A BOOOLEAN POWER AND A DIRECT PRODUCT OF ABELIAN GROUPS

By

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A group means an abelian group in this paper. A Boolean power and a direct product of groups consist of all global sections of groups in some Boolean extensions $V^{(B)}$. We shall study about a homomorphism $h$ whose domain is a group consisting of all the global sections of a group in $V^{(B)}$. We investigate two cases: one of them is that the range of $h$ is a slender group, which is related to a torsion-free group, and the other is that the range of $h$ is an infinite direct sum, which is related to a torsion group. We extend a few theorems which have been obtained in [4] and [5]. As in [5], we not only extend theorems, but improve them and give a good standing point of view.

We refer the reader to [9] or [1], for a Boolean extension $V^{(B)}$. We shall use notations and terminologies in [5], [6] and [7]. Throughout this paper, $B$ is a complete Boolean algebra and $\mathcal{S}$ is the set of all countably complete maximal filters on $B$. We do not mention these any more. $\hat{x}$ is the element of $V^{(B)}$ such that dom $\hat{x} = \{\hat{y} : y \in x\}$ and range $\hat{x} \subseteq \{1\}$. As noted in [5], "$\hat{x}$" in [1] means our "$\hat{x}$". $\hat{x} = \{y : [y \in x] = 1$ and $y \in V^{(B)}\}$ for $x \in V^{(B)}$, where $V^{(B)}$ is separated. For $b \in B$ and a group $A$ in $V^{(B)}$, i.e. $[A$ is a group]$ = 1$, $\hat{A}^b$ is the subgroup of $\hat{A}$ such that $x \in \hat{A}^b$ iff $x \in \hat{A}$ and $-b \equiv [x = 0]$, where 0 is the unit of $A$. By this notation, $\hat{A} = \hat{A}^1$. For $x \in \hat{A}$, $x^b$ is the element of $\hat{A}^b$ such that $b \equiv [x = x^b]$.

1. A general setting about a complete Boolean algebra

Let $\Phi(b)$ be a property of $b \in B$ which satisfies the following conditions:

1. if $\{b_n : n \in \mathbb{N}\}$ is a pairwise disjoint subset of $B$, there exists $k$ such that $\Phi(\bigvee_n b_n)$ and $\Phi(b_n)$ hold for each $n \geq k$;
2. if $b \land c = 0$, $\Phi(b)$ and $\Phi(c)$ hold, then $\Phi(b \lor c)$ holds.

Let $\mathcal{S}$ be the subset of $B$ such that $b \in \mathcal{S}$ iff $\Phi(b)$ does not hold and $c \land c' = 0$ implies $\Phi(c)$ or $\Phi(c')$ for any $c, c' \leq b$.
LEMMA 1. Let $F^b$ be the subset of $B$ defined by: $c \in F^b$ iff $\Phi(b \wedge c)$ does not hold. Then, $F^b \subseteq U$ for every $b \in S$.

PROOF. We prove only the countable completeness. Let $b_n \in F^b$ for $n \in N$. Let $c_0=0$ and $c_{n+1}=\bigvee_{k=1}^n b_k-b_{n+1}$. Then, $b_1=\bigvee_{n \in N} c_n \wedge b_n$. By the condition (1) and (2) of $\Phi$ and the property of $S$, $\Phi(b \wedge \bigvee_{n \in N} c_n)$ and so $\Phi(b \wedge \bigvee_{n \in N} b_n)$ does not hold.

LEMMA 2. Let $M$ be a maximal pairwise disjoint subfamily of $S$. Then, $M$ is finite and $\Phi(c)$ holds for any $c$ such that $c \wedge \bigvee M=0$.

PROOF. By the condition of $\Phi$, $M$ is finite. Suppose that there exists $c$ such that $\Phi(c)$ does not hold and $c \wedge \bigvee M=0$. By the maximality of $M$, there is no element of $S$ below $c$. So, there are $b_0, c_0 \subseteq c$ such that $b_0 \wedge c_0=0$ and $\Phi(b_0)$ or $\Phi(c_0)$ does not hold. Then, take $b_1, c_1 \subseteq c_0$ with the same property of $b_0$ and $c_0$. In such a way, we obtain a pairwise disjoint family $\{b_n; n \in N\}$ such that $\Phi(b_n)$ does not hold for any $n \in N$, which is a contradiction.

2. $\text{Hom}(\hat{A}, G)$

Let $F$ be a maximal filter on $B$. For a group $A$ in $V^B$, $\hat{A}/F$ is the quotient of $\hat{A}$ by the equivalence relation $\sim_F$ such that $x \sim_F y$ iff $[x=y] \in F$. In the case $A=\hat{X}$, $\hat{A}$ is known as a Boolean power $X^{(B)}$ and $\hat{A}/F$ is a Boolean ultrapower $X^{(B)}/F$. (Ref. [8]) In the case that $B=\mathcal{P}(I)$ and $\hat{A}=\prod_{i \in I} A_i$, where $A$ is defined by a natural way, $\hat{A}/F$ is known as an ultraproduct $\prod_{i \in I} A_i/F$. (Ref. [2]) However, the following fact is enough to read the main part of this paper. Let $K$ be the subgroup of $\hat{A}$ defined by: $x \in K \iff [x=0] \in F$. Then, $\hat{A}/F \cong \hat{A}/K$, where the right part is the quotient group.

THEOREM 1. Let $A$ be a group in $V^B$ and $G$ a slender group. Then, $\text{Hom}(\hat{A}, G) \cong \bigoplus_{F \in G} \text{Hom}(\hat{A}/F, G)$ holds.

PROOF. Let $h$ be a homomorphism from $\hat{A}$ to $G$ and $\Phi(b)$ the property $"h^*\hat{A}^b=0"$. Let $\{b_n; n \in N\}$ be a pairwise disjoint subset of $B$ and $x_n \in \hat{A}^{b_n}$ for each $n \in N$. Think of the homomorphism $g: \mathbb{Z}^N \rightarrow \hat{A}$ such that $g(\sum_{n \in N} a_n e_n) = \sum_{n \in N} a_n x_n$, where $x=\sum_{n \in N} a_n x_n$ is the element of $\hat{A}^b$ such that $b=\bigvee_{n \in N} b_n$ and $b_n \subseteq [x=a_n x_n]$ for each $n \in N$, and apply the slenderness of $G$ to $h \cdot g$, then $h \cdot g(e_n) = 0$ and so $h(x_n)=0$ for all almost all $n$. Hence, there exists $k$ such that $\Phi(b_n)$ for any $n \geq k$ and $h(\sum_{n \geq k} x_n)=0$, by Specker's theorem. (Ref. Prop. 1 of [5] or
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Lem. 94.1 of [7]

Therefore, Φ satisfies the conditions (1) and (2) of §1. Hence, Lem. 1 and Lem. 2 hold for this Φ. Now, let \( M = \{ b_1 \cdots b_n \} \) and \( b_0 = 1 - \forall M \). Let \( h_i: \mathcal{A}/F_{b_i} \to G \) be defined by: \( h_i([x]_i) = h(x_{b_i}) \), where \([x]_i\) is the equivalence class containing \( x \) with respect to \( F_{b_i} \), for each \( 1 \leq i \leq n \). Since \([x]_i \in F_{b_i} \) implies \( h(x_{b_i}) = 0 \) for \( x \in \mathcal{A} \), \( h(x) = h(\sum_{i=0}^{n} x_{b_i}) = \sum_{i=0}^{n} h(x_{b_i}) = \sum_{i=0}^{n} h_i([x]_i) \). The linear independence of \( \{ \text{Hom}(\mathcal{A}/F, G) \mid F \in \mathcal{G} \} \) is clear. Now, the proof is completed.

In view of the paragraph preceding Th. 1, Th. 1 includes Th. 2 of [5] and Th. 94.4 of [7]. We express these as corollaries.

**Corollary 1.** Let \( A \) be a group and \( G \) a slender group. Then, \( \text{Hom}(A^{(b)}, G) \cong \bigoplus_{F \in \mathcal{G}} \text{Hom}(A^{(b)}/F, G) \).

**Corollary 2.** Let \( A_i \) be a group for each \( i \in I \) and \( G \) a slender group. Then, \( \text{Hom}(\prod_{i \in I} A_i, G) \cong \bigoplus_{F \in \mathcal{G}} \text{Hom}(\prod_{i \in I} A_i/F, G) \).

If the cardinality of \( A \) is less than the least measurable cardinal \( M_c \) or \( B \) satisfies \( M_c - \text{c. c.}, A^{(b)}/F \cong A \) holds, so Cor. 1 is an extended form of Th. 2 of [5]. If the cardinality of \( I \) is less than \( M_c \), then every \( F \in \mathcal{G} \) is principal. Therefore, \( \text{Hom}(\prod_{i \in I} A_i, G) \cong \bigoplus_{F \in \mathcal{G}} \text{Hom}(A_i, G) \), which is a famous theorem. If the cardinalities of the \( A_i \) are bounded below \( M_c \), then \( \prod_{i \in I} A_i/F \cong A_i \) for some \( i \), which was used in the proof of Cor. 2 of [5].

By Cor. 2, we can calculate a dual group of \( \prod_{i \in I} \bigoplus_{n=1}^{\infty} Z \). Now, we shall do it in a simple case. Let \( j_F: V \to M_F \) be the elementary embedding, where \( F \) is a countably complete maximal filter on \( P(\lambda) \) and \( M_F \) is the transitive model which is isomorphic to \( V^2/F \). (Ref. [10]) Let \( B = P(\lambda_0) \), then

\[
\text{Hom}(\prod_{i \in I} Z, Z) \cong \bigoplus_{F \in \mathcal{G}} \text{Hom}(\prod_{i \in I} Z/F, Z)
\]

\[
\cong \bigoplus_{F \in \mathcal{G}} \text{Hom}(\prod_{j \in \mathbb{N}} Z, Z)
\]

\[
\cong \bigoplus_{F \in \mathcal{G}} \prod_{j \in \mathbb{N}} Z.
\]

In the calculation, we have used the absoluteness of direct sums. Unfortunately, direct products are not absolute among transitive models. So, for the calculation of \( \text{Hom}(\prod_{i \in I} \bigoplus_{n=1}^{\infty} Z, Z) \), we must prepare a proposition which is obtained by modifying Cor. 2. That can be done, if we notice the fact that only the count-
ably completeness of $B$, not the full completeness, has been used in the proof of Th. 1.

In this paper, we deal with the case that $B$ is a complete Boolean algebra. Therefore, unless $B$ is very large, every element of $\mathcal{F}$ is principal. Concerning a Boolean power, a countably complete Boolean algebra can give us interesting groups, for there can be a non-principal c.c. max-filter on a non-complete but countably complete and small Boolean algebra.

3. A homomorphism into an infinite sum

In this section, we shall extend some results of [4]. We do not prove the next lemma, because the proof is in [3] and [4], and the essential idea of it will be developed in the proof of Lem. 5. For $X \subseteq I$, we identify $\prod_{i \in X} A_i$ with the subgroup of $\prod_{i \in I} A_i$ such that $x \in \prod_{i \in X} A_i$ iff $x \in \prod_{i \in I} A_i$ and $x(i) = 0$ for each $i \in X$. Similarly, we do $\bigoplus_{i \in I} A_i$ with the subgroup of $\bigoplus_{i \in I} A_i$.

**Lemma 3.** (Chase [3]) Let $h : \prod_{i \in N} A_i \to \bigoplus_{j \in J} G_j (=G)$ be a homomorphism. Then, there exist an integer $n > 0$ and finite subsets $F \subseteq N$ and $J' \subseteq J$ such that

$$h^n \prod_{i \in N - F} A_i \subseteq \bigoplus_{j \in J'} G_j + \bigcap_{n \in N} nG. \tag{1}$$

**Theorem 2.** Let $A$ be a group in $V(m)$ and $h : \hat{A} \to \bigoplus_{j \in J} G_j (=G)$ a homomorphism. Then, there exist $F_1, \ldots, F_m \subseteq \mathcal{G}$, an integer $n^* > 0$ and a finite subset $J^*$ of $J$ that satisfy the following condition: Let $K$ be the subgroup of $\hat{A}$ such that $x \in K$ iff $[x = 0] \in F_i$ for each $1 \leq i \leq m$, then $h^{n^*} K \subseteq \bigoplus_{j \in J^*} G_j + \bigcap_{n \in N} nG. \tag{2}$

Let $\Phi(b)$ be the property "There exist an integer $n > 0$ and a finite subset $J'$ of $J$ such that $h^n \hat{A} \subseteq \bigoplus_{j \in J'} G_j + \bigcap_{n \in N} nG."$

**Lemma 4.** This $\Phi$ satisfies the conditions (1) and (2) in §1.

**Proof.** Let $b = \bigvee_{n \in N} b_n$, for a pairwise disjoint family $\{b_n; n \in N\}$. Then, $\hat{A} = \bigoplus_{n \in N} A b_n$. $b \subseteq c$ and $\Phi(c)$ imply $\Phi(b)$. Hence, $\Phi$ satisfies the condition (1), by virtue of Lem. 3. $\Phi$ satisfies the condition (2) clearly.

**Lemma 5.** There exist an integer $n^* > 0$ and a finite subset $J^*$ of $J$ such that, for any $b$ which satisfies $\Phi(b)$, $h^{n^*} \hat{A} \subseteq \bigoplus_{j \in J^*} G_j + \bigcap_{n \in N} nG.$

(*) Here we admit $m = 0$ and in such a case $K = \hat{A}$.
Proof. Suppose the negation of the conclusion. Let \( \pi_j: \bigoplus_{j \in J} G_j \to G_j \) be the projection for \( j \in J \). We construct \( b_k \in B \), \( a_k \in \hat{A} \), \( n_k \in N \), \( j_k \in J \) and a finite subset \( J_k \) of \( J \) satisfying the following conditions:

1. \( \langle b_k; k \in N \rangle \) are pairwise disjoint and \( \Phi(b_k) \) for \( k \in N \);
2. \( a_k \in n_{k-1}! \hat{A}^{b_k} \) and \( \pi_{j_k} h(a_k) \in n_k! G_{j_k} \) and \( \pi_{j_k} h(a_k) = 0 \) for each \( i < k \);
3. \( h^n n_{k-1}! \hat{A}^{b_k} \subseteq \bigoplus_{j \in J_k} G_j + \bigcap_{n \in N} nG \), where \( b = \bigoplus_{i=1}^{k-1} b_i \);
4. \( j_k \in J_k \) and \( j_k \notin J_i \) for \( i < k \);
5. \( \langle n_k; k \in N \rangle \) and \( \langle J_k; k \in N \rangle \) are increasing.

Suppose that we have already defined \( b_i \), \( a_i \), \( n_i \), \( j_i \) and \( J_i \) for \( i \leq k \) satisfying the above conditions. By the hypothesis, there exists \( b_{k+1} \) such that \( b_{k+1} \land \bigvee_{i=1}^{k} b_i = 0 \), \( \Phi(b_{k+1}) \) and \( h^n n_{k+1}! \hat{A}^{b_{k+1}} \subseteq \bigoplus_{j \in J_{k+1}} G_j + \bigcap_{n \in N} nG \). So, there exists \( a_{k+1} \in n_k! \hat{A}^{b_{k+1}} \) such that \( h(a_{k+1}) \in \bigoplus_{j \in J_{k+1}} G_j + \bigcap_{n \in N} nG \). Hence, there are \( j_{k+1} \in J_k \) and \( n > n_k \) such that \( \pi_{j_{k+1}} h(a_{k+1}) \in n! G_{j_{k+1}} \). Let \( J' = J_k \cup \{ j; \pi_j h(a_{k+1}) = 0 \} \). By the property of \( b_{k+1} \), there exist \( n_{k+1} \) and a finite subset \( J_{k+1} \) such that \( n < n_{k+1} \) and \( J' \subseteq J_{k+1} \) and \( h^n n_{k+1}! \hat{A}^{b_{k+1}} \subseteq \bigoplus_{j \in J_{k+1}} G_j + \bigcap_{n \in N} nG \). 

Suppose that \( a \) exists in \( \hat{A} \) and so let it be \( a \). Then, \( a - \sum_{i=1}^{k} a_i \in n_k! \hat{A} \) and \( \pi_{j_k} h(a_k) \in n_k! G_{j_k} \) and \( \pi_{j_k} h(a_k) = 0 \) for each \( i < k \). Hence, \( \pi_{j_k} h(a) = \pi_{j_k} h(a - \sum_{i=1}^{k} a_i) + \pi_{j_k} h(a_k) = 0 \) for each \( k \). Since \( k \neq k' \) implies \( j_k \neq j_k' \), it is a contradiction.

Proof of Th. 2. By Lem. 1, Lem. 2 and Lem. 4, \( M \) is finite and so let \( M = \{ b_1, \ldots, b_m \} \) and \( b_0 = 1 - \bigvee M \). Let \( F_i = F^{b_i} \) for \( 1 \leq i \leq m \). Now, the theorem is clear by Lem. 5 and the fact that \( x \in K \) implies \( x \in \hat{A}^b \) for some \( b \) which satisfies \( \Phi(b) \).

For a Group \( A \), \( \hat{A} \) denotes the corresponding Hausdorff group \( A/\bigcap_{n \in N} nA \).

Lemma 6. For a group \( A \) in \( V^{(\mathcal{B})} \), \( \hat{A} \cong \hat{A} \).

Proof. By the absoluteness of \( N \), \( \bigcap_{n \in N} \hat{A} \cong \hat{\bigcap_{n \in N} nA} \). Hence, \( \hat{A} \cong \hat{A}/\bigcap_{n \in N} nA \cong \hat{A} \).

Let \( F \) be a maximal filter on \( B \) and \( K_{\hat{A}} \) the subgroup of \( \hat{A} \) such that \( x \in K_{\hat{A}} \) iff \( \lceil x = 0 \rceil \in F \).

Lemma 7. \( nx \in K_{\hat{A}} \) implies \( nx \in nK_{\hat{A}} \), where \( n \) is an integer.

Proof. Let \( b = \lceil nx = 0 \rceil \). Let \( x' \) be the element of \( \hat{A} \) such that \( -b \leq \lceil x' = x \rceil \).
and \( b \leq [x' = 0] \). Then, \( x' \in K \hat{\beta} \) and \( n x' = nx \).

**Lemma 8.** Let \( \pi : \hat{A} \rightarrow \hat{A} (\equiv \tilde{A}) \) be the canonical homomorphism. Then, \( \pi^* K \hat{\beta} = K \hat{\beta} \).

**Proof.** \( \pi^* K \hat{\beta} \subseteq K \hat{\beta} \) is obvious. Let \( x \in K \hat{\beta} \). Then, there exists \( y \) in \( \hat{A} \) such that \( \pi(y) = x \). So, there exists \( b \) such that \( b \in F \) and \( b \leq [x = 0] \). Let \( y' \) be the element of \( \hat{A} \) such that \( -b \leq [y' = y] \) and \( b \leq [y' = 0] \). Then, \( \pi(y') = \pi(y) \) and \( y' \in K \hat{\beta} \).

**Lemma 9.** Let \( A \) be a torsion group in \( V(\mathcal{B}) \), then \( \hat{A} / F \) is also a torsion group for \( F \subseteq \mathcal{F} \).

**Proof.** Let \( a \in \hat{A} \), then \( \bigvee \{ [na = 0] = \exists n \in \mathbb{N} : (na = 0) \} = 1 \). By the countable completeness of \( F \), \( [na = 0] \subseteq F \) for some \( n \in \mathbb{N} \). So, \( \hat{A} / F \) is a torsion group.

**Theorem 3.** Let \( A \) be a torsion group in \( V(\mathcal{B}) \). Then, for each direct sum decomposition \( \bigoplus_{j \in J} G_j \) of \( \hat{A} \), \( G_j \) is a torsion group for almost all \( j \in J \).

**Proof.** Applying Th. 2 directly, we have \( F_1, \ldots, F_m \subseteq \mathcal{F} \), an integer \( n \) and a finite subset \( J' \) of \( J \) such that \( n K \subseteq \bigoplus_{j \in J'} G_j + \bigcap_{n \in \mathbb{N}} n G \), where \( K \) and \( G \) are the same as Th. 2. Let \( \pi : G \rightarrow \tilde{G} \) be the canonical homomorphism. Then, \( \pi^* G_j \equiv \tilde{G}_j \) for each \( j \in J \) and \( n \pi^* K \subseteq \bigoplus_{j \in J'} \pi^* G_j \).

Let \( \psi : \tilde{G} (\equiv \hat{A}) \rightarrow \tilde{G} / \pi^* K \) be the canonical homomorphism. Then, the restriction \( \phi \) to \( \bigoplus_{j \in J - J'} \pi^* G_j \) is a monomorphism, by Lem. 6, 7 and 8. On the other hand, \( \tilde{G} / \pi^* K \equiv \hat{A}^{n_1} / F_1 \oplus \cdots \oplus \hat{A}^{n_m} / F_m \equiv \hat{A} / F_1 \oplus \cdots \oplus \hat{A} / F_m \), by virtue of Lem. 6, 7 and 8 and the fact: \( K = \bigoplus_{j \in J' \setminus J_0} \hat{A}^{n_j} / F_m \). Therefore, it is a torsion group by Lem. 9 and hence \( \bigoplus_{j \in J - J'} \tilde{G}_j \) is a torsion group.

Let \( A_i \) be a torsion group for each \( i \in I \). In view of the first paragraph of §2, we can take a torsion group \( A \) in \( V(\mathcal{P}(\mathcal{I})) \) such that \( \hat{A} \cong \prod_{i \in I} A_i \). So, Th. 3 is an improvement of Lem. 8 of [4], even in the case of a direct product, i.e. dropping the cardinality hypothesis for \( I \). Hence, we have Th. 9 of [4] without the cardinality hypothesis for \( I \).

**Acknowledgement**

The author would like to thank Prof. K. Honda for his kind teaching in the preparation of this paper.
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References


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