ON ISOMORPHISMS OF A BRAUER
CHARACTER RING ONTO ANOTHER

Dedicated to Professor Hiroyuki Tachikawa

By
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1. Introduction

Throughout this paper $G$, $Z$ and $Q$ denote a finite group, the ring of rational integers and the rational field respectively. Moreover we write $\mathbb{Z}$ to denote the ring of all algebraic integers in the complex numbers and $\overline{Q}$ to denote the algebraic closure of $Q$ in the field of complex numbers. For a finite set $S$, we denote by $|S|$ the number of elements in $S$.

Let $\text{Irr}(G)=\{\chi_1,\ldots,\chi_h\}$ be the complete set of absolutely irreducible complex characters of $G$. Then we can view $\chi_1,\ldots,\chi_h$ as functions from $G$ into the complex numbers. We write $\mathbb{Z}R(G)$ to denote the $\mathbb{Z}$-algebra spanned by $\chi_1,\ldots,\chi_h$. For two finite groups $G$ and $H$, let $\lambda$ be a $\mathbb{Z}$-algebra isomorphism of $\mathbb{Z}R(G)$ onto $\mathbb{Z}R(H)$. Then we can write

$$\lambda(\chi_i) = \sum_{j=1}^{h} a_{ij} \chi'_j, \quad (i=1,\ldots,h)$$

where $a_{ij} \in \mathbb{Z}$ and $\text{Irr}(H)=\{\chi'_1,\ldots,\chi'_h\}$. In this case we write $A$ to denote the $h \times h$ matrix with $(i,j)$-entry equal to $a_{ij}$ and say that $A$ is afforded by $\lambda$ with respect to $\text{Irr}(G)$ and $\text{Irr}(H)$.

As is well known, concerning the isomorphism $\lambda$, we have the following two results, which seem to be most important. (For example see Theorem 1.3 (ii) and Lemma 3.1 in [5])

(i) $|c_G(c_i)|=|c_H(c'_i)|$, $(i=1,\ldots,h)$ where $\{c_1,\ldots,c_h\}$ and $\{c'_1,\ldots,c'_h\}$ are complete sets of representatives of the conjugate classes in $G$ and $H$ respectively and $c_i \xrightarrow{\lambda} c'_i$ $(i=1,\ldots,h)$. (Concerning a symbol "$c_i \xrightarrow{\lambda} c'_i$", see the definition in [5] and also the definition in section 2 in this paper)

(ii) $A$ is unitary where $A$ is the matrix afforded by $\lambda$ with respect to $\text{Irr}(G)$ and $\text{Irr}(H)$.

In this paper our main objective is to give a necessary and sufficient condition
under which the above statements (i) and (ii) hold, concerning an isomorphism $\lambda$ of a Brauer character ring onto another, and to state a generalization of theorems of Saksonov and Weidman about character tables of finite groups. (See Theorem 2, Corollary 2.1 in [3] and Theorem 3 in [4])

From now on, when we consider homomorphisms from an algebra to another, unless otherwise specified, we shall only deal with algebra homomorphisms.

2. Preliminaries

We fix a rational prime number $p$ and use the following notation with respect to a finite group $G$.

$G_o$: the set of all $p$-regular elements of $G$

$Cl(G_o) = \{ [c_i] = \{1, \ldots, [c_i]\}$: the complete set of $p$-regular conjugate classes in $G$

$\{c_i = 1, \ldots, c_r\}$: a complete set of representatives of $[c_1], \ldots, [c_r]$ respectively

$IBr(G) = \{ \phi_1, \ldots, \phi_r\}$: the complete set of irreducible Brauer characters of $G$, which can be viewed as functions from $G_o$ into the complex numbers.

For any subring $R$ of the field of complex numbers such that $1 \in R$, we write $RBR(G)$ to denote the ring of linear combinations of $\phi_1, \ldots, \phi_r$ over $R$. That is, $RBR(G)$ is the $R$-algebra spanned by $\phi_1, \ldots, \phi_r$. In particular we use the notation $BR(G)$ instead of $ZBR(G)$ and say that $BR(G)$ is the Brauer character ring of $G$. Moreover we add the following notation.

$G(\overline{Q}/Q)$: the Galois group of $\overline{Q}$ over $Q$

If $A = (a_{ij})$ is a matrix over $\overline{Q}$, then for $\sigma \in G(\overline{Q}/Q)$ we write $A^\sigma$ to denote the matrix $(a_{ij}^\sigma)$. We use the common notation $X^*$ for the conjugate transpose of a matrix $X$.

Now we define characteristic class functions on $G_o$.

DEFINITION 2.1. We define class functions $f_i$ on $G_o$ $(i = 1, \ldots, r)$ as follows

$$f_i(c_i) = 1, \quad f_i(c_j) = 0 \quad (i \neq j).$$

In this case we say that these class functions are the characteristic class functions on $G_o$ and that $f_i$ corresponds to $[c_i]$ or $[c_i]$ corresponds to $f_i$ $(i = 1, \ldots, r)$.

Now we prove an easy lemma concerning characteristic class functions on $G_o$.

LEMMA 2.2. Let $\{f_1, \ldots, f_r\}$ be the complete set of characteristic class functions on $G_o$. Then we have
\[ f_i \in \overline{Q}BR(G), \quad (i = 1, \ldots, r). \]

**Proof.** Let \( \hat{f}_i \) be a characteristic class function of \( G \) such that \( \hat{f}_i|_{G_a} = f_i \) where \( \hat{f}_i|_{G_a} \) indicates the restriction of \( f_i \) to \( G_a \). Then each \( \hat{f}_i \) is written as a \( \overline{Q} \)-linear combination of \( \chi_1, \cdots, \chi_h \). That is,

\[
\hat{f}_i = \sum_{j=1}^{h} (|G|/|G_a|) \chi_j(c_i) \chi_j, \quad (i = 1, \ldots, r)
\]

For each absolutely irreducible complex character \( \chi_i \) of \( G \), \( \chi_i|_{G_a} \) is written as a \( \mathbb{Z} \)-linear combination of \( \varphi_1, \cdots, \varphi_r \). That is,

\[
\chi_i|_{G_a} = \sum_{j=1}^{r} d_j \varphi_j, \quad (i = 1, \ldots, h)
\]

where \( (d_j) \) is the decomposition matrix of \( G \).

By virtue of the formulas (2.1) and (2.2), we can conclude that \( f_i \in \overline{Q}BR(G) \), \( (i = 1, \ldots, r) \) as required. \( \mathsf{Q.E.D.} \)

We are given two finite groups \( G \) and \( H \). For \( G \) and \( H \) we assume that there exists an isomorphism \( \lambda \) of \( \overline{Z}BR(G) \) onto \( \overline{Z}BR(H) \). Then it follows that the rank of \( BR(G) = \text{rank of } BR(H) \) and \( |Cl(G)| = |Cl(H)| \). We also can extend \( \lambda \) to an isomorphism \( \hat{\lambda} \) of \( \overline{Q}BR(G) \) onto \( \overline{Q}BR(H) \) by linearity. By Lemma 2.2 we have \( f_i \in \overline{Q}BR(G) \). Here we use the following additional notation.

\[
Cl(H_a) = [\mathbb{C}_1', \cdots, \mathbb{C}_s']
\]

\[ \{c'_1, \cdots, c'_s\} : \text{a complete set of representatives of } \mathbb{C}_1', \cdots, \mathbb{C}_s', \text{ respectively} \]

\[ \{f'_1, \cdots, f'_s\} : \text{the complete set of characteristic class functions on } H_a \text{ where } f'_i \text{ corresponds to } \mathbb{C}_i', \quad (i = 1, \ldots, r). \]

\[
IBr(H) = \{\varphi'_1, \cdots, \varphi'_s\}
\]

We now show a lemma which is actually the key step in the proof of Lemma 2.4.

**Lemma 2.3.** In the above situation, \( \hat{\lambda}(f_i) \) is a characteristic class function on \( H_a \), \( (i = 1, \ldots, r) \).

**Proof.** Since \( \overline{Q}BR(G)f_i = \overline{Q}f_i \equiv \overline{Q}, \overline{Q}BR(G)f_i \) is a minimal ideal of \( \overline{Q}BR(G) \) and so \( f_i \) is a (central) primitive idempotent, \( (i = 1, \ldots, r) \). Since \( \hat{\lambda}(f_i) \in \overline{Q}BR(H) \), we can write

\[
\hat{\lambda}(f_i) = \sum_{j=1}^{s} a_j f'_j, \quad a_j \in \overline{Q}
\]

Since \( f_i^2 = f_i \) and \( f_i f'_j = 0 \) \( (i \neq j) \), by the formula (2.3) we have

\[
\hat{\lambda}(f_i) = \sum_{j=1}^{s} a_j^2 f'_j.
\]
Thus $a_j^2 = a_j$, $(j = 1\ldots,r)$. Hence $a_j = 0$ or $a_j = 1$, $(j = 1\ldots,r)$. It follows that $\hat{\lambda}(f_j) = f_j'$ for some $j \in \{1,\ldots,r\}$, because $f_j$ is a primitive idempotent, hence the result.

Q.E.D.

Now we define a bijection from $Cl(G_o)$ to $Cl(H_o)$ through the isomorphism $\lambda$ as follows. For a $p$-regular conjugate class $\mathcal{C}_i$ of $G$, $\mathcal{C}_i$ corresponds to a characteristic class function $f_i$ on $G_o$. Since by Lemma 2.3 $\hat{\lambda}(f_i)$ is also a characteristic class function $f'_i$ on $H_o$, $\hat{\lambda}(f_i) = f'_i$ corresponds to a $p$-regular conjugate class $\mathcal{C}'_{i'}$ of $H$. Here we assign $\mathcal{C}'_{i'}$ to $\mathcal{C}_i$ ($i = 1,\ldots,r$). Thus we get a one-to-one correspondence between $Cl(G_o)$ and $Cl(H_o)$:

$$c_i \in \mathcal{C}_i \rightarrow f_i \rightarrow \hat{\lambda}(f_i) = f'_i \rightarrow \mathcal{C}'_{i'}$$

where $i \rightarrow i''$ ($i = 1\ldots,r$) is a permutation. In this case we write $\mathcal{C}_i \xrightarrow{\lambda} \mathcal{C}'_{i'}$ or $c_i \xrightarrow{\lambda} c'_{i'}$ ($i = 1,\ldots,r$).

Keeping the above notation, we give the following lemma concerning the Brauer character table of $G$. This lemma plays a fundamental role in the proof of Theorem 3.1. The proof is the same as that of Theorem 2.2 in [5] and so we omit its proof.

**Lemma 2.4.** $(\varphi_i(c_j)) = (\lambda(\varphi_i)(c'_{j''}))$ ($r \times r$ matrices) where $c_j \xrightarrow{\lambda} c'_{j''}$, $(j = 1,\ldots,r)$.

**3. Main theorems**

Let $G$ and $H$ be two finite groups with Cartan matrices $C$ and $C'$ respectively. Let $\lambda$ be an isomorphism of $\overline{ZBR}(G)$ onto $\overline{ZBR}(H)$ and $A = (a_{ij})$ be the matrix afforded by $\lambda$ with respect to $IBr(G) = \{\varphi_1,\ldots,\varphi_r\}$ and $IBr(H) = \{\varphi'_1,\ldots,\varphi'_r\}$. We set $Cl(G_o) = \{\mathcal{C}_1,\ldots,\mathcal{C}_s\}$ and $Cl(H_o) = \{\mathcal{C}'_1,\ldots,\mathcal{C}'_s\}$ and assume that $c_i \in \mathcal{C}_i$, $c'_i \in \mathcal{C}'_i$, and $c_i \xrightarrow{\lambda} c'_{i''}$ where $i \rightarrow i''$ ($i = 1,\ldots,r$) is a permutation. We write $m$ to denote the vector with $i$-th entry equal to $|C_i(c_i)|$ and $m'$ to denote the vector with $i$-th entry equal to $|C'_{i''}(c'_{i''})|$, $(i = 1,\ldots,r)$. Then we have the following two theorems.

**Theorem 3.1.** With the above notation, $m = m'$ iff $A'CA = C'$. This necessarily happens if $CA = AC'$, in which case $A$ is clearly unitary.

**Proof.** To prove this theorem, we introduce some simplifying notation: Write $P$ to denote the $r \times r$ matrix with $(i,j)$-entry equal to $\varphi_i(c_j)$ and similarly write $P'$ for the matrix with $(i,j)$-entry equal to $\varphi'_i(c'_{j''})$.

Since $\lambda(\varphi_i) = \sum_{k=1}^r a_k \varphi'_k$ where $A = (a_{ij})$, by Lemma 2.4 we have
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\[ \varphi_i(c_i) = \lambda(\varphi_i)(c'_{i'}) = \sum_{j=1}^r a_{ij} \varphi'_j(c'_{i'}) \]

This implies that \( P = AP' \). Also, if \( B \) is the diagonal matrix with \((i,i)\)-entry equal to \( |C_G(c_i)| \), it follows that \( P^*CP = B \) by Theorem 60.5 in [2]. Similarly \((P')^*C'P' = B'\), where \( B' \) is the diagonal matrix with \((i,i)\)-entry equal to \( |C_H(c'_i)| \). Here we note that \( B = B' \iff m = m' \). Since \( P^* = (P')^*A^* \), we have the two equations

\[(P')^*A^*CAP' = B \quad \text{and} \quad (P')^*C'P' = B'.\]

It is now obvious that \( B = B' \iff A^*CA = C' \).

Now suppose \( CA = AC' \). Then we show that \( A \) is unitary. If we write \( J = A^*A \), then we have \((P')^*JC'P' = B \). Thus \((B')^{-1}B = (P')^{-1}(C')^{-1}JC'P' \). This is a diagonal matrix with rational entries and this shows that \( J \) has rational eigenvalues. But \( J \) has algebraic integer entries, and so must have integer eigenvalues. Thus \((B')^{-1}B \) is a diagonal matrix with positive integer diagonal entries. Also, \( A \) is invertible over \( \mathbb{Z} \) and thus \( A^* \) is too. It follows that \( \det(J) = \det((B')^{-1}B) = 1 \) and so \((B')^{-1}B \) is the identity matrix \( I \). It follows that \( J = A^*A = I \) and so \( A \) is unitary, as required.

Q.E.D.

THEOREM 3.2. If \( CA = AC' \), then we have

(i) \( \lambda(\varphi_i) = \varepsilon_i \varphi'_i \) where the \( \varepsilon_i \) are roots of \( 1 \) and \( i \rightarrow i' \) \( (i = 1, \ldots, r) \) is a permutation.

(ii) The Brauer character tables of \( G \) and \( H \) are the same.

PROOF. (i) Now we pay attention to the fact that if \( \alpha \in \overline{\mathbb{Z}} \) and \( |\alpha^\sigma| \leq 1 \) (an absolute value) for all \( \sigma \in G(\overline{\mathbb{Q}}/\mathbb{Q}) \), then \( \alpha = 0 \) or \( \alpha \) is a root of \( 1 \).

If we use the same notation as in the proof of Theorem 3.1, then we have \( A = P(P')^{-1} \) and so \( A \) has entries that lie in a field with an abelian Galois group. Thus \( (A^\sigma)^{\sigma} = (A^{\sigma'}) \) for all \( \sigma \in G(\overline{\mathbb{Q}}/\mathbb{Q}) \). Since \( A \) is unitary by Theorem 3.1, \( A^\sigma \) is automatically unitary for all \( \sigma \in G(\overline{\mathbb{Q}}/\mathbb{Q}) \). Hence we have the equation with respect to the \( i \)-th row of \( A^\sigma \).

\[ \sum_{j=1}^r a_{ij}^\sigma \overline{a_{ij}^\sigma} = \sum_{j=1}^r |a_{ij}^\sigma|^2 = 1, \quad (i = 1, \ldots, r) \]

Hence we have \( |a_{ij}^\sigma| \leq 1 \) for all \( \sigma \in G(\overline{\mathbb{Q}}/\mathbb{Q}) \). This implies that \( a_{ij} = 0 \) or \( a_{ij} \) is a root of \( 1 \) because of the above attention. Thus it follows that for each \( i \in \{1, \ldots, r\} \), there exists \( i' \in \{1, \ldots, r\} \) such that \( a_{ii'} \) is a root of \( 1 \) and \( a_{ij} = 0 \) \( (j \neq i') \). Hence \( \lambda(\varphi_i) = \varepsilon_i \varphi'_i \) where \( \varepsilon_i = a_{ii'} \) is a root of \( 1 \) and \( i \rightarrow i' \) \( (i = 1, \ldots, r) \) is a permutation.

(ii) We state a one-to-one correspondence \( \mu \) between \( IBr(G) \) and \( IBr(H) \).
through the isomorphism $\lambda$ as follows. By (i) of this theorem, we have $\lambda(\varphi_i) = \varepsilon_i \varphi'_i$ ($i = 1, \cdots, r$) where the $\varepsilon_i$ are roots of 1. Here we assign $\varphi'_i$ to $\varphi_i : \mu(\varphi_i) = \varphi'_i$ ($i = 1, \cdots, r$). Then $\mu$ can be extended to an isomorphism of $BR(G)$ onto $BR(H)$ by linearity. (See the proof of Lemma 3.2 in [5]) By Lemma 2.4 we have $\lbrack \varphi_i(c_j) \rbrack = \lbrack \varphi'_i(c'_{j'}) \rbrack$ ($r \times r$ matrices) where $c_j \rightarrow c'_{j'}$ ($j = 1, \cdots, r$). That is, $G$ and $H$ have the same Brauer character table. Thus the result follows. Q.E.D.

**Remark.** If the condition $m = m'$ in Theorem 3.1 holds, then we can easily prove $|G| = |H|$. But we can give examples such that for two finite groups $G$, $H$ with $|G| \neq |H|$, a matrix $A$ is unitary where $A$ is afforded by an isomorphism of $BR(G)$ onto $BR(H)$. Actually, such an example is given by taking $G$ and $H$ to be any two $p$-groups of different orders. Another example can be found in [1]. ($p = 2$, $G =$ the symmetric group $S_4$ on 4 symbols and $H =$ the dihedral group $D_6$ of order 12. See the examples of section 91 in [1]).

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**References**


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