' r_{g_3}, q_{s'} g_3 s_{g_3}, f q_{s'} s_{g_3} g_3, S'.$$

Since $s t W' < W'$, we have the conclusion.

We conclude the section, inspecting Taylor's map $F : X \to Q$ ([DW], [T]).

**Example 3.8 (Taylor's map):** The map $F$ is obtained as the inverse limit of the following level map $f = \{f_n\}$:

\[
\begin{array}{cccccccc}
L & \xleftarrow{\beta} & \Sigma^1 L & \xleftarrow{\Sigma \beta} & \Sigma^2 L & \xleftarrow{\Sigma^2 \beta} & \cdots & \xleftarrow{\Sigma^{n-1} \beta} & \Sigma^n L & \xleftarrow{\Sigma^n \beta} \\
* & \downarrow{f_0} & \downarrow{f_1} & \downarrow{f_2} & \cdots & \downarrow{f_n} \\
\rho_0 & \xleftarrow{I^1} & \rho_1 & \xleftarrow{I^2} & \cdots & \xleftarrow{I^n} & \rho_n
\end{array}
\]

where $I = [-1, 1]^r$ ($r$ is a fixed positive integer), $\Sigma^n L$ is the $nr$-th suspension of a compact polyhedron $L$ and $\beta$ is a map for which the composition $g : \Sigma^n \beta \cdots \Sigma^2 \beta \beta \beta = 0$ for each $n \geq 0$, $f_n$ is induced from the projection $L \times I^n \to I^n$ and $\rho_n$ is the projection. Note that, by [Ma], Theorem 8, $f$ is an ANR-resolution of $F$. The map $F$ is an example of CE-maps which are not shape equivalences ([T]), and moreover, $F$ is approximately invertible ([A2]). Note that each $f_n$ has a section.

We now show that $F$ does not have the ASEP. To see this, on the contrary, suppose $F$, hence $f$ has the ASEP. Then there exist $n \geq 0$ and $\epsilon > 0$ which satisfy 3.1, (*) w.r.t. the index 0 in $f$. Since $\Sigma^n L$ is a finite dimensional compact metric space, we can find an embedding $i : \Sigma^n L \to I^n$ for some $m \geq 0$. Define an embedding $j : \Sigma^n L \to I^{n+m} = I^n \times I^m$ by $j(x) = (f_a(x), i(x))$ ($x \in \Sigma^n L$) and let $A = j(\Sigma^n L)$, $s = j^{-1} : A \to \Sigma^n L$. Since $f_n s = \rho_{n+m} | A$, by the choice of $n$, there exist $k \geq n + m$ and a map $S : I^k \to L$ such that $S$ coincides with the composition $\beta \cdots \Sigma^{-1} \beta \beta \cdots \Sigma \beta = 0$. Since $S \subseteq 0$ and $s \rho_{n+m} : A \times I^{k-n-m} \to \Sigma^n L$ is a homotopy equivalence,
This contradicts the choice of $\beta$. One can embed $X$ into $Q$ and extend $F$ to the CE-map $G: Q \to Q \cup_s F$. It is known that $Q \cup_s F$ does not have the trivial shape ([DS]), hence $G$ is not a weak domination. This implies that $G$ is not approximately invertible.

References


Institute of Mathematics
University of Tsukuba
Sakuramura, Ibaraki, 305 Japan.