APPROXIMATIVE SHAPE IV
—UVⁿ-MAPS AND THE VIETORIS-SMALE THEOREM—

Dedicated to Professor Yukihiro Kodama on his sixtieth birthday

By
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§ 0. Introduction.

This paper is a continuation of [35–37]. We introduced approximate shape in [35], discussed approximative shape properties of spaces and generalized ANRs in [36], and fixed point theorems in [37]. In this paper we investigate approximative shape properties of maps and show the Vietoris-Smale theorem in shape theory.

Many mathematicians studied UVⁿ-maps. See the references of Lacher [18] for their studies. Smale [30] gave a Vietoris type theorem for homotopy groups and UVⁿ-maps, called the Vietoris-Smale theorem. Kozlowski [13] gave a factorization theorem for UVⁿ-maps. Borsuk introduced approximatively n-connected spaces. This is a basic notion in shape theory. Various Vietoris-Smale theorems in shape theory were given by Bogatyi [2, 3], Dydak [4–7], Kodama [11, 12], Kuperberg [16], Kozlowski-Segal [15] and Morita [27, 28].

In this paper we discuss the following topics: In § 1 we introduce the approximative lifting property and investigate its properties. In § 2 we prove restriction and product theorems for the approximative lifting property. In § 3 we introduce approximatively n-connected maps and give their characterizations. We show the Vietoris-Smale theorem and the Whitehead theorem for approximatively n-connected maps. In § 4 we introduce the approximative extension property. We characterize approximatively n-connected spaces by this property. In § 5 we introduce partial realizations for decomposition spaces. We introduce the approximative full extension property and investigate its properties. In § 6 we show that our approximatively n-connected maps and usual UVⁿ-maps are equivalent. Hence by using results in § 3 we show the Vietoris-Smale theorem and the Whitehead theorem in shape theory for closed UVⁿ-maps between paracompacta.

We assume that the reader is familiar with theory of ANRs and shape theory. As reference books we use Hu [10] for theory of ANRs and Mardešić.

and Segal [23], quoted by MS [23], for shape theory. For undefined notations and terminology see these books. We use the same notations and terminology as in [35-37]. We quote results in [35-37] as follows: For example (I.3.3), (II.5.5) and (III.1.1) denote theorem (3.3) in [35], theorem (5.5) in [36] and theorem (1.1) in [37], respectively.

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§1. Approximative lifting property.

In this section we introduce the approximative lifting property for maps and discuss its properties.

Let \( \mathcal{J} \) be a collection of pairs of spaces. Let \( C \) be a subcategory of \( \text{TOP} \). Let \( (X, \mathcal{U}) = \{(X_a, \mathcal{U}_a), p_a, a, A\} \) and \( (\mathcal{V}) = \{(Y_b, \mathcal{V}_b), q_b, h, B\} \) be approximative inverse systems in \( C \). Let \( f = \{f, f_b : b \in B\} : (X, \mathcal{U}) \rightarrow (\mathcal{V}) \) be an approximative system map in \( C \). We say that \( f \) has the approximative lifting property, in notation ALP, with respect to \( \mathcal{J} \) provided it satisfies the following condition:

(ALP) For each admissible pair \((a, b)\) of \( f \) there exists an admissible pair \((a_b, b_b) > (a, b)\) with the following property; for each \((K, K_b) \in \mathcal{K} \) and for any maps \( g : K \rightarrow Y_{b_b} \), \( h : K_b \rightarrow X_{a_b} \) with \((f_{b_b} p_{a_b} f_{c_b} h, g|K_b) < s t\mathcal{V}_{b_b}\), then there exists a map \( H : K \rightarrow X_a \) such that \((p_{a_b} a, H|K_b) < s t\mathcal{U}_a\) and \((q_{b_b} g, f_{b_b} p_{a_b} f_{c_b} H) < s t\mathcal{V}_{b_b}\).

(1.1) LEMMA. Let \( f, g : (X, \mathcal{U}) \rightarrow (\mathcal{V}) \) be approximative system maps in \( C \). If \( f = : g \) and \( f \) has ALP with respect to \( \mathcal{K} \), then so does \( g \).

PROOF. We put \( g = \{g, g_b : b \in B\} \). We may assume \( f = : g \) without loss of generality. Take any admissible pair \((a, b)\) of \( g \). By (A13) there exists \( b_1 > b \) such that \( q_{b_1} \mathcal{V}_{b_1} > s t\mathcal{V}_{b_1} \). Since \( f = : g \) and \( b_1 > b \), there exists \( a_1 > a, f(b), f(b_1) \) such that

1. \((g_{b_1} p_{a_1}, g(b_1), f_{b_1} p_{a_1}, f(b_1)) < \mathcal{V}_{b_1} \) and
2. \((f_{b_1} p_{a_1}, f(b_1), q_{b_1} h f_{b_1} p_{a_1}, f(b_1)) < \mathcal{V}_{b_1} \).

By the assumption there exists an admissible pair \((a_2, b_2) > (a_1, b_1)\) satisfying ALP with respect to \( \mathcal{K} \) for \( f \) and \((a_1, b_1)\). By (A13) there exists \( b_2 > b_2 \) such that \( q_{b_2} \mathcal{V}_{b_2} > s t\mathcal{V}_{b_2} \). Since \( f = : g \), there exists \( a_2 > f(b_2), g(b_2) \) such that

3. \((f_{b_2} p_{a_2}, f(b_2), g_{b_2} p_{a_2}, g(b_2)) < \mathcal{V}_{b_2} \).
We show that \((a_2, b_2)\) is the required admissible pair for \(g\). Take any \((K, K_0) \in \mathcal{K}\) and any maps \(s : K \rightarrow Y_{b_2}, h : K_0 \rightarrow X_{a_2}\) such that

\[(g_{b_2}p_{a_2}, f(b_2)h, q_{b_2}b_2s | K_0) < s \in \mathcal{C}V_{b_2} \]  

By (3)-(5) and the choice of \(b_3, (f_{b_3}p_{a_3}, f(b_3)h, q_{b_3}b_3s | K_0) < s \in \mathcal{C}V_{b_3}\). Thus by the choice of \((a_3, b_3)\) there exists a map \(H : K \rightarrow X_{a_3}\) such that

\[(p_{a_3}a_3H, H | K_0) < s \in \mathcal{U}_{a_3} \text{ and } (f_{b_3}p_{a_3}, f(b_3)h, q_{b_3}b_3s | K_0) < s \in \mathcal{C}V_{b_3}. \]

By (1), (2), (6) and the choice of \(a_2, b_2\) there exists a map \(H : K \rightarrow X_{a_2}\) such that

\[(p_{a_2}a_2H, H | K_0) < s \in \mathcal{U}_{a_2} \text{ and } (f_{b_2}p_{a_2}, f(b_2)h, q_{b_2}b_2s | K_0) < s \in \mathcal{C}V_{b_2}. \]  

By the assumption there exists an admissible pair \((b_3, c_3)\) satisfying ALP for \(g\) and \((b_2, u(c_3))\). Take any \(c_3 > c_2\) such that \(r_{c_3, c_2}W_{c_2} > s \in \mathcal{W}_{b_2}\). Hence \(g\) has ALP with respect to \(\mathcal{K}\).

Thus by (1.1) we may say that \([f]\) has ALP with respect to \(\mathcal{K}\) provided that \(f\) has ALP with respect to \(\mathcal{K}\).

(1.2) **Lemma.** Let \(f : (X, U) \rightarrow (\mathcal{C}, \mathcal{V})\) and \(g : (\mathcal{Y}, \mathcal{V}) \rightarrow (Z, W)\) be approximative system maps in \(C\). If \([f]\) and \([g]\) have ALP with respect to \(\mathcal{K}\), then so does \([gf]\).

**Proof.** We put \(g = \{g, g_c : c \in C\} : (\mathcal{Y}, \mathcal{V}) \rightarrow (Z, W) = \{(Z_c, W_c), r_c, c, C\}\) and take a 1-refinement function \(u : C \rightarrow C\) of \((Z, W)\). We show that \(r(u)(gf)\) has ALP. Take any admissible pair \((a, c)\) of \(r(u)(gf)\) and take \(c_1 > c\) such that \(r_{c_1, c}W_{c} > s \in \mathcal{W}_{c_1}\). Since \(r(u)(gf)\) is an approximative system map, there exists \(a_1 > a, fg(u(c_1))\) such that

\[(r_{u(c_1), e}g_{u(c_1)}f_{u(c_1)}p_{a_1}, f_{u(c_1)}e_{u(c_1)}f_{u(c_1)}p_{a_1}, e_{u(c_1)}f_{u(c_1)}p_{a_1}) < \mathcal{C}V_{c_1}. \]

By the assumption there exists an admissible pair \((a_3, b_1) > (a_1, g_u(c_1))\) satisfying ALP for \(f\) and \((a_1, g_u(c_1))\). Take any \(b_3 > b_1\) such that \(q_{b_3}b_1 < s \in \mathcal{V}_{b_3}\). By the assumption there exists an admissible pair \((b_3, c_3) > (b_2, u(c_1))\) satisfying ALP for \(g\) and \((b_2, u(c_1))\). Take any \(c_3 > c_2\) such that \(r_{c_3, c_2}W_{c_2} > s \in \mathcal{W}_{b_2}\). By (AM2) there exist \(b_3 > b_2, g_u(c_3)\) and \(a_2 > a_1, f_{u(c_3)}g_u(c_3), f(b_3)\) such that

\[(g_{c_3}b_{c_3}, c_{c_3}), r_{u(c_3), e_{c_3}}g_{u(c_3)}e_{c_3}g_{u(c_3)}q_{b_{c_3}}e_{c_3}g_{u(c_3)}r_{c_3, c_2}W_{c_2}, \]

\[(f_{u(c_3)}p_{a_3}, f_{u(c_3)}e_{u(c_3)}q_{b_{c_3}}e_{c_3}g_{u(c_3)}f_{u(c_3)}p_{a_1}, f_{u(c_3)}e_{u(c_3)}f_{u(c_3)}p_{a_1}) < \mathcal{C}V_{e_{c_3}} \text{ and } \]

\[(f_{b_3}p_{a_4}, f_{b_3}e_{b_3}q_{b_3}, b_{b_3}f_{b_3}p_{a_4}, f_{b_3}e_{b_3}f_{b_3}p_{a_4}) < \mathcal{C}V_{b_3}. \]

We show that the admissible pair \((a_3, c_3)\) of \(r(u)(gf)\) has the required property. Take any \((K, K_0) \in \mathcal{K}\) and maps \(s : K \rightarrow Z_{c_3}, h : K_0 \rightarrow X_{a_3}\) such that
(r_{u(c_2)}, c_2 g_{u(c_2)} f_{u(c_2)} p_{a, f_{u(c_2)} h}, s | K_0) < st' W_{c_2}. Then by (2), (3), (AM1) for \( g \) and the choice of \( c_2 \) (\( g e g_0, g c_2 f_{u(c_2)} h, r_{c_2}, c_2 s | K_0) < st' W_{c_2}. Thus by the choice of \( (b_2, c_2) \) there exists a map \( H_1 : K \rightarrow Y_{b_2} \) such that

\[
(q_{b_1, b_2 f_{b_2}} p_{a, f_{(c_0)} h}, H_1 | K_0) < st' U_{b_2} \quad \text{and}
\]

\[
(g u(c_1) g_{b_2, g u(c_1)} H_1, r_{c_2, u(c_1)} s) < st' W_{u(c_1)}.
\]

By (4), (5) and the choice of \( b_2 \) (\( f_{b_2} p_{a, f_{(c_0)} h}, q_{b_2, b_1} H_1 | K_0) < st' U_{b_1}. Then by the choice of \( (a_3, b_1) \) there exists a map \( H_2 : K \rightarrow X_{a_1} \) such that

\[
(p_{a_2, a_1} h, H_2 | K_0) < st' U_{a_1}
\]

and

\[
(f g_{u(c_1)} p_{a_2, f_{u(c_1)} H_2}, q_{b_2, u(c_1)} H_1) < st' U_{u(c_1)}.
\]

By (1), (6), (8) and the choice of \( c_1 \) (\( p_{a_2, a_1} h, H_1 | K_0) < st' U_{a_1}. Thus \( r(u)(gf) \) has ALP with respect to \( \mathcal{K} \) and hence \([g][f] \) has the required property ALP.

(1.3) Lemma. If \([f'] : (\mathcal{X}, \mathcal{U}) \rightarrow (\mathcal{Y}, \mathcal{V})\) is an isomorphism in Appro-\( \mathcal{C} \), then \([f'] \) has ALP with respect to any collection of pairs.

(1.3) follows from (1.2.16).

Let \( X \) and \( Y \) be spaces and \( f : X \rightarrow Y \) a map. Let \( p : X \rightarrow (\mathcal{X}, \mathcal{U}), p' : X \rightarrow (\mathcal{X}, \mathcal{U})' \), \( q : Y \rightarrow (\mathcal{Y}, \mathcal{V}), q' : Y \rightarrow (\mathcal{Y}, \mathcal{V})' \) be approximative AP-resolutions. Let \( f : (\mathcal{X}, \mathcal{U}) \rightarrow (\mathcal{Y}, \mathcal{V}) \) and \( f' : (\mathcal{X}, \mathcal{U})' \rightarrow (\mathcal{Y}, \mathcal{V})' \) be approximative resolutions of \( f \) with respect to \( p, q \) and with respect to \( p', q' \), respectively. By (1.5.1) \([f'] = [1_y]_{q'} [f'] [1_x]_{p'} \). Since \([1_x]_{p', p} \) and \([1_y]_{q, q'} \) are isomorphisms in Appro-AP by (1.5.1), we have the following from (1.2) and (1.3):

(1.4) Lemma. If \([f'] \) has ALP with respect to \( \mathcal{K} \), then so does \([f'] \).

Let \( p = \{p_a : a \in A\} : X \rightarrow (X_a, p_{a', a}, A) \) and \( q = \{q_b : b \in B\} : Y \rightarrow (Y_b, q_{b', b}, B) \) be AP-resolutions. Let \( f = \{f, f_b : b \in B\} : \mathcal{X} \rightarrow \mathcal{Y} \) be a resolution of \( f : X \rightarrow Y \) with respect to \( p \) and \( q \). We say that \((f, p, q)\) has the approximative lifting property, in notation ALP, with respect to \( \mathcal{K} \) provided it satisfies the following condition:

(\( \text{ALP}^* \)) For each admissible pair \((a, b)\) of \( f \) and for any \( \mathcal{U} \in C_0(X_a), \mathcal{V} \in C_0(Y_b) \) there exist an admissible pair \((a_0, b_0)\rangle (a, b) \) and \( \mathcal{U}_0 \in C_0(Y_{b_0}) \) with the following property; for any \((K, K_0) \in \mathcal{K} \) and any maps \( h : K \rightarrow X_{a_0}, g : K \rightarrow Y_{b_0} \) with \((f_{b_0} p_{a_0, f_{(0)}} h, g | K_0) < \mathcal{V}_0 \), there exists a map \( H : K \rightarrow X_a \) such that \((p_{a_0, a} h, H | K_0) < \mathcal{U} \) and \((f_b p_{a, f_{(0)}} H, q_{b_0, b} g) < \mathcal{V}.\)
Lemma. Let \((f, p, q)\) and \((g, r, s)\) be AP-resolutions of \(f : X \to Y\). If \((f, p, q)\) has ALP with respect to \(x\), then so does \((g, r, s)\).

Proof. We put \(r = \{r_c : c \in C\} : X \to \mathcal{R} = \{R_c, r'_{c, c}, C\}\), \(s = \{s_d : d \in D\} : Y \to S = \{S_d, s'_d, d, D\}\) and \(g = \{g, g_d : d \in D\}\). We assume that \((f, p, q)\) satisfies ALP and show that \((g, r, s)\) satisfies ALP.

Take any admissible pair \((c, d)\) of \(g\) and \(\mathcal{W} \subseteq Cov(R_c), \mathcal{L} \subseteq Cov(S_d)\). There exist \(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4 \subseteq Cov(S_d)\) such that \(st \mathcal{L}_1 < \mathcal{L}, \mathcal{L}_2 \) satisfies (R2) for \(p\) and \(\mathcal{L}_1, \mathcal{L}_3 \) satisfies (R2) for \(s\) and \(\mathcal{L}_1, \mathcal{L}_4 \) satisfies (R2) for \(r\) and \(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4 \) satisfies (R2) for \(r\) and \(\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3 \subseteq Cov(R_c)\) such that \(st \mathcal{W}_1 < \mathcal{W}, \mathcal{W}_2 \) satisfies (R2) for \(r\) and \(\mathcal{W}_1, \mathcal{W}_3 \subseteq Cov(S_d)\) such that \(st \mathcal{W}_3 < \mathcal{W}_1 \wedge \mathcal{W}_2 \wedge (g_d r_{c, d, a, d})^{-1} \mathcal{L}_4\). By (R1) there exist \(b_1 \subseteq B\) and a map \(t : Y_{b_1} \to S_d\) such that

\[(s_d, tq) < \mathcal{L}_4\]

and there exist \(a_1 > f(b_1)\) and a map \(u : X_{a_1} \to R_c\) such that

\[(r, u) < \mathcal{W}_3\]

Then by the choice of \(\mathcal{W}_3\)

\[(g r_{c, d, a, a, a}, t f_{b_1, a, s}) < \mathcal{L}_4\]

By the assumption there exist an admissible pair \((a_3, b_3) > (a_1, b_1)\) and \(\mathcal{W}' \subseteq Cov(Y_{b_3})\) satisfying (ALP)* for \(f\), \((a_2, b_2)\), \((u p_{a_2, a_2})^{-1} \mathcal{W}_1 \subseteq Cov(X_{a_2})\) and \(t^{-1} \mathcal{L}_4 \subseteq Cov(Y_{b_3})\). There exist \(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3 \subseteq Cov(Y_{b_3})\) such that \(st \mathcal{V}_2 < \mathcal{V}'\), \(\mathcal{V}_2\) satisfies (R2) for \(r\) and \(\mathcal{V}_1\), and \(st \mathcal{V}_2 \wedge (t q_{b_2, b_1})^{-1} \mathcal{L}_4\). By (R1) there exist \(d_1 > d\) and a map \(v : S_{a_1} \to Y_{b_3}\) such that

\[(v s_{a_1}, q) < \mathcal{V}'\]

By (5) and the choice of \(\mathcal{V}_2\), \((t q_{b_2, b_1} v s_{a_1}, t q_{b_1}) < \mathcal{L}_4\) and then by (1), \((t q_{b_2, b_1} v s_{a_1}, s_{a_1} a s_{a_1}) < st \mathcal{L}_4 < \mathcal{L}_3\). By the choice of \(\mathcal{L}_3\) there exists \(d_2 > d_1\) such that

\[(t q_{b_2, b_1} v s_{a_1}, s_{a_1} a s_{a_1}) < \mathcal{L}_1\]

By (R1) there exist \(c_1 > c\) and a map \(w : R_{c_1} \to X_{a_3}\) such that

\[(w r_{c_1}, p_{a_2}) < (u p_{a_2, a_2})^{-1} \mathcal{W}_1 \wedge (f_{b_2, b_1} f_{b_1, a, s})^{-1} \mathcal{V}_2\].
By (5), \((\mathcal{V}_{d_2}, f, q_{b_2})^{\sim} \mathcal{V}'_2\). By (7), \((f_{b_2}P_{a_2}, f(a_2))^{\sim} \mathcal{V}'_2\). Since \((f, p, q)\) and \((g, r, s)\) are resolutions of \(f, f_{b_2}P_{a_2}, f(a_2) = q_{b_2}f\) and \(s_{a_2}, f = s_{a_2}, d_1, g_{a_2}, f(a_2)\). Then \((f_{b_2}P_{a_2}, f(a_2))^{\sim} \mathcal{V}'_2\). By the choice of \(\mathcal{V}'_2\) there exists \(c_2 > c_1\) such that

\[(8) \quad (f_{b_2}P_{a_2}, f(a_2))^{\sim} \mathcal{V}'_2 < \mathcal{V}'_2.\]

By (7), \((u_{a_2}, a_1)^{\sim} \mathcal{W}_3\) and by (2), \((u_{a_2}, a_1)^{\sim} \mathcal{W}_3 < \mathcal{W}_3.\)

By the choice of \(\mathcal{W}_3\), there exists \(c_3 > c_2\) such that

\[(9) \quad (u_{a_2}, a_1)^{\sim} \mathcal{W}_3 < \mathcal{W}_3.\]

We show that the admissible pair \((c_3, d_2)\) for \(g\) and \((u_{a_2}, a_1)^{\sim} \mathcal{V}'_2 \in \mathcal{C}_{\mathcal{V}_3}(S_{d_2})\) are the required ones. Take any \((K, K_0) \in \mathcal{K}\) and any maps \(k : K_0 \rightarrow R_{d_1}, h : K \rightarrow S_{d_2}\) such that \((g_{a_2}r_{c_2}, g(a_2)k, h \mid K_0) < (u_{a_2}, a_1)^{\sim} \mathcal{V}'_2\), and then \((u_{a_2}, a_1)^{\sim} \mathcal{W}_3 < \mathcal{W}_3.\)

By (8), \((f_{b_2}P_{a_2}, f(a_2))^{\sim} \mathcal{V}'_2\). By the choice of \((a_2, b_2)\) there exists a map \(H : K \rightarrow X_{d_2}\) such that

\[(10) \quad (u_{a_2}, a_1)^{\sim} \mathcal{W}_3 < \mathcal{W}_3.\]

By (10), \((u_{a_2}, a_1)^{\sim} \mathcal{W}_3 < \mathcal{W}_3.\) and by (9), \((u_{a_2}, a_1)^{\sim} \mathcal{W}_3 < \mathcal{W}_3.\)

By (10), \((u_{a_2}, a_1)^{\sim} \mathcal{W}_3 < \mathcal{W}_3.\) and by (9), \((u_{a_2}, a_1)^{\sim} \mathcal{W}_3 < \mathcal{W}_3.\)

Then

\[(12) \quad (r_{c_3}, k, u_{a_2}, a_1)^{\sim} \mathcal{W}_3 < \mathcal{W}_3.\]

By (11), \((tq_{b_2}, a_1)^{\sim} \mathcal{L}_1 < \mathcal{L}_1.\) By (4), \((g_{a_2}r_{c_3}, a_1)^{\sim} \mathcal{W}_3 < \mathcal{L}_1.\) Thus

\[(13) \quad (g_{a_2}r_{c_3}, a_1)^{\sim} \mathcal{W}_3 < \mathcal{L}_1.\]

Hence, by (12) and (13), \(g\) satisfies \((ALP)^*\) with respect to \(\mathcal{K}\).

By (1.4.9) there exist approximative ANR-resolutions \(p : X \rightarrow (\mathcal{X}, \mathcal{U}), q : Y \rightarrow (\mathcal{Y}, \mathcal{V})\) and an approximative resolution \(f : (\mathcal{X}, \mathcal{U}) \rightarrow (\mathcal{Y}, \mathcal{V})\) of \(f\) with respect to \(p\) and \(q\) such that \((f, p, q)\) is an ANR-resolution of \(f\).

(1.6) LEMMA. Under the above conditions, \(f\) satisfies \((ALP)\) with respect to \(\mathcal{K}\) iff \((f, p, q)\) satisfies \((ALP)^*\) with respect to \(\mathcal{K}\).

PROOF. We assume that \(f\) satisfies \((ALP)\) and show that it satisfies \((ALP)^*\).

Take any admissible pair \((a, b)\) of \(f, \mathcal{U} \in \mathcal{C}_{\mathcal{V}_3}(X_a)\) and \(\mathcal{V} \in \mathcal{C}_{\mathcal{V}_3}(Y_a)\). There exist \(b_1 > b, \ a_1 > a, \ f(b_1)\) such that \(p_{a_1}^{-1}(x) > st \mathcal{U}_{a_1}, q_{b_1}^{-1}(\mathcal{V}) > st \mathcal{V}_{b_1}\) and \(fP_{a_2}, f(b_1) = q_{b_1}^{-1}(\mathcal{V})^{-1} \mathcal{V}_{b_1}\). There exists an admissible pair \((a_2, b_2) > (a_1, b_1)\) satisfying
(ALP) for \((a_1, b_1)\). Thus it is easy to show that \((a_2, b_2)\) and \(sU_{b_2}\) satisfy (ALP)* for \((a, b)\), \(U\) and \(U'\).

Next we assume that \(f\) satisfies (ALP)* and show that it satisfies (ALP). Take any admissible pair \((a, b)\) of \(f\). Then there exist an admissible pair \((a_1, b_1)\geq(a, b)\) and \(U'\subseteq Cov(Y_{b_1})\) satisfying (ALP)* for \((a, b)\), \(U_a\) and \(U_{b_1}\). Take \(b_1>b_2\) and \(a_2>a_1\), \(f(b_2)\) such that \(q_{b_2}^{-1} U'\subseteq sU_{b_2}\) and \(f_1, p_{\alpha_2}, f(\alpha_1) = q_{b_2}^{-1} f_2 p_{\alpha_2}, f(\alpha_2)\). It is easy to show that \((a_2, b_2)\) satisfies (ALP) for \((a, b)\).

**1.7** Theorem. Let \(\mathcal{K}\) be a class of pairs. Let \(f: X\to Y\) be a map. Then the following statements are equivalent:

(i) Any/some approximative AP-resolution of \(f\) has ALP with respect to \(\mathcal{K}\).

(ii) Any/some AP-resolution of \(f\) has ALP with respect to \(\mathcal{K}\).

(1.7) follows from (1.4)-(1.6).

Thus by (1.7) we say that a map \(f: X\to Y\) has the approximative lifting property, in notation ALP, with respect to \(\mathcal{K}\) provided that it satisfies any one of the conditions in (1.7).

**1.8** Corollary. Let \(f: X\to Y\) and \(g: Y\to Z\) be maps.

(i) If \(CT(f): CT(X)\to CT(Y)\) is a homeomorphism, then \(f\) has ALP with respect to any \(\mathcal{K}\).

(ii) If \(f\) is a homeomorphism, then \(f\) has ALP with respect to any \(\mathcal{K}\).

(iii) If \(f\) and \(g\) have ALP with respect to \(\mathcal{K}\), then so does \(gf\).

**1.9** Corollary. A map \(f: X\to Y\) has ALP with respect to \(\mathcal{K}\) iff so does \(CT(f): CT(X)\to CT(Y)\).

(i) in (1.8) follows from (1.7.10) and (1.3). (ii) follows from (i). (iii) follows from (1.2). (1.9) follows from (1.6.9), (1.6.11), (1.7.10), (1.2) and (1.3.).

Let \(\text{TOP}(\text{ALP}, \mathcal{K})\) be the subcategory of \(\text{TOP}\) consisting of all spaces and all maps having ALP with respect to \(\mathcal{K}\). Thus by (1.8) we have the following:

**1.10** Theorem. \(\text{TOP}(\text{ALP}, \mathcal{K})\) forms a subcategory of \(\text{TOP}\) including all homeomorphisms for any collection \(\mathcal{K}\) of pairs.

Let \(X_0\) be a subspace of a space \(X\). We say that \((X, X_0)\) is a closed pair provided that \(X_0\) is closed in \(X\). We recall that coverings are always normal open coverings (see [1]). We say that \(X_0\) is \(P\)-embedded in \(X\) provided that for any covering \(U_0\) of \(X_0\) there exists a covering \(U\) of \(X\) such that \(U|_{X_0} =\)
\[ \{ U \cap X_\delta : U \subseteq \mathcal{U} \} < \mathcal{U}_\delta. \] In MS [23] "normally embedded" is used instead of "P-embedded".

Let \( \text{TOP}_{\text{pairs}} \) be the category consisting of all pairs and all map between pairs. \( \text{TOP}_{\text{C-pairs}} \) and \( \text{TOP}_{\text{P-pairs}} \) denote the full subcategories of \( \text{TOP}_{\text{pairs}} \) consisting of all closed pairs and all P-embedded pairs, respectively. \( \text{TOP}_{\text{cp-pairs}} \) denotes the full subcategory consisting of all closed P-embedded pairs. \( \text{TOP}_{\text{pairs}}^n \) and \( \text{TOP}_{\text{pairs}}^\ast \) denote the full subcategories of \( \text{TOP}_{\text{pairs}} \) consisting of all pairs \((X, X_\delta)\) with \( \dim X \leq n \) and \( \dim(X - X_\delta) \leq n \), respectively.

We say that \((X, X_\delta)\) is a polyhedral pair provided that there exists a simplicial complex \( K \) and a subcomplex \( K_\delta \) of \( K \) such that \( |K| = X \) and \( |K_\delta| = X_\delta \). Here \( |K| \) denotes the realization of \( K \) endowed with CW-topology. We say that \((X, X_\delta)\) is an ANR-pair provided that \( X_\delta \) is closed in \( X \) and \( X, X_\delta \) are ANRs. \( \text{POL}_{\text{pairs}} \) and \( \text{ANR}_{\text{pairs}} \) denote the full subcategories of \( \text{TOP}_{\text{cp-pairs}} \) consisting of all polyhedral pairs and ANR-pairs, respectively. Similarly, we may define categories \( \text{POL}_{\text{pairs}}^n \), \( \text{POL}_{\text{pairs}}^\ast \), \( \text{ANR}_{\text{pairs}}^n \), \( \text{ANR}_{\text{pairs}}^\ast \) and so on.

Let \( K_1 \) and \( K_2 \) be subcategories of \( \text{TOP}_{\text{pairs}} \). We say that \( K_1 \) is expandable by \( K_2 \), in notation \( K_1 <_e K_2 \), provided that any \((X, X_\delta) \in \text{Ob} K_1 \) admits a \( K_2 \)-resolution. We say that \( K_1 \) and \( K_2 \) are expansively equivalent, in notation \( K_1 =_e K_2 \), provided that both \( K_1 <_e K_2 \) and \( K_2 <_e K_1 \). We say that \( f : (X', \mathcal{U}) \to (\mathcal{Y}, \mathcal{V}) \) and \( f : X \to Y \) have ALP with respect to \( K_1 \) provided that they have ALP with respect to \( \text{Ob} K_1 \), respectively. \[(1.3.15) \text{ and (1.3.16) (see [21] and [23]) mean the following:}\]

\begin{equation}
(1.11) \text{LEMMA.} \quad \text{TOP}_{\text{pairs}} =_e \text{TOP}_{\text{P-pairs}} =_e \text{POL}_{\text{pairs}} =_e \text{ANR}_{\text{pairs}}, \quad \text{TOP} =_e \text{POL} =_e \text{ANR}, \quad \text{TOP}_{\text{pairs}}^n =_e \text{TOP}_{\text{P-pairs}}^n =_e \text{POL}_{\text{pairs}}^n \text{ and } \text{TOP}^n =_e \text{POL}^n. \quad \blacksquare
\end{equation}

\begin{equation}
(1.12) \text{THEOREM.} \quad \text{Let } K_1 \text{ and } K_2 \text{ be subcategories of } \text{TOP}_{\text{pairs}}. \quad \text{We assume that } K_1 <_e K_2. \quad \text{If a map } f \text{ has ALP with respect to } K_2, \text{ then } f \text{ has ALP with respect to } K_1.
\end{equation}

To prove (1.12) we need the following lemma.

\begin{equation}
(1.13) \text{LEMMA.} \quad \text{Let } Z \text{ be an AP. Then for each } \mathcal{U} \subseteq \text{Cov}(Z) \text{ there exists } \mathcal{U}' \subseteq \text{Cov}(Z) \text{ satisfying (R2) for any resolution and } \mathcal{U}.
\end{equation}

\begin{proof}
For any \( \mathcal{U} \subseteq \text{Cov}(Z) \) we take \( \mathcal{U}' \subseteq \text{Cov}(Z) \) with \( s \mathcal{U}' < \mathcal{U} \). We show that \( \mathcal{U}' \) has the required one. Take any resolution \( p : X \to \mathcal{X} \) of any space \( X \). Take any \( a \in A \) and any maps \( g, h : X_a \to Z \) such that \( (g \circ p_a, h \circ p_a) < \mathcal{U}' \). Put \( \mathcal{W} = g^{-1} \mathcal{U}' \setminus h^{-1} \mathcal{U}' \subseteq \text{Cov}(X_a) \).
\end{proof}
Approximative Shape IV

(1) \( (g \mid st(p_a(X), \mathcal{W}), h \mid st(p_a(X), \mathcal{W})) < \mathcal{U} \).

Take any \( y \in st(p_a(X), \mathcal{W}) \) and then there exist \( U_1', U_2' \subseteq \mathcal{U} \) and \( x \in X \) such that \( p_a(x) \in p_a(X) \cap g^{-1} U_1' \cap h^{-1} U_2' \) and \( y \in g^{-1} U_1' \cap h^{-1} U_2' \). Thus \( g p_a(x), g(y) \subseteq U_1', h p_a(x), h(y) \subseteq U_2' \). Since \( (g p_a, h p_a) < \mathcal{U} \), there exists \( U_3' \subseteq \mathcal{U} \) such that \( g p_a(x), h p_a(x) \subseteq U_3' \). Thus \( g(y), h(y) \subseteq st(U_3', \mathcal{W}) \subseteq U \) for some \( U \in \mathcal{U} \). Hence we have (1).

By (1.3.3) \( p \) satisfies (B4) and then there exists \( a' > a \) such that \( p_{a', a}(X_{a'}) \subseteq st(p_a(X), \mathcal{W}) \). Hence by (1) \( (g p_{a', a}, h p_{a', a}) < \mathcal{U} \).

Proof of (1.12). Take any approximative \( \text{AP-resolution} \ p : X \to \mathcal{X} \) and \( q : Y \to \mathcal{Y} \). Let \( f : (\mathcal{X} \times \mathcal{Y}) \to (\mathcal{X} \times \mathcal{Y}) \) be an approximative resolution of \( f \) with respect to \( p \) and \( q \). We assume that \( f \) has ALP with respect to \( K_2 \) and we show that it has ALP with respect to \( K_1 \).

Take any admissible pair \( (a, b) \) of \( f \). Then there exist \( b_1 > b \) such that \( q_{a, b'} \circ \mathcal{V}_{a} < st \mathcal{V}_{a_1} \). By (AM1)-(AM2) there exist \( a_2 > a_1 > a, f(b_1) \) such that \( (f a_{a_1} \circ f(b_1), q_{a_1, b_1} \circ f a_{a_1} \circ f(b_1)) < \mathcal{V}_{b_1} \) and \( p_{a_2, a_1} \circ U_{a_2} < st \mathcal{V}_{a_2} \). By the assumption there exists an admissible pair \( (a_1, b_1) > (a_2, a) \) satisfying (ALP) for \( K_2 \) and \( (a_3, b_3) \). By (1.13) there exists \( \mathcal{V} \subseteq C_{00}(Y_{a_0}) \) satisfying (R2) for any resolution and \( \mathcal{V}_{b_1} \). Take any \( b_1 > b_2 \) and \( a_3 > a_2, f(b_1) \) such that \( q_{a_2, a_1} \circ \mathcal{V}_{a_3} \circ U_{a_2} \) and \( (f a_{a_3} \circ f(b_1), q_{a_3, b_2} \circ f a_{a_3} \circ f(b_1)) \).

We show that the admissible pair \( (a_3, b_2) \) has the required property. Take any \( (Z, Z_0) \in K_1 \) and maps \( g : Z \to X_{a'}, h : Z_0 \to X_{a_2} \) such that

(1) \( (f a_{a'}, f(b_1), h \mid Z_0, g \mid Z_0) < st \mathcal{V}_{a_1} \).

By the assumption there exists a \( K_0 \)-resolution \( r = \{ r_c : c \in C \} : (Z, Z_0) \to (\mathcal{X}, \mathcal{Y}) = \{ (Z_0, Z_{0c}), r_{c, c}, C \} \) of \( (Z, Z_0) \). Then by Theorems 2-3 of Mardesić [21] \( r_{a_2} = \{ r_c : c \in C \} : Z \to Z = (Z_0, r_{c, c}, C) \) and \( r_{a_3} = \{ r_c : c \in C \} : Z_0 \to Z_0 = (Z_{0c}, r_{c, c}, C) \) form resolutions. Here \( r_{c, c} : Z_0 \to Z_{0c} \) and \( r_{c, c} : Z_{0c} \to Z_{0c} \) are induced by \( r_{c, c} \) and \( r_{c, c} \). By (R1) there exist \( c_1 \subseteq C \) and maps \( h' : Z_{0c_1} \to X_{a'}, g' : Z_{0c_1} \to X_{a_2} \) such that

(2) \( (h, h' \circ r_{c_1}) < U_{a_2} \) and \( (g, g' \circ r_{c_1}) < \mathcal{V}_{b_2} \).

By (1), (2) and the choice of \( b_2 \) \( (q_{a_2, b_2} \circ f a_{a'}, f(b_1), h' \circ r_{c_1} \circ f a_{a_1}, q_{a_1, b_1} \circ g' \circ r_{c_1} \circ f a_{a_1}) \) and \( (f a_{a_3} \circ f(b_1), q_{a_3, b_2} \circ g' \circ r_{c_1} \circ f a_{a_3} \circ f(b_1)) \).

Thus \( f a_{a_3} \circ f(b_1), q_{a_3, b_2} \circ g' \circ r_{c_1} \circ f a_{a_3} \circ f(b_1) \).

By the choice of \( (a_3, b_3) \) there exists a map \( H : Z_{0c_2} \to X_{a_2} \) such that

(3) \( (H, Z_{0c_2}, p_{a_2, a_2} \circ h' \circ r_{c_1}, q_{a_2, b_2} \circ g' \circ r_{c_1} \circ f a_{a_1} \circ f(b_1)) \) and

(4) \( (f a_{a_3} \circ f(b_1), H, q_{a_3, b_2} \circ g' \circ r_{c_1} \circ f a_{a_3} \circ f(b_1)) \).

By (2), (3) and the choice of \( a_2 \).
By (2), (4) and the choices of \( a_1, b_1 \)

\[(5) \quad p_{a_2, a} H_c z, p_{a_4, a} h \succ st U_a.\]

By (5) and (6) \((a_4, b_4)\) satisfies the required conditions. Hence \( f \) has ALP with respect to \( K_1 \).

(1.14) COROLLARY. Let \( K_1 \) and \( K_2 \) be subcategories of \( \operatorname{TOP}_P\text{-pairs} \) and \( K_1 \ Completes (5) and (6).

\[(6) \quad (f \circ p_{a_2, f} H_c z, q_{b_4, b} g) \succ st C V_b.\]

(1.15) COROLLARY. The following statements are equivalent:

(i) A map \( f \) has ALP with respect to \( \operatorname{TOP}_P\text{-pairs} \).

(ii) \( f \) has ALP with respect to \( \operatorname{ANR}_P\text{-pairs} \).

(iii) \( f \) has ALP with respect to \( \operatorname{POL}_P\text{-pairs} \).

(1.16) COROLLARY. For each integer \( n \) the following statements are equivalent:

(i) A map \( f \) has ALP with respect to \( \operatorname{TOP}_P\text{-pairs} \).

(ii) \( f \) has ALP with respect to \( \operatorname{POL}_P\text{-pairs} \).

(1.15) and (1.16) follow from (1.11) and (1.14).

(1.16) REMARK. Let \( \mathcal{K} \) be a collection of pairs of spaces. Let \( t(\mathcal{K}) \) be the full subcategory of \( \operatorname{TOP}_P\text{-pairs} \) consisting of \( \mathcal{K} \) as objects. Then \( \mathcal{K} \) uniquely determines \( t(\mathcal{K}) \). Since \( \operatorname{Ob}(\mathcal{K}) = \mathcal{K} \), \( t(\mathcal{K}) \) uniquely determines \( \mathcal{K} \). Hence we may identify \( \mathcal{K} \) and \( t(\mathcal{K}) \). For example, for collections \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) of pairs of spaces we say that \( \mathcal{K}_1 \) is expandable by \( \mathcal{K}_2 \), in notation \( \mathcal{K}_1 \prec t(\mathcal{K}_2) \), provided that \( t(\mathcal{K}_1) \prec t(\mathcal{K}_2) \). Similarly we can define \( \mathcal{K}_1 \prec t(\mathcal{K}_2) \). Thus (1.12) and (1.14) hold for collections.

(1.17) REMARK. Approximative homotopy lifting property was introduced for \( \text{maps between compacta by Mardesic and Rushing} \) [22] and for \( \text{maps between arbitrary spaces by Mardesic} \) [20]. In the sequel we shall show that the approximative homotopy lifting property is a special case of our approximative lifting property.
§ 2. Restrictions and products of the approximative lifting property.

Let \((X, X_0)\) be a pair of spaces. Let \(p = \{p_a : a \in A\} : X \rightarrow (\mathcal{X}, \mathcal{U}) = \{(X_a, \mathcal{U}_a), p_a : a, A\}\) be an approximative resolution. We put \(X_{\mathcal{U}_a} = \text{st}(p_a(X_a), \mathcal{U}_a)\) and \(\mathcal{U}_{X_{\mathcal{U}_a}} = \mathcal{U}_a \cap X_{\mathcal{U}_a}\) for \(a \in A\). Then \(p_a : X \rightarrow X_a\) and \(p_a : X_a \rightarrow X_a\) induce maps \(p_a : X \rightarrow X_{\mathcal{U}_a}\) and \(p_a : X_a \rightarrow X_{\mathcal{U}_a}\) for \(a' \geq a\).

(2.1) Lemma. If \(X_0\) is \(P\)-embedded in \(X\) and all \(X_a\) are paracompact, then \(p^a = \{p_a : a \in A\} : X \rightarrow (\mathcal{X}, \mathcal{U}) = \{(X_{\mathcal{U}_a}), p_a : a, A\}\) is an approximative resolution.

Proof. First we show that \((\mathcal{X}_0, \mathcal{U}_0)\) forms an approximative inverse system. \((\text{AIa})\) and \((\text{AI2})\) for \((\mathcal{X}_0, \mathcal{U}_0)\) follows from \((\text{AIa})\) and \((\text{AI2})\) for \((\mathcal{X}, \mathcal{U})\).

We show \((\text{AI3})\). Take any \(a \in A\) and any \(\mathcal{V} \in \mathcal{C}_a(X_a)\). Since \(\overline{p_a(X_a)} \subset X_{\mathcal{U}_a}\) and \(X_a\) is paracompact, \(\mathcal{U} = \mathcal{V} \cup \{X_a - \overline{p_a(X_a)}\}\) is an open normal covering of \(X_a\). Take any \(\mathcal{V}' \in \mathcal{C}_a(X_a)\) such that \(st(\mathcal{V}') < \mathcal{V}\). By \((\text{AI3})\) for \((\mathcal{X}, \mathcal{U})\) there exists \(a' > a\) such that \(\overline{p_a(X_{\mathcal{U}_a})} \cap \mathcal{V}' = 0\). We thus have the required property. Hence \((\mathcal{X}_0, \mathcal{U}_0)\) forms an approximative inverse system.

Next we show that \(p^a : X \rightarrow (\mathcal{X}_0, \mathcal{U}_0)\) is an approximative resolution. We show \((\text{AR1})\). Take any \(\mathcal{V} \in \mathcal{C}_a(X_a)\). Since \(X_0\) is \(P\)-embedded, there exists \(\mathcal{U} \subset \mathcal{C}_a(X)\) such that \(\mathcal{V} > \mathcal{U} \cap X_0\). By \((\text{AR1})\) for \(p\) there exists \(a \in A\) such that \(p_a^{-1} \mathcal{U}_a < \mathcal{V}\). It is easy to show that \((p_a^{-1})^{-1} \mathcal{U}_a < \mathcal{V}\). We show \((\text{AR2})\). Take any \(a \in A\) and any \(\mathcal{V} \in \mathcal{C}_a(X_a)\). Put \(\mathcal{U}, \mathcal{V}' \in \mathcal{C}_a(X_a)\) and \(a' > a\) as in the second paragraph. Since \((p_a^{-1})^{-1} \mathcal{U} = \mathcal{V}_a \cap \mathcal{V}^a \subset \mathcal{U}_a, p_a : X_a \rightarrow X_{\mathcal{U}_a}\) for \(a' > a\). Hence \(p^a\) is an approximative resolution.

(2.2) Corollary. If \(p : X \rightarrow (\mathcal{X}, \mathcal{U})\) is an approximative ANR-resolution and \(X_0\) is \(P\)-embedded in \(X\), then \(p^a : X \rightarrow (\mathcal{X}_0, \mathcal{U}_0)\) is an approximative ANR-resolution.

(2.3) Corollary. Any space \(X\) admits an approximative ANR-resolution \(p : X \rightarrow (\mathcal{X}, \mathcal{U})\) such that \(\text{st}(p_a(X), \mathcal{U}_a) = X_a\) for all \(a \in A\).
Let \((X, X_0), (Y, Y_0)\) be \(P\)-pairs and \(f: (X, X_0)\to (Y, Y_0)\) a map. Then \(f\) induces a map \(f_0: X_0\to Y_0\). Let \(q = \{q_b: b \in B\}: Y \to (q, C) = \{Y_0, \mathcal{C}_V, q', B\}\) be an approximative resolution. We assume that all \(X_a\) and all \(Y_a\) are paracompact. We put \(Y_{o0} = \text{st}(q_b(Y_0), C_{V_b})\) and \(C_{V_o} = C_{V_b} | Y_{o0} \subseteq C_{V_0}(Y_{o0})\) for all \(b \in B\). Then \(q_b: Y \to Y_0\) and \(q_{B', b}: Y_{B'} \to Y_b\) induce maps \(q_b^0: Y_{o0} \to Y_{o0}\) and \(q_{B', b}^0: Y_{o0} \to Y_{o0}\) for \(b' > b\). By (2.1) \(q_b^0 = \{q_b^0: b \in B\}: Y_0 \to (q_{b_0}, C_{V_0}) = \{Y_{o0}, \mathcal{C}_{V_{o0}}, q_{b_0}', B\}\) is an approximative resolution. Let \(f = \{f, f_b: b \in B\}: (X, \mathcal{U}) \to (q, C_V)\) be an approximative resolution of \(f\) with respect to \(p\) and \(q\). Let \(k: B \to B\) be a 1-refinement function of \((q, C_V)\). We put \(f_b^k = q_{k(b), b}^k f_{k(b), b}^k \vert X_{k(b), b} \to Y_{o0}\) for \(b \in B\). In a straightforward way we can easily show the following:

(2.4) \textbf{Lemma.} Under the above conditions \(f_o = \{f_k, f_b^k: b \in B\}: (X_o, U_o) \to (q, C_V)\) is an approximative resolution of \(f_o: X_0 \to Y_0\) with respect to \(p^o\) and \(q^o\).

(2.5) \textbf{Theorem.} Let \(f: (X, X_0) \to (Y, Y_0)\) be a map and \(\mathcal{K}\) a collection of pairs of spaces. We assume the following two conditions:
\begin{enumerate}
\item \(X_a\) and \(Y_a\) are \(P\)-embedded in \(X\) and \(Y\), respectively.
\item For each \(U \subseteq C_{V_0}(X)\) there exists \(\mathcal{C} \subseteq C_{V_0}(Y)\) such that \(f^{-1}(\text{st}(Y_a, \mathcal{C})) \subseteq \text{st}(X_a, \mathcal{U})\).
\end{enumerate}

If \(f: X \to Y\) has \(\text{ALP}\) with respect to \(\mathcal{K}\), then so does \(f_o: X_0 \to Y_0\). Here \(f_o\) is induced by \(f\).

\textbf{Proof.} By (2.3) there exist approximative \(\text{ANR}\)-resolutions \(p: X \to (X, \mathcal{U})\) and \(q: Y \to (q, C_V)\) satisfying \(X_a = \text{st}(p_a(X), \mathcal{U}_a)\) and \(Y_a = \text{st}(q_b(Y), \mathcal{C}_V)\) for \(a \in A\) and \(b \in B\). By (i) and (2.2) \(p^o: X_0 \to (X_0, U_0)\) and \(q^o: Y_0 \to (q, C_V)\) are approximative \(\text{ANR}\)-resolutions. Let \(f: (X, \mathcal{U}) \to (q, C_V)\) be an approximative resolution of \(f\) with respect to \(p\) and \(q\). By (2.4) \(f_o: (X_o, U_o) \to (q, C_V)\) is an approximative resolution of \(f_o: X_0 \to Y_0\) with respect to \(p^o\) and \(q^o\).

By the assumption \(f\) has \(\text{ALP}\) with respect to \(\mathcal{K}\). We show that also \(f_o\) has \(\text{ALP}\) with respect to \(\mathcal{K}\). Take any admissible pair \((a, b)\) of \(f_o\), that is, \(a > f_b(b)\). Here \(k: B \to B\) is a 1-refinement function of \((q, C_V)\). By (A13) there exists \(a_1^o > a\) such that \(p_{a_1}^o \subseteq U_o > \text{st}(U_a)\). By (ii) there exist \(\mathcal{C} \subseteq C_{V_0}(Y)\) such that \(f^{-1}(\text{st}(Y_a, \mathcal{C})) \subseteq \text{st}(X_a, \mathcal{U}_a)\). By (A1) and (A13) there exist \(b_2 > b \) such that \(q_{b_1}^o \subseteq \mathcal{C} \) and \(q_{b_1}^o \subseteq \mathcal{C} \). By (A2) there exists \(a_2 > a_o\) such that \(f_{k(b_1)}(a_2) > (a_2, k(b_1))\) such that

\begin{equation}
(f_{k(b_1)}(a_2), f_{k(b_1)}(a_2), k(b_1), k(b_1), f_{k(b_1)}(a_2), f_{k(b_1)}(a_2)) \subseteq \mathcal{C}_{k(b_1)}.
\end{equation}

By the assumption there exists an admissible pair \((a_2, b_2)\) satisfying (ALP) for \(f\) and \((a_2, k(b_2))\). Take \(b_1 > b_2\) such that \(q_{b_1}^o \subseteq \mathcal{C} \) and \(q_{b_1}^o \subseteq \mathcal{C} \). By (A2)
there exists \( a_4 > a_3 \), \( f(k(b_4)) \) such that

\[
(2) \quad (f_0, p_0, f_0(k(b_4)), q_0(k(b_4)), b_0, f_0(k(b_4))) < \mathcal{C} \mathcal{V}_{b_4} \quad \text{and} \quad (q_0(k(b_4)), b_0, f(k(b_4)), p_0, f(k(b_4))) < \mathcal{C} \mathcal{V}_{b_4}.
\]

We show that \((a_4, b_4)\) has the required properties. Clearly \((a_4, b_4)\) is an admissible pair of \( f_0 \) and \( a_4 > a_3 \). Take any \((K, K_0) \in \mathcal{K} \) and take maps \( g : K_0 \rightarrow X_{a_4}, \quad h : K \rightarrow Y_{a_4} \) such that \( (h | K_0, f_0, p_0, f(k(b_4))), g \mathcal{S} \mathcal{C} \mathcal{V}_{a_4} \). By the choice of \( b_4 \), \((q_0(k(b_4)), b_0, f(k(b_4)), p_0, f(k(b_4))), g \mathcal{S} \mathcal{C} \mathcal{V}_{b_4} \). Here \( j_1 : Y_{a_4} \rightarrow Y_{b_4} \) and \( j_2 : X_{a_4} \rightarrow X_{b_4} \) are inclusion maps. By \((2)\) \((f_0, p_0, f_0(k(b_4))), g \mathcal{S} \mathcal{C} \mathcal{V}_{b_4} \) and then \((f_0, p_0, f_0(k(b_4))), g \mathcal{S} \mathcal{C} \mathcal{V}_{b_4} \). By the choice of \((a_4, b_4)\) there exists a map \( H : K \rightarrow X_{a_4} \) such that

\[
(3) \quad (H | K_0, p_0, a_3, g \mathcal{S} \mathcal{C} \mathcal{V}_{a_4} \quad \text{and} \quad (f_0, p_0, f_0(k(b_4))), g \mathcal{S} \mathcal{C} \mathcal{V}_{b_4} \).
\]

We show that

\[
(5) \quad p_{a_4} : H(K) \subset X_{a_4}.
\]

Take any \( k \in K \). Since \( st(p_{a}(X), U_{a}) = X_{a} \) for all \( a \in A \), there exist \( x \in X \) and \( U_x \in U_{a} \) such that

\[
(6) \quad H(k), p_{a_4}(x) \in U_1.
\]

By \((AM1)\) and \((A12)\) there exists \( V_1 \in \mathcal{C} \mathcal{V}_{b_4} \) such that \((f_0, p_0, f_0(k(b_4))), g \mathcal{S} \mathcal{C} \mathcal{V}_{a_4} \). Then by \((6)\)

\[
(7) \quad f_0, p_0, f_0(k(b_4)), H(k), f_0(k(b_4)), p_0, f_0(k(b_4)), x \in V_1.
\]

Since \( f \) is an approximative resolution of \( f \), there exists \( V_2 \in \mathcal{C} \mathcal{V}_{b_4} \) such that

\[
(8) \quad f_0, p_0, f_0(k(b_4)), x \in V_2.
\]

By \((4)\) there exists \( V_3 \in \mathcal{C} \mathcal{V}_{b_4} \) such that \( f_0, p_0, f_0(k(b_4)), H(k), q_0, k(b_4), f_0, k(b_4), x \in st(V_2, \mathcal{C} \mathcal{V}_{b_4}) \). Since \( q_0, k(b_4), f_0, k(b_4), x \in Y_{b_4} \) and \( V_4 \in \mathcal{C} \mathcal{V}_{b_4} \) such that \( q_0, k(b_4), f_0, k(b_4), x \in V_4 \). Thus by \((7)\) and \((8)\)

\[
(9) \quad q_0, k(b_4), f_0, k(b_4), y_0 \in st(st(V_2, \mathcal{C} \mathcal{V}_{b_4}), st \mathcal{C} \mathcal{V}_{b_4})
\]

Since \( k \) is a 1-refinement function of \((q, \mathcal{C})\), by the choice of \( b_4 \), \( q_0, k(b_4), b_4, \mathcal{C} \mathcal{V}_{b_4} \)

\[
(10) \quad q_0, k(b_4), y_0 \in st(st(V_2, \mathcal{C} \mathcal{V}_{b_4}), st \mathcal{C} \mathcal{V}_{b_4})
\]

Then by \((9)\) there exists \( V' \in \mathcal{C} \mathcal{V}_{b_4} \) such that \( q_0, f(x), q_0, y_0 \in V' \). By the choice of \( b_4 \) there exists \( V \in \mathcal{C} \mathcal{V} \) such that \( f(x), y_0 \in st(V, \mathcal{C} \mathcal{V}) \). Since \( y_0 \in Y_{a_4}, f(x) \in st(Y_{a_4}, \mathcal{C} \mathcal{V}) \). By the choice of \( \mathcal{C} \mathcal{V}, x \in f^{-1} st(Y_{a_4}, \mathcal{C} \mathcal{V}) \subset st(X_{a_4}, p_{a_4}^{-1} U_{a_4}) \) and then there exist \( x_0 \in X_{a_4} \) and \( U_3 \in U_{a_4} \) such that \( p_{a_4}(x), p_{a_4}(x_0) \in U_3 \). By \((6)\) there
exists $U_s \in \mathcal{U}_{a_1}$ such that $p_{a_2, a_1} H(k)$, $p_{a_2}(x) \in U_s$. Thus $p_{a_2, a_1} H(k)$, $p_{a_1}(x) \in \text{st}(U_s, \mathcal{U}_{a_1})$ and then by the choice of $a_1$ there exists $U \in \mathcal{U}_a$ such that $p_{a_2, a_1} H(k)$, $p_{a_1}(x_0) \in U$. This means $p_{a_2, a_1} H(k) \subseteq \text{st}(p_{a}(X_0), \mathcal{U}_a) = X_{a_0}$. Hence we have (5).

By (3) and the choice of $a_1$

(10) \[ (p_{a_2, a_1} H(K), p_{a_2, a_1} g) \subseteq \mathcal{U}_{a_0} \subseteq \text{st}(U_{a_0}). \]

Since $k$ is a 1-refinement function, by (1) and (4) it is easy to show that

(11) \[ (f_0 p_{a_1}, f_k p_{a_2}, a H, q_{b_1}, b_2 h) \subseteq \mathcal{U}_{a_0}. \]

By (5) this means $p_{a_2, a_1} H(K) \subseteq X_{a_0}$ has the required properties. Then $f_0$ has ALP with respect to $\mathcal{K}$ and hence so does $f_a$. ■

(2.6) Lemma. Let $f : X \rightarrow Y$ be a map and let $Y$ be a normal space. Let $X_a$ and $Y_a$ be subspaces of $X$ and $Y$, respectively. If $f$ is a closed map and $f^{-1}(Y_a) = X_a$, then for each $U \in \mathcal{C}_W(X)$ there exists $C \subseteq \mathcal{C}_W(Y)$ such that $f^{-1}\text{st}(Y_a, C \subseteq X_a)$.

Proof. Take any $U \in \mathcal{C}_W(X)$. Since $\text{st}(X_a, U) = \text{st}(X_a, U) = \text{st}(f^{-1}(Y_a), U)$ and $f$ is closed, $W = Y - f(X - \text{st}(X_a, U))$ is an open set in $Y$ and $W \supseteq Y_a$. Since $Y$ is normal, $C \subseteq \{W, Y - Y_a\}$ forms a normal open covering of $Y$ by Theorem 1 of MS [23, p. 324]. Hence $f^{-1}\text{st}(Y_a, C \subseteq X_a) = f^{-1}\text{st}(Y_a, C \subseteq X_a)$.

(2.7) Corollary. Let $(X, X_a)$ and $(Y, Y_a)$ be P-pairs and $f : (X, X_a) \rightarrow (Y, Y_a)$ a map. Let $f : X \rightarrow Y$ be a closed map, $Y$ a normal space and $f^{-1}(Y_a) = X_a$. If $f : X \rightarrow Y$ has ALP with respect to $\mathcal{K}$, then so does $f_0 : X_0 \rightarrow Y_a$. Here $f_0$ is induced by $f$.

(2.8) Corollary. Let $f : X \rightarrow Y$ be a closed map. We assume any one of the conditions (i)-(iii) below. If $f$ has ALP with respect to $\mathcal{K}$, then so does $f | f^{-1}(Y_a) : f^{-1}(Y_a) \rightarrow Y_a$ for any closed subset $Y_a$ of $Y$.

(i) $X$ and $Y$ are collection-wise normal.

(ii) $X$ and $Y$ are paracompact.

(iii) $X$ and $Y$ are pseudo-compact normal.

(2.7) follows from (2.5) and (2.6). (2.8) follows from (2.7) and the fact that every closed subset of a space satisfying one of the conditions (i)-(iii) is P-embedded (see Corollary 15.7, Theorem 15.11 and Corollary 15.15 of Aló-Shapiro [1]). ■
(2.9) Theorem. Let all \( f_c : X_c \to Y_c \) be maps between compact spaces for \( c \in C \). Then \( f = \{ f_c : c \in C \} : X = \bigcup X_c : c \in C \to Y = \bigcup Y_c : c \in C \) has ALP with respect to \( \mathcal{K} \) if and only if all \( f_c \) have ALP with respect to \( \mathcal{K} \).

Proof. We use the same notations as in the proof of (III.2.1).

First we assume that all \( f_c \) have ALP with respect to \( \mathcal{K} \). Then all \( f^c \) satisfy (ALP) with respect to \( \mathcal{K} \). We show that \( f = \{ f, f_h : h \in F(B) \} : (X, U) \to (\mathcal{Y}, V) \) has ALP with respect to \( \mathcal{K} \). Take any admissible pair \( (g, h) \) of \( f \), that is \( g \supset f(h) \). Put \( h : m = \{ c_1, c_2, \ldots, c_k \} \to B \) and \( g : m' = \{ c_k+1, \ldots, c_{k'} \} \to A \). Since \( g \supset f(h) \), \( g(c_i) \supset f^c h(c_i) \) and then \( (g(c_i), h(c_i)) \) is an admissible pair of \( f^c_i \) for \( 1 \leq i \leq k \). By the assumption there exist admissible pairs \( (a_i, b_i) \) of \( f^c_i \), \( a_i \supset g(c_i), b_i \supset h(c_i) \), satisfying (ALP) for \( f^c_i \) and \( (g(c_i), h(c_i)) \) with respect to \( \mathcal{K} \), \( 1 \leq i \leq k \). We define choice functions \( h' : m \to B \) and \( g' : m' \to A \) as follows: \( h'(c_i) = b_i \) for \( 1 \leq i \leq k \), and \( g'(c_i) = a_i \) for \( 1 \leq i \leq k', g'(c_i) = g(c_i) \) for \( k+1 \leq i \leq k' \). Then \( g' \supset g \), \( h' \supset h \) and \( (g', h') \) is an admissible pair of \( f \).

We show that \( (g', h') \) has the required properties. Take any \( (K, K_0) \in \mathcal{K} \) and maps \( u = (u_{c_1}, u_{c_2}, \ldots, u_{c_k}) : K \to Y_{g'} \), \( v = (v_{c_1}, v_{c_2}, \ldots, v_{c_k}) : K \to X_{g'} \) such that \( (f_{h'}, f_{h'}) v, u | K_0 \subset \mathcal{V}_{h'} \). Then \( (f_{h'}^i(c_i), f_{h'}^i(c_i), f_{h'}^i(c_i), v_{c_i}) : u_i | K_0 \subset \mathcal{V}_{h'} \) for \( 1 \leq i \leq k \). By the choice of \( (a_i, b_i) \) for \( i, 1 \leq i \leq k \), there exists a map \( U_{c_1} : K \to X_{g'} \) satisfying


We define \( U = (U_{c_1}, U_{c_2}, \ldots, U_{c_k}, v_{c_k+1}, \ldots, v_{c_{k'}}) : K \to X_{g'} \). By (1) and (2) it is easy to show that \( (f_h v, u) | K_{0} \subset \mathcal{V}_{h} \) and \( (f_{h'} v, f_{h'} u) | K_{0} \subset \mathcal{V}_{h'} \). Then \( f \) has ALP with respect to \( \mathcal{K} \).

Next we assume that \( f \) has ALP with respect to \( \mathcal{K} \) and show that all \( f^c \) have ALP with respect to \( \mathcal{K} \). Take any \( c \in C \) and any admissible pair \( (a, b) \) of \( f^c \). We define choice functions \( g = m = \{ c \} \to A \) and \( h = m \to B \) by \( g(c) = a \) and \( h(c) = b \). Then \( (g, h) \) is an admissible pair of \( f \). By the assumption there exists an admissible pair \( (g', h') \) of \( f \), \( g' \supset g \), \( h' \supset h \), satisfying (ALP) for \( f \) and \( (g, h) \). Put \( h' : m_1 \to B \) and \( g' : m_2 \to A \). Since \( g' \supset g \), \( h' \supset h \) and \( g' \supset f(h') \), \( m \subset m_1 \), \( m \subset m_2 \) and \( m_1 \subset m_2 \). Then we may put \( m_1 = \{ c, c_1, c_2, \ldots, c_k \} \) and \( m_2 = m_1 \cup \{ c_{k+1}, \ldots, c_{k'} \} \).

We put \( a' = g'(c) \) and \( b' = h'(c) \). Trivially \( (a', b') \) is an admissible pair of \( f^c \). We show that it has the required property. Take any \( (K, K_0) \in \mathcal{K} \) and maps \( v : K_0 \to X_{g'}, u : K \to Y_{h'} \) such that


Take any point \( x = (x_e, x_{e_1}, \ldots, x_{e_k}) \in X_{e'} \) and put \( f_{h'} \cdot p_{e', h'}(x) = (y_e, y_{e_1}, \ldots, y_{e_k}) \in Y_{e'} \). We define \( v': \text{Ko} \to \text{Xg} \) and \( u': \text{Ko} \to \text{Yh} \) as follows: \( v'(z) = (v(z), x_{c_1}, x_{c_2}, \ldots, x_{c_k}) \) and \( u'(z) = (u(z), y_{c_1}, y_{c_2}, \ldots, y_{c_k}) \) for \( z \in \text{Ko} \). By (3) it is easy to show that \((f_{h'} \cdot p_{e', h'}, u'_{\text{Ko}}) < \text{st}\text{CV}_{h'}\). By the choice of \((g', h')\) there exists a map \( U: \text{Ko} \to \text{Xg} \) such that \((p_{g', f_{c}} \cdot U_{\text{Ko}}, q_{h', h} U) < \text{st}\text{CV}_{h'}\). This means that \((p_{g', a}, U_{\text{Ko}}) < \text{st}\text{CV}_{a} \) and \((f_{h} \cdot p_{e, f_{c}} U, q_{h'} h U) < \text{st}\text{CV}_{h'}\). Thus \( f^e \) has ALP with respect to \( \mathcal{K} \) and hence so does each \( f_i \). □

§ 3. Approximatively \( n \)-connected maps.

Using the approximative lifting property we introduce approximatively \( n \)-connected maps and investigate their properties.

Let \( \mathcal{K} \) be a collection of pairs of spaces. Let \( (\mathcal{X}, \mathcal{U}) = \{(X_a, U_a), p_a, a, A\} \) and \( (\mathcal{Y}, \mathcal{V}) = \{(Y_b, V_b), q_b, b, B\} \) be approximative inverse systems. Let \( f = \{(f, f_a, b \in B): (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V})\} \) be an approximative system maps.

We say that \( f: (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V}) \) has the \( * \)-approximative lifting property, in notation ALP\(*\), with respect to \( \mathcal{K} \) provided that it satisfies the following:

(ALP\(*\)) For each admissible pair \((a, b)\) of \( f \) there exists an admissible pair \((a_0, b_0) > (a, b)\) with the following property; for each \((K, K_0) \in \mathcal{K} \) and any maps \( g: K \to Y_{b_0}, h: K_0 \to X_{a_0} \) with \((f_{b_0} \cdot p_{a_0, f_{b_0}} h, g \cdot K_0) < \text{st}\text{CV}_{b_0}\), there exists a map \( H: K \to X_a \) such that \( p_{a_0, a} h = H \cdot K_0 \) and \((q_{b_0, b} g, f_{b} \cdot p_{a, f_{c}} h) < \text{st}\text{CV}_{b}\).

We say that \( f \) has the \( ** \)-approximative lifting property, in notation ALP\(**\), with respect to \( \mathcal{K} \) provided that it satisfies the following:

(ALP\(**\)) For each admissible pair \((a, b)\) of \( f \) there exists an admissible pair \((a_0, b_0) > (a, b)\) with the following property; for each \((K, K_0) \in \mathcal{K} \) and any maps \( g: K \to Y_{b_0}, h: K_0 \to X_{a_0} \) with \( f_{b_0} \cdot p_{a_0, f_{b_0}} h = g \cdot K_0 \), there exists a map \( H: K \to X_{a_0} \) such that \( p_{a_0, a} h = H \cdot K_0 \) and \((q_{b_0, b} g, f_{b} \cdot p_{a, f_{c}} h) < \text{st}\text{CV}_{b}\).

(3.1) Lemma. Let \( \mathcal{K} \) be a collection of polyhedral pairs. Let \( (\mathcal{X}, \mathcal{U}) \) and \( (\mathcal{Y}, \mathcal{V}) \) be approximative inverse systems in POL or in ANR.

(i) An approximative system map \( f: (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{V}) \) has ALP with respect to \( \mathcal{K} \) iff \( f \) has ALP\(*\) with respect to \( \mathcal{K} \).

(ii) Let \( f \) be commutative. \( f \) has ALP\(*\) with respect to \( \mathcal{K} \) iff \( f \) has ALP\(**\) with respect to \( \mathcal{K} \).

Proof. We show that (ALP) implies (ALP\(*\)). Take any admissible pair \((a, b)\) of \( f \). Since \( X_a \) is a polyhedron or an ANR, there exists \( U \in C_{ov}(X_a) \) satisfying (*) in (1.5.5) for \( U_a \). There exist \( b_1 > b \) and \( a_i > a, f(b_i) \) such that
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\[ g_{b_1}e_{CV_b} > st\langle CV_{b_1} \rangle, \text{ st}\{U\}_{a_1} > st\langle U\rangle_{a_1} \text{ and } \]

(1) \[ (f_{a_2,b_2} p_{a_2,f(a_2)}, q_{a_2,b_2}) < CV_b. \]

By the assumption there exists an admissible pair \((a_2, b_2) > (a_1, b_1)\) satisfying \((ALP)\) for \(f\) and \((a_1, b_1)\).

We show that \((a_2, b_2)\) has the required property. To do so take any \((K, K_0) \in \mathcal{K}\) and maps \(g: K \rightarrow Y_{b_2}\), \(h: K_0 \rightarrow X_{a_2}\) such that \((f_{b_2} p_{a_2,f(c_{b_2})}, g|K_0) < st\langle CV_{b_2}\rangle\). By the choice of \((a_2, b_2)\) there exists a map \(H: K \rightarrow X_{a_1}\) such that

(2) \[ (p_{a_2, a_1} h, H|K_0) < st\langle U\rangle_{a_1} \text{ and } \]

(3) \[ (f_{a_2,b_2} p_{a_2,f(c_{b_2})} H, q_{b_2,b_2} g) < st\langle CV_{b_1}\rangle. \]

By the choice of \(a_1\) and (2) \((p_{a_2, a_1} h, p_{a_2, a_1} H|K_0) < st\langle U\rangle.\) By the choice of \(U\) there exists a homotopy \(F: K \times I \rightarrow X_a\) such that \(F\) is a \(U_a\)-homotopy, \(F_0 = p_{a_1, a} H|K_0\) and \(F_1 = p_{a_2, a} H.\) By the homotopy extension theorem (see Th. 3 of MS [23, p. 291]) there exists a homotopy \(\bar{F}: K \times I \rightarrow X_a\) such that \(\bar{F}|U \times I: U \times I \rightarrow X_a\) is a \(U_a\)-homotopy. Take a map \(v: K \rightarrow I\) such that \(v(z) = 1\) for \(z \in K_0\) and \(v(z) = 0\) for \(z \in K \setminus U\). We define a map \(\bar{H}: K \rightarrow X_a\) as follows: \(\bar{H}(z) = F(z, v(z))\) for \(z \in K.\) It is easy to show that

(4) \[ \bar{H}|K_0 = p_{a_2, a_1} h \text{ and } \]

(5) \[ (p_{a_1, a_1} H, \bar{H}) < st\langle U\rangle_a. \]

By (1), (3), (5) and the choice of \(b_1\) we have that \(\langle f_{b_1} p_{a_1,f(c_{b_1})}, q_{b_1,b_1} g\rangle < st\langle CV_{b_1}\rangle.\) Thus by this and (4) \(f\) satisfies \((ALP^*)\). Thus \((ALP)\) implies \((ALP^*)\). The converse is trivial and hence we have (i).

We assume that \(f\) is commutative and satisfies \((ALP^{**})\). We show that \(f\) satisfies \((ALP^{**})\). Take any admissible pair \((a, b)\) of \(f\). Since \(f\) is commutative, there exist \(b_2 > b\) and \(a_1 > a, f(b_1)\) such that \(q_{b_1,a_1} e_{CV_y} > st\langle CV_{b_1}\rangle\) and \(f_{a_1,b_2} p_{a_1,f(c_{b_1})} = q_{b_2,b_1} e_{CV_y} > st\langle CV_{b_2}\rangle.\) There exists an admissible pair \((a_2, b_2) > (a_1, b_1)\) satisfying \((ALP^{**})\) for \(f\) and \((a_1, b_1)\). Since \(Y_{b_2}\) is a polyhedron or an ANR, there exists \(\mathcal{C} \subseteq Cov(Y_{b_2})\) satisfying \((*)\) in (1.5.5) for \(\mathcal{C} \subseteq Y_{b_2}\). There exist \(b_3 > b_2\) and \(a_3 > a_2, f(b_1)\) such that \(q_{b_2,b_3} e_{CV_y} > st\langle CV_{b_2}\rangle\) and \(f_{a_3,b_3} p_{a_3,f(c_{b_3})} = q_{b_2,b_1} e_{CV_y} > st\langle CV_{b_2}\rangle.\)

We show that \((a_3, b_3)\) has the required properties. Take any \((K, K_0) \in \mathcal{K}\) and any maps \(g: K \rightarrow Y_{b_1}\), \(h: K_0 \rightarrow X_{a_2}\) such that \(\langle f_{b_2} p_{a_2,f(c_{b_2})}, g|K_0 \rangle < st\langle CV_{b_2}\rangle.\) Then by the choices of \(a_3\) and \(b_3, (f_{b_3} p_{a_3,f(c_{b_3})}, q_{b_3,b_2} g|K_0) < CV.\) Then by the choice of \(\mathcal{C}\) there exists a \(\mathcal{C} \subseteq Y_{b_3}\)-homotopy \(G: K \times I \rightarrow Y_{b_2}\) such that \(G_0 = q_{b_2,b_3} g|K_0\) and \(G_1 = f_{b_3} p_{a_3,f(c_{b_3})} h.\) By the same way as (i) there exist a homotopy \(\tilde{G}: K \times I \rightarrow Y_{b_3},\) an open neighborhood \(V\) of \(K_0\) in \(K\) and a map \(v: K \rightarrow I\) such
that $\overline{\partial} = q_{b_2, b_2} g$, $\overline{\partial} | K_0 \times I = G$, $\overline{\partial} | V \times I : V \times I \to Y_{b_2}$ is a $\mathcal{V}_{b_2}$-homotopy and $v(z) = 0$ for $z \in K-V$, $v(z) = 1$ for $z \in K_0$. We define a map $\overline{g} : K \to Y_{b_2}$ by $\overline{g}(z) = \overline{G}(z, v(z))$ for $z \in K$. It is easy to show that

$$
\overline{g} | K_0 = f_{b_2} p_{a_2, \phi_{(b_2)}} h \quad \text{and} \quad (\overline{g}, q_{b_2, b_2} g) < \mathcal{V}_{b_2}.
$$

By (6) and the choice of $(a_2, b_2)$ there exists a map $H : K \to X_{a_1}$ such that $H | K_0 = p_{a_2, a_1} h$ and $(f_{b_2} p_{a_2, \phi_{(b_2)}} H, q_{b_2, b_2} g) < \mathcal{V}_{b_2}$. Thus $p_{a_1, a} H | K_0 = p_{a_2, a_1} h$. By the choices of $a_1$ and $b_1$, $(f_{b_2} p_{a_2, \phi_{(b_2)}} H, q_{b_2, b_2} g) < \mathcal{V}_{b_2}$ and then by (6), $(f_{b_2} p_{a_2, \phi_{(b_2)}} H, q_{b_2, b_2} g) < \mathcal{V}_{b_2}$. Thus $f$ satisfies (ALP*). Hence (ALP**) implies (ALP*). Trivially (ALP*) implies (ALP**). Hence we have (ii).}\end{proof}

Let $\Delta^k$ be the standard $k$-simplex. $\partial \Delta^k$ denotes the boundary of $\Delta^k$. We put $(D, S)_n = \{(\Delta^k, \partial \Delta^k) : k = 0, 1, \ldots, n+1\}$ and $(D, S)_n = \{(\Delta^k, \partial \Delta^k) : k = 0, 1, 2, \ldots\}$ for each $n = -1, 0, 1, 2, \ldots$.\end{equation}

Lemma 3.2. The following statements are equivalent for each $n$, $n = -1, 0, 1, 2, \ldots$:

(i) $\mathbf{f}$ has ALP* with respect to $(D, S)_n$.

(ii) $\mathbf{f}$ has ALP* with respect to $n^\mathbf{POLpairs}$.

Proof. Trivially (ii) implies (i). By induction on $n$ we will show that (i) implies (ii).

First we take $n = -1$. Take any admissible pair $(a, b)$ of $\mathbf{f}$. There exist $b_1 > b$ and $a_1 > a$, $f(b_1)$ such that $q_{b_1}^{-1} \mathcal{V}_{b_1} > st \mathcal{V}_{b_1}$ and $(f_{b_1} p_{a_1, \phi_{(b_1)}} q_{b_1}, b_1 g) < \mathcal{V}_{b_1}$. There exists an admissible pair $(a_2, b_2) > (a_1, b_1)$ satisfying (ALP*) with respect to $(S, D)_{-1}$ for $(a_1, b_1)$. There exists $a_2 > a_1$ such that $(f_{b_2} p_{a_2, \phi_{(b_2)}} q_{b_2}, b_2 g) < \mathcal{V}_{b_1}$.

We show that the admissible pair $(a_2, b_2)$ has the required property. Take any $(P, P_0) \in n^\mathbf{POLpairs}$ and maps $g : P \to Y_{b_2}$, $h : P_0 \to X_{a_2}$ such that $(f_{b_2} p_{a_2, \phi_{(b_2)}} g, g | P_0) < \mathcal{V}_{b_2}$. There exist a simplex complex $K$ and a subcomplex $L$ of $K$ such that $|K| = P$, $|L| = P_0$ and $\dim(K-L) \leq 0$. Take any vertex $v \in K-L$. Since $(\Delta^k, \partial \Delta^k) = (\Delta^0, \phi)$, by the choice of $(a_2, b_2)$ there exists a map $g_\phi : \{v\} \to X_{a_2}$ such that $(f_{b_1} p_{a_1, \phi_{(b_1)}} g_\phi, q_{b_1}, b_1 g_\phi | \{v\}) < \mathcal{V}_{b_1}$. Now we define a map $H : P \to X_{a_1}$ as follows: $H(z) = p_{a_2, a} h(z)$ for $z \in |L|$ and $H(v) = p_{a_1, a} g_\phi(v)$ for $v \in |K| - |L|$. Clearly $H$ is continuous. It is easy to show that $H | P_0 = p_{a_2, a} h$ and $(f_{b_2} p_{a_2, \phi_{(b_2)}} H, q_{b_2}, b_2 g) < \mathcal{V}_{b_2}$. Hence $f$ has the required property.

Next we assume that (i) implies (ii) for each $k$, $k \leq n$, and show it for $n+1$. Take any admissible pair $(a, b)$ of $\mathbf{f}$. There exist $b_1 > b$ and $a_1 > a$, $f(b_1)$ such that $q_{b_1}^{-1} \mathcal{V}_{b_1} > st \mathcal{V}_{b_1}$ and $(f_{b_1} p_{a_1, \phi_{(b_1)}} q_{b_1}, b_1 g) < \mathcal{V}_{b_1}$. There exists an
admissible pair \((a_2, b_2) > (a_1, b_1)\) satisfying (ALP*) with respect to \((S, D)_{n+1}\) for \((a_1, b_1)\). There exists \(a_3 > a_2\) such that \((f_{a_1}p_{a_1}f_{b_1}, q_{b_2}b_1f_{b_2}p_{b_2}) < \text{CV}_b\). By the inductive assumption there exists an admissible pair \((a_4, b_4) > (a_3, b_2)\) satisfying (ALP*) with respect to \(n+1\) pairs for \((f_{b_2}p_{a_2}, q_{b_2}b_2)\).

We show that \((a_4, b_4)\) is the required one. Take any \((P, P_0) \in n+1\) pairs and maps \(g: P \to Y, h: P_0 \to X_{a_3}\) such that \((f_hp_{a_3}f_{b_3}h, q_{b_4}b_4g) < \text{CV}_b\). There exists an admissible pair \((a_3, b_2)\) satisfying (ALP*) with respect to \(n+1\) pairs for \((f_{b_2}p_{a_2}, q_{b_2}b_2)\).

We say that an approximative system map \(f: (X, \mathcal{U}) \to (Y, \mathcal{V})\) is approximatively \(n\)-connected, in notation \(f \in AC^n\), provided \(f\) has ALP with respect to \((D, S)_n\). We say that \(f\) is approximatively \(\infty\)-connected, in notation \(f \in AC^\infty\), provided \(f \in AC^n\) for each integer \(n \geq -1\). We say that \(f\) is approximatively \(\infty\)-connected, in notation \(f \in AC^{\infty}\), provided \(f\) has ALP with respect to \((D, S)_\infty\).

We say that a map \(f: X \to Y\) is approximatively \(n\)-connected, in notation}\]
\[ f \in AC^n, \] provided that \( f \) has \( \text{ALPP} \) with respect to \( (D, S)_n \). Thus \( f \in AC^n \) iff it satisfies any one of the conditions (i)-(iv) in (3.3). Similarly we may define approximative \( \infty \)-connectedness, in notation \( f \in AC^{\infty} \), and approximative \( \infty \cdot \infty \)-connectedness, in notation \( f \in AC^{\infty \cdot \infty} \).

(3.4) **Theorem.** If a map \( f : X \to Y \) is approximatively \( n \)-connected, then for each \( x \in X \) \( \text{pro-} \pi_n(f) : \text{pro-} \pi_n(X, x) \to \text{pro-} \pi_n(Y, f(x)) \) is an isomorphism in pro-groups for \( k \leq n \) and an epimorphism for \( k = n+1 \).

**Proof.** By (1.4.9) there exist approximative ANR-resolutions \( p : X \to (\mathcal{X}, \mathcal{U}), \) \( q : Y \to (\mathcal{Y}, \mathcal{V}) \) and an approximative resolution \( f \) of \( f \) with respect to \( p \) and \( q \) satisfying (RM1) and (RM2). Take any point \( x \in X \). Then by (1.3.16) \( p : (X, x) \to (\mathcal{X}, x, \mathcal{U}_x) = (X_a, x_a, \mathcal{U}_a, p_{a} \cdot A) \) and \( q : (Y, f(x)) \to (\mathcal{Y}, f(x), \mathcal{V}_b) = (Y_b, f(x)_b, \mathcal{V}_b, q_{b} \cdot B) \) form approximative ANR-resolutions. Here \( x_a = p_a(x) \) and \( f(x)_b = q_{b}(f(x)) \) for \( a \in A \) and \( b \in B \). By (RM2) \( f_b : X_{f(b)} \to Y_b \) induces \( f_b : (X_{f(b)}, x_{f(b)}) \to (Y_b, f(x)_b) \) for \( b \in B \). Then \( f : (\mathcal{X}, x, \mathcal{U}) \to (\mathcal{Y}, f(x), \mathcal{V}) \) forms an approximative resolution of \( f : (X, x) \to (Y, f(x)) \). By (1.5.10) \( H(p) : (X, x) \to H(\mathcal{X}, x) \) and \( H(q) : (Y, f(x)) \to H(\mathcal{Y}, f(x)) \) form \( \text{HANR}_0 \)-expansions and \( H(f) : H(\mathcal{X}, x) \to H(\mathcal{Y}, f(x)) \) forms a \( \text{HANR}_0 \)-expansion of \( f : (X, x) \to (Y, f(x)) \). Here \( \text{ANR}_0 \) is the pointed category of ANRs. By (1.4.4) and (1.5.5) we may assume that each \( \mathcal{V}_b \) has the property:

\( (\ast) \) if \( r, s : (Z, z) \to (Y_b, f(x)_b) \) are s\text{VCV}-near, then \( r \simeq s \text{ rel. } z \) for any pointed space \( (Z, z) \).

By (3.2) and (3.3) \( f \) has \( \text{ALPP} \) with respect to \( \text{POL}^{\ast \text{pairs}} \). Take any admissible pair \((a, b)\) of \( f \). Then there exists an admissible pair \((a_1, b_1) \succ (a, b)\) satisfying (ALPP) with respect to \( \text{POL}^{\ast \text{pairs}} \) for \((a, b)\).

When we take the polyhedral pair \((D^{k+1}, \partial D^{k+1})\), by the choice of \((a_1, b_1)\) it is easy to show that

\( (1) \quad \text{Ker} \pi_k(f_b : p_{a_1} \cdot f_{a_1} \to Z_{a_1, e}) \subseteq \text{Ker} \pi_k(p_{a_1, e}) \quad \text{for} \quad k \leq n. \)

When we take a polyhedral pair \( (\partial D^{k+1}, v) \) where \( v \) is a vertex of \( \partial D^{k+1} \), by the choice of \((a_1, b_1)\) and \((\ast)\) it is easy to show that

\( (2) \quad \text{Im} \pi_k(q_{b_1, v}) \subseteq \text{Im} \pi_k(f_b : p_{a_1} \cdot f_{a_1} \to Z_{a_1, e}) \quad \text{for} \quad k \leq n+1. \)

Here \( \text{Ker}(h) \) and \( \text{Im}(h) \) denote the kernel of \( h \) and the image of \( h \) for any homomorphism \( h \).

By Th. 2 of MS [23, p. 108] (1) means that \( \text{pro-} \pi_k(f) \) is a monomorphism for \( k \leq n \). Also by Th. 4 of MS [23, p. 112] (2) means that \( \text{pro-} \pi_k(f) \) is an epimorphism for \( k \leq n+1 \). Hence by Th. 6 of MS [23, p. 114] \( \text{pro-} \pi_k(f) \) is an
isomorphism for $k \leq n$. ■

(3.5) Corollary. Let $f : X \to Y$ be a map. If $f \in AC^n$, then $\text{pro-}\pi_k(f) : \text{pro-}\pi_k(X, x) \to \text{pro-}\pi_k(Y, f(x))$ is an isomorphism for all $k$ and for any $x \in X$. ■

(3.6) Corollary. Let $f : X \to Y$ be a map.
(i) If $f \in AC^n$, then $\xi_n(f) : \xi_n(X, x) \to \xi_n(Y, f(x))$ is an isomorphism of shape groups for any $x \in X$.
(ii) If $f \in AC^\infty$, then $\xi_k(X, x) \to \xi_k(Y, f(x))$ is an isomorphism for all $k$ and for any $x \in X$. ■

For a space $X$, $sdX$ denotes the shape dimension of $X$. This notion was introduced by Dydak and he showed that $sdX \leq \dim X$ and $sd(X, x) = sdX$ (see Th. 7 of MS [23, p. 103]).

(3.7) Theorem. Let $f : X \to Y$ be a map and $n+1 = \text{Max}(sdY, sdX+1) < \infty$. If $f \in AC^n$, then for each $x \in X$ $f : (X, x) \to (Y, f(x))$ induces a shape isomorphism.

Proof. Let $p : X \to (X, \xi)$, $q : Y \to (Y, \xi)$ and $f : (X, \xi) \to (Y, \xi)$ be as in the proof of (3.4). Then $(\xi, \xi)$ satisfies (*) in the proof of (3.4). By (3.2) and (3.3) $f$ has ALP* with respect to $^{n+1}\text{POL}_{pairs}$. Take any admissible pair $(a, b)$ of $f$ and then there exists an admissible pair $(a_1, b_1) > (a, b)$ satisfying (ALP*) with respect to $^{n+1}\text{POL}_{pairs}$ for $(a, b)$. Since $sd(X, f(x)) = sdY$, by Theorem 2 of MS [23, p. 96] there exist $b_2 > b_1$, a pointed polyhedron $(P, p)$ and maps $r_1 : (P, p) \to (Y_{b_1}, f(x)_{b_1})$, $s_1 : (X_{a_2}, f(x)_{a_2}) \to (P, p)$ such that

(1) $\dim P \leq sdY$ and $r_1s_1 \equiv q_{b_2} S_1 rel. f(x)_{b_2}$.

Since $f$ satisfies (RM2), there exists $a_2 > a_1$, $f(b_2)$ such that

(2) $f_{b_2}p_{a_2, f(b_2)} \equiv q_{b_2} S_1 rel. f_{b_2}$.

Since $sd(X, x) = sdX$, by Theorem 2 of MS [23, p. 96] there exist $a_2 > a_3$, a pointed polyhedron $(Q, q)$ and maps $r_2 : (Q, q) \to (X_{a_2}, x_{a_2})$, $s_2 : (X_{a_3}, x_{a_3}) \to (Q, q)$ such that

(3) $\dim Q \leq sdX$ and $r_2s_2 \equiv p_{a_2, a_3} rel. x_{a_2}$.

By the simplicial approximation theorem (see [32, 33]) there exist simplicial complexes $K, L$, vertexes $q \in K$, $p \in L$, and a simplicial map $k : K \to L$ such that $|K| = Q$, $|L| = P$ and

(4) $s_1f_{b_2}p_{a_2, f(b_2)}r_2 \equiv k rel. q$. 

Let $M(k)$ be the mapping cylinder of $k$. Let $u : |K| \times I \rightarrow |M(k)|$ be the identification map. We identify $(x, 1)$ and $k(x)$ for $x \in |K|$. Let $T = |K| \times \{0\} \cup \{q\} \times I$. Since $k$ is simplicial, $(M(k), u(T))$ is a polyhedral pair by Theorem 6 of MS [23, p. 295]. By (1) and (3) $(M(k), u(T)) \in \mathcal{POL}$.

By (1) and (4) there exist pointed homotopies $H^1 : Q \times I \rightarrow P$ and $H^2 : Y_b \times I \rightarrow Y_{b_1}$ such that $H_1 = k$, $H_2 = s_1 f_{b_2} p_{a_2, f(a_2)} r_2$, $H'(q \times I) = p$ and $H_1 = r_1 s_1$, $H_2 = q_{b_2, b_1}$ and $H'(f(x)_{b_2} \times I) = f(x)_{b_1}$. We define a map $H : |K| \times I \rightarrow Y_{b_1}$ as follows: $H(x, t) = H'(f_{b_2} p_{a_2, f(a_2)} r_2(x), 2t)$ for $(x, t) \in |K| \times [0, 1/2]$ and $H(x, t) = r_1 H'(x, 2t - 1)$ for $(x, t) \in |K| \times [1/2, 1]$. Then $H \oplus r_1 : |K| \times I \times L \rightarrow Y_{b_1}$ induces a map $H' : M(k) \rightarrow Y_{b_1}$ such that $H \oplus r_1 = H' u$. We define a map $h : u(T) \rightarrow X_a$ as follows: $h(u(x, 0)) = p_{a_2, a_1} r_1(x)$ for $(x, 0) \in T$ and $h(u(q, t)) = x_{a_1}$ for $(q, t) \in T$. Since $u(|K| \times \{0\}) : |K| \times \{0\} \rightarrow u(|K| \times \{0\})$ is a homeomorphism, $h$ is well-defined and continuous. By (2) $f_{b_2} p_{a_2, f(a_2)} h = H' u(T)$. By the choice of $(a_1, b_1)$ there exists a map $G : M(k) \rightarrow X_a$ such that

$$(5) \quad G \cdot u(T) = p_{a_1, a_1} h \text{ and } (f_{b_2} p_{a_2, f(a_2)} G, q_{b_2, b_1} H') \in \mathcal{STC}_b.$$  

Let $m = Gu |L : |L| \rightarrow X_a$. By the definition of $H'$ and (5) $(f_{b_2} p_{a_2, f(a_2)} m, q_{b_2, b_1} r_1) \in \mathcal{STC}_b$. Then by (*) in the proof of (3.4) $f_{b_2} p_{a_2, f(a_2)} m s_1 = q_{b_2, b_1} r_1 s_1$ rel. $f(x)_{b_2}$ and hence by (1)

$$(6) \quad f_{b_2} p_{a_2, f(a_2)} m s_1 = q_{b_2, b_1} r_1 \text{ rel. } f(x)_{b_2}.$$  

The homotopy $Gu |K| \times I : |K| \times I \rightarrow X_a$ gives that $p_{a_2, a_1} r_3 = mk$ rel. $q$ and hence by (3) and (4)

$$(7) \quad p_{a_2, a_1} = m s_1 f_{b_2} p_{a_2, f(a_2)} \text{ rel. } x_{a_2}.$$  

By Morita's diagonal lemma (see MS [23, p. 112]) (6) and (7) mean that $H(f) : H(Y, x) \rightarrow H(Y, f(x))$ induces an isomorphism in pro-$\mathcal{HANR}_0$. Hence $f : (X, x) \rightarrow (Y, f(x))$ induces a shape isomorphism. ■

(3.8) **COROLLARY.** Let $f : X \rightarrow Y$ be a map, $sd X < \infty$ and $sd Y < \infty$. If $f \in AC^n$, then $f : (X, x) \rightarrow (Y, f(x))$ induces a shape isomorphism for each $x \in X$. ■

In the same way as in (3.7) without the dimension condition, we can show the following:

(3.9) **LEMMMA.** If $f : X \rightarrow Y$ has ALP with respect to $\mathcal{POL}$, then $f : (X, x) \rightarrow (Y, f(x))$ induces a shape isomorphism for each $x \in X$. ■

(3.10) **REMARK.** (i) When $X$ and $Y$ are connected, (3.7) follows from (3.4) and Th. 7 of MS [23, p. 152]. (ii) Since any pointed shape equivalence induces
a shape equivalence, (3.7)-(3.9) hold for unpointed case.

Kozlowski [14] introduced the notation of hereditary shape equivalence. We say that a map $f : X \to Y$ is a hereditary shape equivalence provided that for each closed subset $Y_0$ of $Y$, $f \mid _{f^{-1}(Y_0)} : f^{-1}(Y_0) \to Y_0$ is a shape equivalence.

By (2.8), (3.9) and (ii) of (3.10) we have the following:

\[(3.11)\] Theorem. Let $f : X \to Y$ be a closed map between paracompact spaces. If $f$ has ALP with respect to POL pairs, then $f$ is a hereditary shape equivalence.

\section*{§ 4. Approximative extension property.}

We introduce the approximative extension property, and investigate relations between it and approximative $n$-connectedness.

Let $\mathcal{K}$ be a collection of pairs of spaces. Let $(\mathcal{X}, \mathcal{U}) = \{(X_a, U_a), p_{a', a}, A\}$ be an approximative inverse system in TOP. We say that $(\mathcal{X}, \mathcal{U})$ has the approximative extension property, in notation AEP, with respect to $\mathcal{K}$ provided it satisfies the following condition:

\[(\text{AEP})\] For each $a \in A$ there exists $a_1 > a$ satisfying that for any $(K, K_0) \in \mathcal{K}$ and for any maps $h : K_0 \to X_{a_1}$, there exists a map $H : K \to X_a$ such that $(H|K_0, p_{a_1, a} h) < U_a$.

Let $\mathcal{X} = \{X_a, p_{a', a}, A\}$ be an inverse system in TOP. We say that $\mathcal{X}$ has the approximative extension property, in notation AEP, with respect to $\mathcal{K}$ provided it satisfies the following condition:

\[(\text{AEP}^*)\] For each $a \in A$ and for each $U \in Cov(X_a)$ there exists $a_1 > a$ such that for any $(K, K_0) \in \mathcal{K}$ and for any map $h : K_0 \to X_{a_1}$, there exists a map $H : K \to X_a$ such that $(H|K_0, p_{a_1, a} h) < U$.

\[(4.1)\] Lemma. Let $(\mathfrak{q}, \mathfrak{v})$ be an approximative inverse system. We assume that $(\mathfrak{q}, \mathfrak{v})$ is dominated by $(\mathcal{X}, \mathcal{U})$ in Appro-TOP. If $(\mathcal{X}, \mathcal{U})$ has ALP with respect to $\mathcal{K}$, then so does $(\mathfrak{q}, \mathfrak{v})$.

\[(4.2)\] Corollary. The property AEP with respect to $\mathcal{K}$ is invariant in Appro-TOP.

\[(4.3)\] Lemma. Let $(\mathcal{X}', \mathcal{X})$ be an approximative inverse system. Then $(\mathcal{X}', \mathcal{U})$ has AEP with respect to $\mathcal{K}$ iff so does $\mathcal{X}$.

By the way similar to (II.1.1) or (II.1.6) we can easily show (4.1) and (4.3). (4.2) follows from (4.1).
We say that \((\mathcal{X}, \mathcal{U})\) and \(\mathcal{X}\) have the extension property, in notation EP, with respect to \(\mathcal{K}\) provided they satisfy the following condition:

\[(\text{EP}) \text{ For each } a \in A \text{ there exists } a_i > a \text{ such that for each } (K, K_0) \in \mathcal{K} \text{ and for any map } h: K_0 \to X_{a_i} \text{ there exists a map } H: K \to X_a \text{ with } H|_{K_0} = r_{a_i, a} h.\]

\[4.4 \text{ LEMMA. Let } \mathcal{K} \text{ be a collection of polyhedral pairs. Let } (\mathcal{X}, \mathcal{U}) \text{ be an approximative inverse system in POL or ANR. Then } (\mathcal{X}, \mathcal{U}) \text{ has AEP with respect to } \mathcal{K} \text{ iff it has EP with respect to } \mathcal{K}.\]

Using (1.5.7), by the way similar to (3.2), we can show (4.4) for approximative inverse systems in POL or ANR.

Let \(p = \{r_a : a \in A\} : X \to (\mathcal{X}, \mathcal{U})\) be an approximative resolution and \(p : X \to \mathcal{X}\) a resolution of a space \(X\). We say that \(p : X \to (\mathcal{X}, \mathcal{U})\) has AEP with respect to \(\mathcal{K}\) provided that \(\langle \mathcal{X}, \mathcal{U} \rangle\) has AEP with respect to \(\mathcal{K}\). Similarly we define EP for \(p : X \to (\mathcal{X}, \mathcal{U})\). Similarly we define AEP and EP for \(p : X \to \mathcal{X}\). By the way similar to (II.1.3) and (II.1.7) we can show the following:

\[4.5 \text{ LEMMA. (i) Let } p : X \to (\mathcal{X}, \mathcal{U}) \text{ and } p' : X \to (\mathcal{X}, \mathcal{U})' \text{ be approximative AP-resolutions. If } p \text{ has AEP with respect to } \mathcal{K}, \text{ then so does } p'.\]

\[\text{(ii) Let } p : X \to \mathcal{X} \text{ and } p' : X \to \mathcal{X}' \text{ be AP-resolutions. If } p \text{ has AEP with respect to } \mathcal{K}, \text{ then so does } p'.\]

From (4.3)-(4.5), by the way similar to (II.1.8) we have the following:

\[4.6 \text{ THEOREM. Let } X \text{ be a space and } \mathcal{K} \text{ a collection of pairs of spaces. Then (i) and (ii) below are equivalent, and moreover if } \mathcal{K} \text{ is a polyhedral pair, then (i)-(iv) below are equivalent.}\]

\[\text{(i) Any/some approximative AP-resolution has AEP with respect to } \mathcal{K}.\]

\[\text{(ii) Any/some AP-resolution has AEP with respect to } \mathcal{K}.\]

\[\text{(iii) Any/some approximative ANR- or POL-resolution has EP with respect to } \mathcal{K}.\]

\[\text{(iv) Any/some ANR- or POL-resolution has EP with respect to } \mathcal{K}.\]

Thus by (4.6) we say that a space \(X\) has the approximative extension property, in notation AEP, with respect to \(\mathcal{K}\) provided it satisfies any one of the conditions in (4.6).
By the way similar to (1.12) we have the following:

(4.7) Lemma. Let \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) be collections of \( P \)-pairs of spaces and \( \mathcal{K}_1 \subset \subset \mathcal{K}_2 \). If a space \( X \) has AEP with respect to \( \mathcal{K}_2 \), then so does \( X \) with respect to \( \mathcal{K}_1 \). ■

Borsuk introduced the notion of approximative \( n \)-connectedness for spaces. We may assume that a space \( X \) is approximatively \( n \)-connected, in notation \( X \in AC^n \), provided \( \text{pro-} \pi_n(X, x) = 0 \) for each \( x \in X \) and for each \( k, 0 \leq k \leq n \). We say that \( X \) is approximatively \( \infty \)-connected, in notation \( X \in AC^\infty \), provided that \( X \in AC^n \) for each \( n, n=0, 1, 2, \ldots \).

(4.8) Lemma. \( \text{pro-} \pi_d(X, x) = 0 \) for some \( x \in X \) iff \( X \) is connected.

Proof. Let \( p : X \to \mathcal{X} \) be an ANR-resolution and \( x \) be a point of \( X \). We put \( x_a = p_a(x) \) for \( a \in A \). Then \( p : (X, x) \to (\mathcal{X}, x) = \{(X_a, x_a), p_a', a : A\} \) is an ANR-resolution of \( (X, x) \).

First we assume that \( X \) is connected and show that \( \text{pro-} \pi_d(X, x) = 0 \). Take any \( a \in A \). Since \( X_a \) is an ANR, \( X_a \) is locally path-connected. Then all path-connected components are open and closed. Since \( X \) is connected, there exists a path-connected component \( T \) of \( X_a \) such that \( \bar{p}_a(X) \subset T \). Since \( T \) is open, by (B3) there exists \( a' > a \) such that \( p_a'(X_a') \subset T \).

We show that \( p_{a',a} \) induces the zero homomorphism \( p_{a',a*} : \pi_d(X_a', x_a') \to \pi_d(X_a, x_a) \). Take any map \( r : (\partial I, 0) \to (X_a', x_a') \). By the choice of \( a' \), \( p_{a',a}(\partial I) \subset T \). Since \( T \) is path-connected, there exists an extension \( r' : I \to T \subset X_a \) of \( p_{a',a}r \). Thus \( p_{a',a} \) is the zero homomorphism and hence \( \text{pro-} \pi_d(X, x) = 0 \).

Next we assume that \( X \) is not connected and show that \( \text{pro-} \pi_d(X, x) \neq 0 \). By the assumption there exist open subsets \( X_0, X_1 \) of \( X \) such that \( X_0 \cup X_1 = X \), \( X_0 \cap X_1 = \emptyset \), \( X_0, X_1 \neq \emptyset \). There exists a map \( f : X \to \partial I = \{0, 1\} \) such that \( f(X_0) = 0 \) and \( f(X_1) = 1 \). By (R1) there exist \( a \in A \) and a map \( g : X_a \to \partial I \) such that \( (f, gp_a) \in \mathcal{U} \), where \( \mathcal{U} = \{(0), (1)\} \subset \mathcal{C}_o(\partial I) \). By the choice of \( \mathcal{U} \), \( f = gp_a \) and then \( p_a(X_0) \subset g^{-1}(0) \) and \( p_a(X_1) \subset g^{-1}(1) \). We assume that \( \text{pro-} \pi_d(X, x) = 0 \). Then there exists \( a' > a \) such that \( p_{a',a} \) induces the zero homomorphism.

We may assume that \( x \in X_a \). Take any point \( x_i \in X_1 \). We define a map \( k : (\partial I, 0) \to (X_a', x_a') \) by \( k(0) = x_a \) and \( k(1) = p_{a'}(x_i) \). By the choice of \( a' \) there exists an extension \( k' : I \to X_a \) of \( p_{a',a}k \). Then \( g k' : I \to \partial I \) is an extension of \( gp_{a',a}k \). Since \( p_{a',a}k(0) = x_a \in g^{-1}(0) \) and \( p_{a',a}k(1) = p_a(x_i) \in g^{-1}(1) \), \( g p_{a',a}k : \partial I \to \partial I \) is the identity map. Thus \( 1_{\partial I} : \partial I \to \partial I \) is extendable to \( g k' : I \to \partial I \). This is a contradiction and hence \( \text{pro-} \pi_d(X, x) \neq 0 \). ■
(4.9) **Lemma.** Let \((X, x)\) be a pointed space. Then \(\text{pro-}\tau_0(X, x) = 0\) iff any **POL**- or **ANR**-resolution of \(X\) has **EP** with respect to \(\{(A^t, \partial A^t)\}\).

**Proof.** We use the same notation as in the proof of (4.8). We assume that \(\text{pro-}\tau_0(X, x) = 0\), and show the property **EP**. Take any \(a \in A\). Since \(\text{pro-}\tau_0(X, x) = 0\), there exists \(a' > a\) such that \(\rho_{a', a^*} : \tau_0(X, x) \to \tau_0(X, x)\) is the zero homomorphism. Take any map \(f : \partial I \to X_{a'}\). Take another point \(2\). We define maps \(f_0 : \{0, 1\} \to X_{a'}\) and \(f_1 : \{0, 1\} \to X_{a'}\) as follows: \(f_0(0) = f(0), f_1(1) = x_{a'} = f_1(1)\) and \(f_1(2) = f(1)\). By the choice of \(a'\) there exist maps \(g_0 : [0, 1] \to X_a\) and \(g_1 : [1, 2] \to X_a\) such that \(g_0\) and \(g_1\) are extensions of \(\rho_{a', a, 0}\) and \(\rho_{a', a, 1}\), respectively. Thus we define a map \(g : I \to X_a\) as follows: \(g(t) = g_0(2t)\) for \(0 \leq t \leq 1/2\) and \(g(t) = g_1(2t)\) for \(1/2 \leq t \leq 1\). It is easy to show that \(g\) is well-defined and is an extension of \(\rho_{a', a, 1}\). Hence \(p\) has the required property. The converse is trivial. \(\blacksquare\)

(4.10) **Lemma.** Let \((X, x)\) be a pointed space and let \(n\) be an integer. Then \(\text{pro-}\tau_0(X, x) = 0\) and \(\text{pro-}\tau_n(X, x) = 0\) iff any **POL**- or **ANR**-resolution of \(X\) has **EP** with respect to \(\{(A^t, \partial A^t)\}, (A^{n+1}, \partial A^{n+1})\}\).

**Proof.** We use the same notations as in the proof of (4.8). We assume that \(\text{pro-}\tau_0(X, x) = 0\) and \(\text{pro-}\tau_n(X, x) = 0\). Take any \(a \in A\). Since \(\text{pro-}\tau_0(X, x) = 0\), there exists \(a_1 > a\) such that \(\rho_{a_1, a^*} : \tau_n(X, x) \to \tau_n(X, x)\) is the zero homomorphism. By (4.9) there exists \(a_2 > a_1\) satisfying (**EP**) with respect to \(\{(A^t, \partial A^t)\}\) for \(p\) and \(a\).

We show that \(a_2\) has the required property. Take any map \(f : \partial A^{n+1} \to X_{a_2}\) and any point \(v \in \partial A^{n+1}\). By the choice of \(a_2\) there exists a map \(h : I \to X_{a_1}\) such that \(h(0) = x_{a_1}\) and \(h(1) = \rho_{a_1, a_2}f(v)\). Put \(T = \partial A^{n+1} \cup \{v\} \cap I \subset \partial A^{n+1} \times I\) We define a map \(k : T \to X_{a_1}\) as follows: \(k(x, 1) = \rho_{a_2, a_1}f(x)\) for \(x \in \partial A^{n+1}\) and \(k(v, t) = h(t)\) for \(t \in I\). By the homotopy extension theorem there exists a map \(K : \partial A^{n+1} \times I \to X_{a_1}\) such that \(K|_{\partial A^{n+1}} = k\). We define a map \(w : (\partial A^{n+1}, v) \to (X_{a_1}, x_{a_1})\) by \(w(x) = K(x, 0)\) for \(x \in \partial A^{n+1}\). By the choice of \(a_1\) there exists a map \(W : A^{n+1} \to X_a\) such that \(W|_{\partial A^{n+1}} = \rho_{a_1, a_1}w\). Put \(S = A^{n+1} \cup \{0\} \cup \partial A^{n+1} \times I \subset A^{n+1} \times I\). We can define a map \(m : S \to X_{a_1}\) as follows: \(m(x, 0) = W(x)\) for \((x, 0) \in A^{n+1} \times \{0\}\) and \(m(x, t) = \rho_{a_1, a_1}K(x, t)\) for \((x, t) \in \partial A^{n+1} \times I\). By the homotopy extension theorem there exists a map \(M : A^{n+1} \times I \to X_{a_2}\) such that \(M|S = m\). We define a map \(g : A^{n+1} \to X_a\) by \(g(x) = M(x, 1)\) for \(x \in A^{n+1}\). It is easy to show that \(g|_{\partial A^{n+1}} = \rho_{a_2, a_2}f\).

By the choice of \(a_2\), for any map \(f : \partial I \to X_{a_2}\), \(\rho_{a_2, a_2}f\) is extendable to \(I\). Hence \(p\) has the required property. The converse is trivial. \(\blacksquare\)
(4.11) Lemma. Let \((X, x)\) be a pointed space. Then \(\text{pro-}\pi_k(X, x) = 0\) for \(k, 0 \leq k \leq n\) iff any POL- or ANR-resolution of \(X\) has EP with respect to \((\Delta^{k+1}, \partial \Delta^{k+1})\): \(k = 0, 1, \ldots, n\).

(4.12) Lemma. Let \((X, x)\) be a pointed space. Then \(\text{pro-}\pi_k(X, x) = 0\) for \(k, 0 \leq k \leq n\) iff any POL- or ANR-resolution of \(X\) has EP with respect to \((D, S)_n\).

By induction on \(n\) we easily show (4.11) from (4.9) and (4.10). (4.12) follows from (4.11) and the fact that any resolution has EP with respect to POL-pairs.

(4.13) Corollary. Let \(X\) be a space and let \(n\) be an integer.

(i) For any points \(x_1, x_2\) of a connected space \(X\), \(\text{pro-}\pi_k(X, x_1) = 0\) iff \(\text{pro-}\pi_k(X, x_2) = 0\).

(ii) For any \(x_1, x_2 \in X\), \(\text{pro-}\pi_k(X, x_1) = 0\) for \(k, 0 \leq k \leq n\) iff \(\text{pro-}\pi_k(X, x_2) = 0\) for \(k, 0 \leq k \leq n\).

(4.13) follows from (4.8)-(4.12).

(4.14) Theorem. For each integer \(n \geq 0\) the following statements are equivalent:

(i) \(\text{pro-}\pi_k(X, x) = 0\) for any \(x \in X\) and for \(k, 0 \leq k \leq n\).

(ii) \(\text{pro-}\pi_k(X, x) = 0\) for some \(x \in X\) and for \(k, 0 \leq k \leq n\).

(iii) \(X\) has AEP with respect to \((D, S)_n\).

(iv) \(X\) has AEP with respect to POL\(^{n+1}\) pairs.

(v) \(X\) has AEP with respect to POL\(^n\) pairs.

(vi) \(X\) has AEP with respect to \(n^+\) POL pairs.

Proof. (i) and (ii) are equivalent by (4.13). (ii) and (iii) are equivalent by (4.6) and (4.12). (iv) and (vi) are equivalent by (1.3.16) and (4.7). Trivially (v)\(\rightarrow\) (iv)\(\rightarrow\) (iii). By the way similar to (4.12) we can show that the property EP with respect to \((D, S)_n\) implies the property EP with respect to POL\(^n\) pairs for polyhedral resolutions. Hence (iii) implies (v).

Approximative \(n\)-connectedness can be defined by (i) in (4.14). Thus it is equivalent to the other conditions in (4.14). We say that \(X\) is approximatively \(\infty\) \(\infty\)-connected, in notation \(X \in AC^\infty\), provided that it has AEP with respect to \((D, S)_\infty\). Also we need to consider spaces having AEP with respect to POL pairs. These properties are characterized as follows:
Theorem. The following statements are equivalent:

(i) A space $X$ has trivial shape.
(ii) $X$ has AEP with respect to $\text{POL}_{\text{pairs}}$.
(iii) $X$ has AEP with respect to $\text{TOP}_{\text{pairs}}$.

(ii) and (iii) are equivalent by (1.3.16) and (4.3). By the way similar to (4.16) below we can show the equivalence of (i) and (ii). □

Theorem. Let $X$ be a space and $\text{sd} X = n < \infty$. Then $X$ is approximatively $n$-connected iff $X$ has trivial shape.

Proof. Let $p : X \to \mathbb{X}$ be an $\text{POL}$-resolution. We assume that $X \subseteq AC^n$. Take any $a \in A$. By (4.4) and (4.14) there exists $a' > a$ satisfying EP with respect to $\text{POL}_{\text{pairs}}^{a+1}$. Since $\text{sd} X = n$, by Theorem 2 of MS [23, p. 96] there exist $a'' > a'$, a polyhedron $P$ and maps $f : X_{a'} \to P$, $g : P \to X_{a'}$ such that $p_{a'-a'} \simeq gf$ and $\dim P = n$. Put $T = P \times \{0, 1\} \subset P \times I$ and take any point $x_{a'} \in X_{a'}$. We define a map $h : T \to X_{a'}$ as follows: $h(x, 0) = g(x)$ and $h(x, 1) = x_{a'}$ for $x \in P$. Since $(P \times I, T) \subseteq \text{POL}_{\text{pairs}}^{a+1}$, by the choice of $a'$ there exists a map $H : P \times I \to X_{a'}$ such that $H|T = p_{a'-a'}$. Thus $p_{a'-a'}g$ is homotopic to a constant map. Since $p_{a'-a'} \simeq gf$, $p_{a'-a'}$ is homotopic to a constant map. Hence $X$ has trivial shape.

We assume that $X$ has trivial shape. We show that $p$ has EP with respect to $\text{POL}_{\text{pairs}}$. By the assumption for any $a \in A$ there exists $a' > a$ such that $p_{a'-a}$ is homotopic to a constant map $k : X_{a'} \to X_a$. We put $k(X_{a'}) = x_a$. Let $H : X_{a'} \times I \to X_a$ be a homotopy such that $H_0 = p_{a'-a}$ and $H_1 = k$. Take any polyhedral pair $(P, P_0)$ and any map $f : P_0 \to X_{a'}$. We define a map $G : P \times \{1\} \cup P_0 \times I \to X_a$ by $G(x, 1) = x_a$ for $x \in P$ and $G(x, t) = H(f(x), t)$ for $(x, t) \in P_0 \times I$. By the homotopy extension theorem (see MS [23, p. 291]) there exists an extension $\tilde{G} : P \times I \to X_a$ of $G$. We put $g : P \to X_a$ by $g(x) = \tilde{G}(x, 0)$ and then $g$ is an extension of $p_{a'-a}$. Thus $p$ has EP with respect to $\text{POL}_{\text{pairs}}$. Hence $X \subseteq AC^n$ by (4.14). □

Corollary. Let $X$ be a space and $\text{sd} X = \infty$. Then $X \subseteq AC^\infty$ iff $X$ has trivial shape. □

§ 5. Partial realizations for decompositions.

In this section we introduced partial realizations for decompositions and the approximative full extension property.
Approximative Shape IV

We say that \((K, L)\) is a simplicial pair provided \(L\) is a subcomplex of a simplicial complex \(K\). SC pairs denotes the category of all simplicial pairs and simplicial maps among them. Let \(n\) be an integer. SC pairs \(SC\) denotes the full subcategory of \(SC\) consisting of all simplicial pairs \((K, L)\) with \(\dim(K-L) \leq n\). SC pairs \(SC^n\) denotes the full subcategory of \(SC\) consisting of all simplicial pairs \((K, L)\) with \(\dim K \leq n\). \(K^n\) denotes the \(n\)-skeleton of \(K\). SC pairs denotes the full subcategory of \(SC\) consisting of all simplicial pairs \((K, L)\) with \(K^0 \subseteq L\). We put \(SC = SC \cap SC\) and \(SC^n = SC \cap SC^n\).

Let \(X\) be a space and \(U\) a collection of subsets of \(X\). Let \((K, L) \in Ob SC\). We say that a map \(g : |L| \rightarrow X\) is a partial realization of \((K, L)\) in \(X\) relative to \(U\) provided for each (closed) simplex \(s\) of \(K\) there exists \(U \in U\) such that \(g([s] \cap |L|) \subseteq U\). In case \(L = K\), we say that \(g\) is a full realization of \((K, L)\) in \(X\) relative to \(U\). Sometimes we identify a simplicial complex \(K\) and its geometrical realization \(|K|\) (endowed with the \(CW\)-topology).

Hereafter we assume that \(Y\) is a paracompact space and \(f : X \rightarrow Y\) a closed onto map with the following property:

\[(\#)\] \(f^{-1}(Y_0)\) is \(P\)-embedded in \(X\) for each closed subset \(Y_0\) of \(Y\).

Let \(\mathcal{K}\) be a subcollection of \(Ob SC\). Let \(\mathbf{p} = \{p_a : a \in A\} : X \rightarrow (\mathcal{X}, \mathcal{U})\) \(= \{(X_a, U_a), p_{a'}, a, A\}\) be an approximative resolution. We put \(D_\mathcal{U} = f^{-1}(Y_0)\), \(D(y, U) = st(p_a(D_\mathcal{U}), U)\) for any \(a \in A\), \(y \in Y\) and \(U \subseteq C_{\mathcal{U}}(X_a)\), and thus \(\partial(U)\) \(= \{D(y, U) : y \in Y\} \subseteq C_{\mathcal{U}}(X_a)\). We put \(D(y, a) = D(y, U_a)\) and \(\partial_a = \partial(U_a)\) for \(y \in Y\) and \(a \in A\).

We say that \(\mathbf{p}\) has the approximative full extension property, in notation \(AP\), with respect to \(\mathcal{K}\) and \(f\) provided it satisfies the following property:

\((AP)\) For each \(a \in A\) there exists \(a' > a\) such that for each \((K, L) \in \mathcal{K}\) and each partial realization \(g : L \rightarrow X_{a'}\) of \((K, L)\) in \(X_{a'}\) relative to \(\partial_{a'}\) there exists a full realization \(G : K \rightarrow X_a\) of \((K, L)\) in \(X_a\) relative to \(\partial_a\) such that \((G|L, p_{a'}, g) < \partial_{a'}\).

We say that \(\mathbf{p} : X \rightarrow (\mathcal{X}, \mathcal{U})\) has the full extension property, in notation \(FEP\), with respect to \(\mathcal{K}\) and \(f\) provided that it satisfies the following condition:

\((FEP)\) For each \(a \in A\) there exists \(a' > a\) such that for each \((K, L) \in \mathcal{K}\) and each partial realization \(g : L \rightarrow X_{a'}\) of \((K, L)\) in \(X_{a'}\) relative to \(\partial_{a'}\) there exists a full realization \(G : K \rightarrow X_a\) of \((K, L)\) relative to \(\partial_a\) such that \((G|L, p_{a'}, g)\).

\((5.1)\) LEMMA. Let \(\mathbf{p}\) and \(\mathbf{p}'\) be approximative \(AP\)-resolutions of \(X\). If \(\mathbf{p}\) has \(AP\) with respect to \(\mathcal{K}\) and \(f\), then so does \(\mathbf{p}'\).
(5.2) Lemma. Let \( p \) be an approximative POL- or ANR-resolution of \( X \). Then \( p \) has AFEP with respect to \( \mathcal{K} \) and \( f \) iff \( p \) has FEP with respect to \( \mathcal{K} \) and \( f \).

Let \( p = \{p_a : a \in A\} : X \to \mathcal{X} = \{X_a, p_{a'}, A\} \) be a resolution. We say that \( p \) has the approximative full extension property, in notation AFEP, with respect to \( \mathcal{K} \) and \( f \) provided it satisfies the following condition:

(AFEP)* For each \( a \in A \) and each \( \mathcal{U} \in \mathcal{C}_o(X_a) \) there exist \( a' > a \) and \( \mathcal{U}' \in \mathcal{C}_o(X_{a'}) \) such that for each \( (K, L) \subseteq \mathcal{K} \) and each partial realization \( g : L \to X_{a'} \) of \( (K, L) \) in \( X_a \) relative to \( \mathcal{D}''(\mathcal{U}') \) there exists a full realization \( G : K \to X_a \) of \( (K, L) \) in \( X_a \) relative to \( \mathcal{D}''(\mathcal{U}) \) such that \( (G|L, p_{a', a}) \ll \mathcal{U} \).

We say that \( p \) has the full extension property, in notation FEP, with respect to \( \mathcal{K} \) and \( f \) provided it satisfies the following property:

(FEP)* For each \( a \in A \) and each \( \mathcal{U} \in \mathcal{C}_o(X_a) \) there exist \( a' > a \) and \( \mathcal{U}' \in \mathcal{C}_o(X_{a'}) \) such that for each \( (K, L) \subseteq \mathcal{K} \) and each partial realization \( g : L \to X_{a'} \) of \( (K, L) \) in \( X_{a'} \) relative to \( \mathcal{D}''(\mathcal{U}') \) there exists a full realization \( G : K \to X_{a'} \) of \( (K, L) \) in \( X_{a'} \) relative to \( \mathcal{D}''(\mathcal{U}) \) such that \( G|L = p_{a', a} \).

(5.3) Lemma. Let \( p \) and \( p' \) be AP-resolutions of \( X \). If \( p \) has AFEP with respect to \( \mathcal{K} \) and \( f \), then so does \( p' \).

(5.4) Lemma. Let \( p : X \to (\mathcal{X}, \mathcal{U}) \) be an approximative AP-resolution. Then \( p : X \to (\mathcal{X}, \mathcal{U}) \) has AFEP with respect to \( \mathcal{K} \) and \( f \) iff \( p : X \to \mathcal{X} \) has AFEP with respect to \( \mathcal{K} \) and \( f \).

(5.5) Lemma. Let \( p \) be an ANR- or POL-resolution of \( X \). Then \( p \) has AFEP with respect to \( \mathcal{K} \) and \( f \) iff \( p \) has FEP with respect to \( \mathcal{K} \) and \( f \).

(5.6) Theorem. The following statements are equivalent:

(i) Any/some approximative AP-resolution of \( X \) has AFEP with respect to \( \mathcal{K} \) and \( f \).

(ii) Any/some AP-resolution of \( X \) has AFEP with respect to \( \mathcal{K} \) and \( f \).

(iii) Any/some approximative ANR- or POL-resolution has FEP with respect to \( \mathcal{K} \) and \( f \).

(iv) Any/some ANR- or POL-resolution has FEP with respect to \( \mathcal{K} \) and \( f \).

Proofs of (5.1)-(5.6). (5.6) follows from (5.1)-(5.5). In a way similar to (4.4) we can show (5.2) and (5.5). (5.4) is an easy consequence of the definitions. (5.1) follows from (1.3.3), (5.3) and (5.4).
We show (5.3). We need an elementary fact:

Claim. Let \( p: X \to R \), \( q: X \to S \) and \( g: R \to S \) be maps. Let \( R \subseteq Cov(R) \) and \( S \subseteq Cov(S) \). If \( g^{-1}(S) \supseteq R \) and \( (gp, q) < S \), then \( g(st(p(Dy), R)) \subseteq st(q(Dy), stS) \) for each \( y \in Y \).

Let \( p: X \to X' \) and \( q = \{ q_b: b \in B \}: X \to q = \{ Y_b, q_{v, b}, B \} \) be AP-resolutions. We assume that \( p \) has AFEP with respect to \( X \) and \( f \), and show that \( q \) has also AFEP.

Take any \( b \in B \) and any \( CV \subseteq Cov(Y_b) \). Then there exist \( CV_1, CV_2, CV_3 \subseteq Cov(Y_b) \) such that \( stCV_1 < CV, CV_2 < CV_1 \), \( stCV_3 < CV_2 \) and \( CV_3 \) satisfies (R2) for \( q \) and \( CV \). By (R1) there exist \( a \in A \) and a map \( g: X_a \to Y_b \) such that

\[
(q_b, gp_a) < CV_1. \tag{1}
\]

By the assumption there exist \( a_i > a \) and \( U \subseteq Cov(X_{a_i}) \) satisfying (AFEP) for \( a \) and \( g^{-1}CV_b \). Take \( U_1 \subseteq Cov(X_{a_i}) \) such that \( stU_1 < U \wedge (gp_{a_i, a})^{-1}CV_b \). By (R1) there exist \( b_i > b \) and a map \( h: Y_{a_i} \to X_1 \) such that

\[
(q_{b_i}, gp_{a_i, a}h_{b_i}) < CV_i \tag{2}.
\]

By the choice of \( U_1 \) and (2) \( (gp_{a_i, a}h_{b_i}) < CV_2 \). Then by (1) \( (q_{b_i}, gp_{a_i, a}h_{b_i}) < stCV_3 < CV_2 \). By the choice of \( CV_3 \) there exists \( b_2 > b_1 \) such that

\[
(q_{b_2}, gp_{a_i, a}h_{b_2}) < CV_2. \tag{3}
\]

We show that \( b_2 \) and \( CV' = (h_{b_2, b_1})^{-1}U_1 \subseteq Cov(Y_{b_2}) \) have the required properties. Take any \( (K, L) \in \mathcal{K} \) and any partial realization \( t: L \to Y_{b_2} \) of \( (K, L) \) in \( Y_{b_2} \) relative to \( \mathcal{D}(CV') \). By (2) and the Claim, \( h_{b_2, b_1}t: L \to X_{a_i} \) is a partial realization of \( (K, L) \) relative to \( \mathcal{D}(stU_1) \) and then relative to \( \mathcal{D}(U) \), because \( stU_1 < U \). By the choice of \( a_i \) and \( U \) there exists a full realization \( T': K \to X_a \) of \( (K, L) \) relative to \( \mathcal{D}(g^{-1}CV_b) \) such that

\[
(T'|L, p_{a_i, a}h_{q_{b_2}, q_1}t) < g^{-1}CV_3. \tag{4}
\]

By (3) and (4) \( (gT'|L, q_{b_2}) < stCV_1 < CV \). By the Claim and (1), \( gT': K \to Y_b \) is a full realization of \( (K, L) \) relative to \( \mathcal{D}(stCV_1) \) and then relative to \( \mathcal{D}(CV) \), because \( stCV_1 < CV \). Hence \( q \) has AFEP.

Thus by (5.6) we may say that \( f \) and the decomposition \( D = \{ Dy: y \in Y \} \) of \( X \) have the approximative full extension property, in notation AFEP, with respect to \( \mathcal{K} \) provided they satisfy any one of the conditions in (5.6). Let \( K \) be a subcategory of \( eSC_{pairs} \). We say that \( f \) and \( D \) have AFEP with respect to \( K \) provided that they have AFEP with respect to \( ObK \), respectively.
Hereafter we assume that \( p : X \to (X, \mathcal{U}) \) is an approximative ANR-resolution such that \( st(p_a(X), \mathcal{U}_a) = X_a \) for all \( a \in A \). By (2.3) any space admits an ANR-resolution satisfying the above condition.

(5.7) Lemma. For each \( a \in A \) there exists \( a_i > a \) such that \( p^{-1}_{a_i} \mathcal{U}_a > st \mathcal{U}_{a_i} \).

Proof. Take any \( a \in A \) and then there exists \( a_i > a \) such that \( p^{-1}_{a_i} \mathcal{U}_a > st \mathcal{U}_{a_i} \).

Since \( f : X \to Y \) is closed, for each \( y \in Y \) there exists an open neighborhood \( V_y \) of \( y \) in \( Y \) such that \( D_y \subseteq f^{-1}(V_y) \subseteq st(D_y, p^{-1}_{a_i}(\mathcal{U}_{a_i})) \). Since \( Y \) is paracompact, \( \mathcal{U} = \{ V_y : y \in Y \} \subseteq C_{C_0}(Y) \) and then there exists \( \mathcal{V}_1 \subseteq C_{C_0}(Y) \) such that \( \mathcal{V} > st \mathcal{V}_1 \).

There exists \( a_2 > a \) such that \( f^{-1}(\mathcal{V}_1) > p^{-1}_{a_2}(\mathcal{U}_{a_2}) \). Put \( \mathcal{U}_{a_2} = \{ U_e : e \in E \} \). Since \( X_{a_2} \) is metrizable, by Theorem 2 of Kuratowski [17, p. 226] for each \( e \in E \) there exists an open \( U(e) \) of \( X_{a_2} \) satisfying

\[
(1) \quad U_e \cap p^{-1}_{a_2}(X) = G(U_e) \cap p^{-1}_{a_2}(X) \quad \text{and} \quad U_e \cap p^{-1}_{a_2}(X) = \emptyset.
\]

By (1) \( \mathcal{W} = \{ X_{a_2} - p^{-1}_{a_2}(X) \} \cap \{ G(U_e) : e \in E \} \) is an open covering of \( X_{a_2} \). Then there exists \( a_3 > a_2 \) such that \( p^{-1}_{a_3}(\mathcal{W}) > \mathcal{U}_{a_3} \).

We show that \( a_3 \) has the required property. Take any \( y \in Y \) and then there exists \( V_y \subseteq \mathcal{V}_1 \) such that \( y \in V_y \). By the choice of \( \mathcal{V}_1 \) there exists \( V_y \subseteq \mathcal{V} \) such that \( st(V_y, st(\mathcal{V}_1)) \subseteq V_y \). Take any \( y' \in Y \) such that \( D(y, a_3) \cap D(y', a_3) = \emptyset \). Thus there exist \( U_1, U_2 \subseteq \mathcal{U}_{a_3} \) such that \( p_{a_3}(D_y) \cap U_1 = \emptyset \), \( p_{a_3}(D_{y'}) \cap U_1 = \emptyset \) and \( U_1 \cap U_2 = \emptyset \). By the choice of \( a_3 \) there exist \( e_1, e_2 \in E \) such that \( p^{-1}_{a_3, a_2}(G(U_{e_1})) \supseteq U_1 \) and \( p^{-1}_{a_3, a_2}(G(U_{e_2})) \supseteq U_2 \). Then

\[
(3) \quad p_{a_2}(D_y) \cap G(U_{e_1}) \neq \emptyset, \quad p_{a_2}(D_{y'}) \cap G(U_{e_2}) \neq \emptyset \quad \text{and} \quad G(U_{e_1}) \cap G(U_{e_2}) \neq \emptyset.
\]

By (2) and (3), \( U_{e_1} \cup U_{e_2} \cap p^{-1}_{a_2}(X) = \emptyset \) and then \( U_{e_1} \cup U_{e_2} \cap p_{a_2}(X) = \emptyset \). This means that

\[
(4) \quad p^{-1}_{a_2}(U_{e_1}) \cap p^{-1}_{a_2}(U_{e_2}) = \emptyset.
\]

By (1) and (3), \( p_{a_2}(D_y) \cap U_{e_1} = p_{a_2}(D_{y'}) \cap p^{-1}_{a_2}(X) \cap U_{e_1} = p_{a_2}(D_y) \cap p^{-1}_{a_2}(X) \cap G(U_{e_1}) = p_{a_2}(D_y) \cap G(U_{e_1}) \neq \emptyset \) and \( p_{a_2}(D_{y'}) \cap U_{e_2} = \emptyset \). This means that \( D_y \cap p_{a_2}(U_{e_1}) \neq \emptyset \) and \( D_{y'} \cap p_{a_2}(U_{e_2}) = \emptyset \). Thus \( f^{-1}(V_1) \supseteq \mathcal{U}_{a_2}(U_{e_1}) \) and \( f^{-1}(V_2) \supseteq \mathcal{U}_{a_2}(U_{e_2}) \). Then by (4), \( D_y \cap f^{-1}(V_1) \neq \emptyset \), \( D_{y'} \cap f^{-1}(V_2) \neq \emptyset \) and \( f^{-1}(V_1) \cap f^{-1}(V_2) \neq \emptyset \). That is, \( f(D_y) \subseteq V_1 \), \( f(D_{y'}) \subseteq V_2 \) and \( V_1 \cap V_2 = \emptyset \). This means that \( y' \in st(y, st(\mathcal{V}_1)) \). Since \( y \in V_y \), \( y' \in st(V_y, st(\mathcal{V}_1)) \subset V_{y'} \). By the choice of \( V_{y'} \), \( D_y \subset f^{-1}(V_{y'}) \subset st(D_{y'}, p^{-1}_{a_3}(\mathcal{U}_a)) \) and then \( D(y', a_3) \subset f^{-1}(V_{y'}) \subset D(y', a_3) \subset \mathcal{V}_1 \).
Let \( \mathcal{K} \) be a collection of polyhedral pairs. Let \( Y_0 \) be a closed subset of \( Y \). Let \( a \in A \) and \( \mathcal{U} \subseteq C_{ov}(X_a) \). We say that \( Y_0 \) has the extension property with respect to \( a, \mathcal{U} \) and \( \mathcal{K} \), in notation \( EP(a, \mathcal{U}, \mathcal{K}) \), provided it satisfies the following condition:

\[
EP(a, \mathcal{U}, \mathcal{K}): \text{There exists } a_1 \succ a \text{ with the property that for each point } y \text{ of } Y_0, \text{for any } (K, K_0) \subseteq \mathcal{K} \text{ and for each map } g : K_0 \to st(p_a(D_y), \mathcal{U}_{a_1}) \text{ there exist a point } y' \text{ of } Y_0 \text{ and a map } G : K \to st(p_a(D_{y'}), \mathcal{U}) \text{ such that } G | K_0 = p_a.g(K_0). 
\]

(5.8) Lemma. Let \( Y_1 \) and \( Y_2 \) be closed subsets of \( Y \). If \( Y_1 \) has \( EP(a, \mathcal{U}, \mathcal{K}) \) and \( Y_2 \subseteq Y_1 \), then \( Y_2 \) has \( EP(a, st(st(p_a(D_y), \mathcal{U}): y \in Y_1), \mathcal{K}) \).

Proof. By the assumption there exists \( a_1 \succ a \) satisfying \( EP(a, \mathcal{U}, \mathcal{K}) \) for \( Y_1 \). By (AI3) there exists \( a_2 \succ a_1 \) such that \( p_{a_2, a} \mathcal{U} \supseteq \mathcal{U}_{a_2} \). We show that \( a_2 \) has the required property. Take any point \( y \) of \( Y_2 \) and any map \( g : K_0 \to st(p_{a_1}(D_y), \mathcal{U}_{a_2}) \). Since \( p_{a_2, a_1}g : K_0 \to st(p_{a_1}(D_y), \mathcal{U}_{a_1}) \), by the choice of \( a_1 \) there exist a point \( y' \) of \( Y_1 \) and a map \( G : K \to st(p_a(D_{y'}), \mathcal{U}) \) such that \( p_{a_2, a}g = G | K_0 \). By the choice of \( a_2 \) \( p_{a_2, a}g : K_0 \to st(p_a(D_y), \mathcal{U}) \). Thus \( G(K) \subseteq st(p_a(D_y), st(st(p_a(D_y), \mathcal{U}): y \in Y_1) \). Hence \( Y_2 \) has the required property.

(5.9) Lemma. Let \( Y_1, Y_2, \ldots, Y_n \) be closed subsets of \( Y \). If all \( Y_i \), have \( EP(a, \mathcal{U}, \mathcal{K}) \), then so does \( Y_\circlearrowright = \cup \{Y_i: i = 1, 2, \ldots, n\} \).

(5.9) is an easy consequence of the definition.

(5.10) Lemma. Let \( \{Y_s: s \in S\} \) be a discrete family of closed subsets in \( Y \). If all \( Y_s \) have \( EP(a, \mathcal{U}, \mathcal{K}) \), then \( Y_\circlearrowright = \cup \{Y_s: s \in S\} \) has \( EP(a, st\mathcal{U}, \mathcal{K}) \).

Proof. Since \( Y \) is paracompact, then by Theorem 5.1.17 of Engelking [9, p. 379] there exists a collection \( \{V_s: s \in S\} \) such that \( Y_s \subset V_s \), \( V_s \) are open subsets of \( Y \), \( V_s \cap V_s' = \emptyset \) for \( s, s' \in S \) with \( s \neq s' \). Since \( \{Y_s: s \in S\} \) is discrete, \( Y_0 \) is closed and then \( \mathcal{U} = \{V_s: s \in S\} \cup \{Y - Y_0\} \subseteq Cov(Y) \). There exists \( a_1 \succ a \) such that \( f^{-1}\mathcal{U} \supseteq p_{a_1}^{-1}\mathcal{U}_{a_1} \). Put \( \mathcal{U}_a = \{U_e: e \in E\} \) and \( E' = \{e \in E: U_e \cap p_{a_1}^{-1}(Y_0) \neq \emptyset\} \). By Theorem 2 of Kuratowski [17, p. 226] there exists a collection \( \{G(U_e): e \in E'\} \) such that
(1) $G(U_e)$ are open in $X_{a_1}$, $G(U_e) \cap \overline{p_{a_2} f^{-1}(Y_e)} = U_e \cap \overline{p_{a_1} f^{-1}(Y_e)}$ for all $e \in E'$ and

(2) for each finite subset $\{e_1, e_2, \ldots, e_n\}$ of $E'$,

$U_{e_1} \cap \cdots \cap U_{e_n} = U \cap \overline{p_{a_1} f^{-1}(Y_e)} = \emptyset$ implies $G(U_{e_1}) \cap \overline{G(U_{e_2}) \cap \cdots \cap G(U_{e_n})} = \emptyset$.

By the choice of $E'$ and (1) $V_1 = \{G(U_e) : e \in E' \} \cup \{X_{a_1} - \overline{p_{a_1} f^{-1}(Y_e)}\} \subseteq Cov(X_{a_1})$.

By (A13) there exists $a_3 > a_1$ such that $p_{a_3}^{-1} U_1 \supseteq U_{a_3}$.

We put $X^*_a = st(p_a f^{-1}(Y_e), U_a)$ for $a \in A$ and $s \in S$. We show that

(3) $X^*_a \cap X^*_a = \emptyset$ for $s \neq s'$.

To prove (3) we assume that for some $s \neq s'$ (3) does not hold. Then there exist $U_1, U_2 \subseteq U_a$ such that $U_1 \cap p_{a_2} f^{-1}(Y_e) \neq \emptyset$, $U_2 \cap p_{a_2} f^{-1}(Y_e) \neq \emptyset$ and $U_1 \cap U_2 \neq \emptyset$. By the choice of $a_2$ there exist $e_1, e_2 \in E'$ such that $G(U_{e_1}) \supset p_{a_2, a_1}(U_1)$ and $G(U_{e_2}) \supset p_{a_2, a_1}(U_2)$. Then $G(U_{e_1}) \cap G(U_{e_2}) \neq \emptyset$, $G(U_{e_1}) \cap p_{a_1} f^{-1}(Y_e) \neq \emptyset$ and $G(U_{e_2}) \cap p_{a_1} f^{-1}(Y_e) \neq \emptyset$. Thus by (1) and (2)

(4) $U_{e_1} \cap p_{a_1} f^{-1}(Y_e) \neq \emptyset$, $U_{e_2} \cap p_{a_1} f^{-1}(Y_e) \neq \emptyset$ and

$U_{e_1} \cap U_{e_2} \cap p_{a_1} f^{-1}(Y_e) = \emptyset$.

By the choice of $cV$ and (4), $V_1 \supset f_p(a_1)(U_{e_1}), V_2 \supset f_p(a_2)(U_{e_2})$ and $f_p(a_1)(U_{e_1}) \cap f_p(a_2)(U_{e_2}) = \emptyset$. Thus $V_1 \cap V_2 \supset f_p(a_1)(U_{e_1}) \cap f_p(a_2)(U_{e_2}) = \emptyset$ and then $s = s'$ by the choice of $\{V_s\}$. However $s \neq s'$ by the assumption. This is a contradiction. Hence we have (3).

We put $X^*_a = st(p_a f^{-1}(Y_e), U_a)$, $U^*_a = U_a | X^*_a$ for $a \in A$. The maps $p_a$ and $p_{a_1, a}$ induce $p^*_a : f^{-1}(Y_e) \to X^*_a$ and $p^*_{a_1, a} : X^*_a \to X^*_a$ for $a' = a$. By (2.1) $p^* = \{p^*_a : a \in A\} : f^{-1}(Y_e) \to (X_a, U_a) = (X^*_a, U^*_a, p^*_{a_1, a}, A)$ is an approximative ANR-resolution. By (1.3.2) $p^* : f^{-1}(Y_e) \to X^*_a$ is an ANR-resolution.

By the assumption for each $s \in S$ there exists $a_s > a$ satisfying $EP(a, U, \mathcal{K})$ for $Y_s$. We put $Z = \bigoplus\{X^*_a \times \{s\} : s \in S\}$ (topological sum). We define maps $q : f^{-1}(Y_e) \to Z$ and $q' : Z \to X^*_a$ as follows: $q(x) = (p_a(x), s)$ for $x \in f^{-1}(Y_e)$ and $q'(z, s) = p_{a, a}(z)$ for $z \in X^*_a \times \{s\}$ Since $\{Y_s : s \in S\}$ is discrete in $Y$, $\{f^{-1}(Y_s) : s \in S\}$ is discrete in $X$. Using Theorem 5.1.17 of Engelking [9, p. 379] we can easily show that $f^{-1}(Y_e) = \bigcup\{f^{-1}(Y_s) : s \in S\} = \bigoplus\{f^{-1}(Y_s) : s \in S\}$ (topological sum). Then $q$ is well-defined and continuous. Also $q'$ is well-defined and continuous, and $q' q = p_a | f^{-1}(Y_e)$.

Since $X^*_a$ is an ANR, there exists $cV \subseteq Cov(X^*_a)$ satisfying (a) in (I.5.7) for $U^*_a$. There exists $cV \subseteq Cov(X^*_a)$ satisfying (R2) for $p^*$ and $cV$. Since $Z$ is a topological sum, there exist $Z^* \subseteq Cov(X^*_a)$, $s \in S$, such that $q^{-1} cV^* \supset Z^* = \bigcup\{Z^* \times \{s\} : s \in S\}$ and $s t Z^* \subseteq U_a | X^*_a$ for $s \in S$. By (R1) for $p^*$ there exist $a_3 > a_2$ and
a map \( g : X_{a_3}^s \to Z \) such that

\[
(q, g p_{a_3}^s) < \mathcal{W}.
\]

By the choice of \( \mathcal{W} \), \((q', q' g p_{a_3}^s) < \mathcal{V}_s \) and then by the choices of \( q \) and \( q' \),
\[
(p_{a_2, a_3} p_{a_3}^s, q' g p_{a_3}^s) < \mathcal{V}_s.
\]

By the choice of \( \mathcal{V}_s \) there exists \( a_4 > a_3 \) such that

\[
(p_{a_2, a_3} p_{a_3}^s, q' g p_{a_3}^s) < \mathcal{V}_1.
\]

Take any \( s \in S \) and any point \( x \in f^{-1}(Y_s) \). Since \( q(x) \in X_{a_4}^s \times \{s\} \), by (5) there exists \( W_1 \in \mathcal{W} \) such that \( q(x), \ g p_{a_3}(x) \in W_1 \times \{s\} \subset X_{a_4}^s \times \{s\} \). Then \( g p_{a_3} f^{-1}(Y_s) \subset X_{a_4}^s \times \{s\} \) and hence \( p_{a_2, a_3} f^{-1}(Y_s) \subset R_s = X_{a_4}^s \cap g^{-1}(X_{a_4}^s \times \{s\}) \), because \( X_{a_4}^s \times \{s\} \) is closed in \( Z \). Since \( a_3 > a_2 \), by (3),

\[
X_{a_4}^s \cap X_{a_3}^s = \emptyset \quad \text{for } s \neq s'.
\]

Since \( X_{a_3}^s \times \{s\} \) is open in \( Z \), by (7)

\[
R_s \text{ is open in } X_{a_3}^s, \quad p_{a_3} f^{-1}(Y_s) \subset R_s \text{ and } R_s \cap R_{s'} = \emptyset
\]

for \( s, s' \in S \) with \( s \neq s' \).

We will show the following:

\[
\mathcal{T} = \{ p_{a_3} f^{-1}(Y_s) : s \in S \} \text{ is a discrete collection in } X_{a_3}^s.
\]

Indeed, let \( u \) be any point of \( X_{a_3}^s \). We will find an open neighborhood \( N \) of \( u \) in \( X_{a_3}^s \) such that \( N \) meets at most one member of \( \mathcal{T} \).

If \( u \in X_{a_3}^s \setminus X_{a_3}^s \), we choose for \( N \) any member of \( \mathcal{U}_s \) which contains \( u \). Then \( N \) misses all members of \( \mathcal{T} \). To see this assume that for a given \( s \in S \) we have \( N \cap p_{a_3} f^{-1}(Y_s) \neq \emptyset \). Then also \( N \cap p_{a_3} f^{-1}(Y_s) \neq \emptyset \) and this implies \( u \in N \subset \delta \{ p_{a_3} f^{-1}(Y_s) \} \), \( \mathcal{U}_s = X_{a_3}^s \subset X_{a_3}^s \), which is a contradiction.

Now assume that \( u \in X_{a_3}^s \). Then \( g(u) \in Z = \bigoplus X_{a_3}^s \times \{s\} \). Therefore, there exists an \( s' \in S \) such that \( g(u) \in X_{a_3}^s \times \{s'\} \). We put \( N = g^{-1}(X_{a_3}^s \times \{s'\}) \). This set is open in \( X_{a_3}^s \) and therefore open in \( X_{a_3}^s \). Moreover, \( u \in N \). Finally, by (8), for all \( s \neq s' \) we have \( \delta p_{a_3} f^{-1}(Y_s) \subset R_s \subset g^{-1}(X_{a_3}^s \times \{s\}) \), which is disjoint from \( N = g^{-1}(X_{a_3}^s \times \{s'\}) \). Consequently, only \( p_{a_3} f^{-1}(Y_s) \) can meet \( N \). Hence we have (9).

By (9) and the definition of collectionwise normality (see Engelking [9, p. 379]), there exists a discrete family of open sets \( \{ R_s' : s \in S \} \) such that \( p_{a_3} f^{-1}(Y_s) \subset R_s' \), \( s \in S \). We may assume that \( \bigcup \{ R_s' : s \in S \} \subset X_{a_3}^s \). By (B3) for \( p^s \), there exists \( a_5 > a_4 \) such that \( p_{a_5, a_3} (X_{a_5}^s) \subset \bigcup \{ R_s' : s \in S \} \). Since \( p_{a_5, a_3} (X_{a_5}^s) \subset X_{a_5}^s \), we see that \( p_{a_5, a_3} (X_{a_5}^s) \subset X_{a_5}^s \).
We will show that $a_6$ has the required property. Take any point $y \in Y_6$ and any map $h: K \to \text{st}(\rho_{a_6}(D_y), U_{a_6})$. Then there exists $s_6 \in S$ such that $y \in Y_{s_6}$, and then $g \rho_{a_6, a_7}(h(K)) \subseteq X_{s_6}^2 \times \{a_6\}$. Take any $t \in K_6$. Then there exist $x \in D_y$ and $U_1 \subseteq U_{a_6}$ such that $h(t), \rho_{a_6}(x) \in U_1$. By the choice of $a_6$ there exists $W_2 \subseteq W_{a_6}$ such that $g \rho_{a_6, a_7}(h(K)) \subseteq X_{s_6}^2 \times \{a_6\}$. By (5) $g \rho_{a_6, a_7}(x) \subseteq X_2 \times \{a_6\}$. Thus, $rg \rho_{a_6, a_7, h}(K_6) \subseteq \text{st}(\rho_{a_6}(D_y), U_{a_6})$. Then by the choice of $W_{a_6}$ there exists $U_3 \subseteq U_{a_6}$ such that $q(x), g \rho_{a_6, a_7}(h(t)) \subseteq U_2 \times \{a_6\}$. Thus, $rg \rho_{a_6, a_7, h}(K_6) \subseteq \text{st}(\rho_{a_6}(D_y), U_{a_6})$. Then by (10), $g'_{y} K_6 = \rho_{a_6, a_7, h} \rho_{a_6, a_7} g'_{y} \rho_{a_6, a_7} h = q'_{g \rho_{a_6, a_7, h}}$

By (6), $(\rho_{a_6, a_7, h}, q'_{g \rho_{a_6, a_7, h}}) < c \mathcal{U}_1$. By the choice of $c \mathcal{U}_1$ there exists a $\mathcal{U}$-homotopy $u: K_6 \times I \to X_{s_6}$ such that $u_0 = q'_{g \rho_{a_6, a_7, h}}$ and $u_1 = \rho_{a_6, a_7, h}$. We define a map $g^*: K \times \{0\} \cup K_6 \times I \to X^2$ as follows: $g^*(t, 0) = g'(t)$ for $(t, 0) \in K \times \{0\}$ and $g^*(t, t') = u(t, t')$ for $(t, t') \in K_6 \times I$. By (10), $g^*$ is well-defined and continuous. Since $g^*(K) \subseteq \text{st}(\rho_{a_6}(D_y), \mathcal{U})$ and $u$ is a $\mathcal{U}$-homotopy, $\text{Im} g^* \subseteq \text{st}(\rho_{a_6}(D_y), st \mathcal{U})$. Since $(K, K_6)$ is a polyhedral pair, by the homotopy extension theorem (see Th. 3 of MS [23, p. 291]) there exists an extension $G^*: K \times I \to \text{st}(\rho_{a_6}(D_y), st \mathcal{U})$ of $g^*$. Then we put $G: K \to \text{st}(\rho_{a_6}(D_y), st \mathcal{U})$ as follows; $G(x) = G^*(x, 1)$ for $x \in K$. Thus $G$ is an extension of $\rho_{a_6, a_7, h}$. Hence $Y_6$ has the required property.

(5.11) LEMMA. If $D_y$ has AEP with respect to $\mathcal{K}$, then for each $a \in A$ there exists an open neighborhood $V_y$ of $y$ in $Y$ such that $\bar{V}_y$ has EP($a$, $U_a$, $\mathcal{K}$).

PROOF. By (1.3.2) and (2.1) $p^a = \{p^a_2: a \in A\}: D_y \to \{D(y, a), p^a_2, a, A\}$ is an ANR-resolution of $D_y$. Here $p^a_2$ and $p^a_2, a$ are induced by $p^a_2$ and $p^a_2, a$, respectively. Since $\mathcal{K}$ is a collection of polyhedral pairs, by (4.4) $p^a$ has EP with respect to $\mathcal{K}$. Thus for each $a \in A$ there exists $a_i > a$ satisfying (EP) with respect to $\mathcal{K}$ for $a$. Since $\bar{p^a_1(D_y)} \subseteq \text{st}(p^a_1(D_y), U_{a_1})$, there exists an open subset $W$ in $X_{a_1}$ such that $\bar{p^a_1(D_y)} \subseteq W \subseteq \bar{W} \subseteq \text{st}(p^a_1(D_y), U_{a_1}) = D(y, a_i)$. Since $p^a_1(W)$ is an open neighborhood of $D_y$ in $X$ and $f: X \to Y$ is a closed map, $Y - f(X - p^a_1(W))$ is an open neighborhood of $y$ in $Y$. Since $Y$ is paracompact, there exists an
open neighborhood $V_y$ of $y$ in $Y$ such that $y \in V_y \subseteq \overline{V}_y \subseteq Y - f(X - p_a^{-1}(W))$, and then

$$\overline{p_a f^{-1}(\overline{V}_y)} \subseteq \overline{W} \subseteq \text{st}(p_a(D_y), U_a) \subseteq \text{st}(p_a f^{-1}(\overline{V}_y), U_a).$$  \hspace{1cm} (1)

By (1.3.2) and (2.1) $p^\ast = \{ p_a^\ast : a \in A \} : f^{-1}(\overline{V}_y) \to \{ \text{st}(p_a f^{-1}(\overline{V}_y), U_a), p_a^\ast, a, A \}$ is an ANR-resolution. Here $p_a^\ast$ and $p_a^\ast, a$ are induced by $p_a$ and $p_a^\ast, a$, respectively. By (1) and (B3) for $p^\ast$ there exists $a_2 > a_1$ such that

$$\overline{p_{a_2, a_1}(\text{st}(p_{a_2 f^{-1}(\overline{V}_y), U_{a_2}) \subseteq \text{st}(p_{a_2}(D_y), U_{a_2}).}$$  \hspace{1cm} (2)

We show that $V_y$ and $a_2$ have required property. Take any $y' \in \overline{V}_y$ and any map $h : K \to \text{st}(p_a(D_y), U_a)$. Since $p_{a_2, a_1} h : K \to \text{st}(p_{a_2}(D_y), U_{a_2})$ by (2), by the choice of $a_2$ there exists a map $G : K \to \text{st}(p_a(D_y), U_a)$ such that $G|K = p_{a_2, a_1} h$. Hence we have the required property. \hfill \blacksquare

(5.12) PROPOSITION. Let $\mathcal{K}$ be a collection of polyhedral pairs. If $D_y = f^{-1}(y)$ has AEP with respect to $\mathcal{K}$ for any $y \in Y$, then any approximative ANR-resolution $p : X \to (\mathcal{X}, \mathcal{U})$ has the following property:

(*) For each $a \in A$ there exists $a_1 > a$ such that for any $y \in Y$, any $(K, K_0) \in \mathcal{K}$ and any map $h : K \to \text{st}(p_a(D_y), U_a)$, there exist $y' \in Y$ and a map $H : K \to \text{st}(p_a(D_y), U_a)$ satisfying $H|K_0 = p_{a_1, a} h$.

To prove (5.12) we need the Michael method as follows: Let $Z$ be a space and $\mathcal{U}$ a collection of subsets of $Z$. We introduce the following notations:

- $\mu^\ast(\mathcal{U}) = \{ A \subseteq Z : A$ is closed in $Z$ and $A$ is a subset of some element of $\mathcal{U} \}$.
- $\sigma^\ast(\mathcal{U}) = \{ A \subseteq Z : A$ is the union of a finite collection of closed subsets of $Z$, whose interiors with respect to $A$ cover $A$, and which are elements of $\mathcal{U} \}$.
- $\Sigma^\ast(\mathcal{U}) = \{ A \subseteq Z : A$ is the union of a discrete collection of closed subsets of $Z$ which are elements of $\mathcal{U} \}$.

(5.13) LEMMA (Michael [24]). If $Z$ is a paracompact space and $\mathcal{U}$ covers $Z$, then $Z \in \sigma^\ast \Sigma^\ast \Sigma^\ast \mu^\ast(\mathcal{U})$.

Proof of (5.12). From any approximative ANR-resolution $p : X \to (\mathcal{X}, \mathcal{U})$, by (2.1) we have an approximative ANR-resolution $p' = \{ p_a' : a \in A \} : X \to (\mathcal{X}, \mathcal{U})' = \{(X_a', \mathcal{U}_a'), p_a'^\ast, a, A)$. Here $X_a' = \text{st}(p_a(X), U_a)$ and maps $p_a', p_a'^\ast, a$ are induced by $p_a, p_a'^\ast, a$ for $a' > a$. Trivially if $p'$ has the property (*), then so does $p$. Thus without loss of generality we may assume that $p$ satisfies $\text{st}(p_a(X), U_a) = X_a$ for all $a \in A$. 

Take any $a \in A$. By (5.7) there exist $a_2 > a_1 > a$ such that $p_{a_1}^{-1} \partial a_2 > st \partial a_1$ and $p_{a_2}^{-1} \partial a_2 > st \partial a_2$. By the assumption and (5.11) there exist open neighborhoods $V_y$ of $y$ such that $V_y$ has EP$(a_2, U_{a_2}, K)$ for any $y \in Y$. Since $Y$ is paracompact, $\mathcal{V} = \{V_y : y \in Y\}$ covers $Y$ and then by (5.13) $Y \in \sigma^* \Sigma^* \sigma^* \mu^*(\mathcal{V})$. By (5.8)-(5.10) any element of $\sigma^* \Sigma^* \sigma^* \mu^*(\mathcal{V})$ has EP$(a_2, st \partial a_2, K)$ and hence so does $Y$. Then there exists $a_3 > a_2$ satisfying EP$(a_2, st \partial a_2, K)$.

We show that $a_3$ is the required index. Take any $y \in Y$, any $(K, K_0) \in \mathcal{K}$ and any map $h : K_0 \to st(p_{a_2}(D_y), U_{a_2})$. By the choice of $a_3$ there exist $y' \in Y$ and a map $H : K \to st(p_{a_2}(D_y), st \partial a_2)$ such that $H|K_0 = p_{a_2} \cdot a \cdot h$. By the choices of $a_1$ and $a_3$, it is easy to show that there exists $y'' \in Y$ such that $p_{a_2} \cdot H(K) \subseteq st(p_{a_2}(D_y), U_{a_2})$. Hence we have the required property. □

(5.14) Theorem. Let $Y$ be a paracompact space and $f : X \to Y$ a closed onto map with (2). Let $n \geq 0$ be an integer. If $f^{-1}(y)$ is approximatively $n$-connected for each $y \in Y$, then $f$ has FEP with respect to $^{n+1} \mathcal{SC}_{pairs}$.

Proof. Let $p : X \to (\mathcal{X}, \mathcal{U})$ be an approximative ANR-resolution such that $st(p_a(X), U_a) = X_a$ for $a \in A$. By (4.14) and $D_y$ has AEP with respect to $^{n+1} \mathcal{POL}_{pairs}$.

We show (5.14) by induction on $n$. First we show this for $n=0$. Take any $a \in A$ and then there exists $a_0 > a$ satisfying (6) in (5.12) for $^{1} \mathcal{POL}_{pairs}$ and $a$. We show that $a_0$ is the required index. Take any $(K, L) \in ^{n} \mathcal{SC}_{pairs}$ and any partial realization $g : L \to X_{a_0}$ of $(K, L)$ relative to $\partial a_0$. Take any 1-simplex $s = [v_0, v_1] \subseteq K-L$. Then $g(\partial s) = g(s \cap L) \subseteq st(p_{a_0}(D_y), U_{a_0})$ for some $y \in Y$. By the choice of $a_0$ there exists a map $g_1 : s \to X_a$ such that $g_1|\partial s = p_{a_0} \cdot g|\partial s$ and $g_1(s) \subseteq st(p_{a_0}(D_y), U_a)$ for some $y' \in Y$. We define a map $G : K \to X_a$ as follows: $G(x) = p_{a_0} \cdot g(x)$ for $x \in L$ and $G(x) = g_1(x)$ for $x \in s \subseteq K-L$. Obviously $G$ is a full realization of $(K, L)$ relative to $\partial a$ and $G|L = p_{a_0} \cdot g$.

We assume that (5.14) holds for $n$ and show it for $n+1$. Take any $a \in A$ and then there exists $a_1 > a$ satisfying (6) in (5.12) for $^{n+1} \mathcal{POL}_{pairs}$ and $a$. By (5.7) there exists $a_2 > a_1$ such that $p_{a_1}^{-1} \partial a_2 > st \partial a_2$. By the inductive assumption there exists $a_3 > a_2$ satisfying (FEP) for $a_2$ and $^{n+1} \mathcal{SC}_{pairs}$.

We show $a_3$ is the required index. Take any $(K, L) \in ^{n+1} \mathcal{SC}_{pairs}$ and any partial realization $g : L \to X_{a_0}$ of $(K, L)$ relative to $\partial a_0$. By the choice of $a_3$ there exists a full realization $g' : K^{n+1} \to X_{a_0}$ of $(K^{n+1} \cup L, L)$ relative to $\partial a_0$ such that $g'|L = p_{a_0} \cdot g$. Take any $(n+2)$-simplex $s \subseteq K-L$. Since $g'(\partial s) \subseteq st(st(p_{a_0}(D_y), U_{a_0}), \partial a_2)$ for some $y \in Y$, by the choice of $a_3$, $p_{a_2} \cdot g'(\partial s) \subseteq st(p_{a_2}(D_y), U_{a_2})$ for some $y' \in Y$. Then by the choice of $a_1$ there exists $y'' \in Y$...
and a map \( g'_s: s \to st(\partial_a(D_y), \mathcal{U}_a) \) such that \( g'_s|\partial s = p_a \cdot g'|\partial s \). Now we define a map \( G: K \to X_a \) as follows: \( G(x) = p_a \cdot g'(x) \) for \( x \in K^{k+1} \cup L \) and \( G(x) = g'(x) \) for \( x \in s \in K - (K^{k+1} \cup L) \). Obviously \( G|L = p_a \cdot g \) and for each simplex \( s \) of \( K \), \( G(s) \subset st(\partial_a(D_y), \mathcal{U}_a) \) for some \( y \in Y \). Hence \( f \) has FEP with respect to \( \mathcal{A} \times \text{SC pairs} \).

§ 6. The Vietoris-Smale theorem in shape.

In this section we shall give a characterization of approximatively \( n \)-connected maps and the Vietoris-Smale theorem in shape theory.

Let \( f: X \to Y \) be a map. Let \( \mathcal{P} = \{ p_a: a \in A \} : X \to (\mathcal{X}, \mathcal{U}) = \{ (x, \mathcal{U}_a), p_a \cdot a, A \} \) and \( \mathcal{Q} = \{ q_b: b \in B \} : Y \to (\mathcal{Y}, \mathcal{Q}) = \{ (y, c\mathcal{N}_b), q_b \cdot b, B \} \) be approximative ANR-resolutions. Let \( f = \{ f, f_b: b \in B \} : (\mathcal{X}, \mathcal{U}) \to (\mathcal{Y}, \mathcal{Q}) \) be an approximative resolution of \( f \) with respect to \( \mathcal{P} \) and \( \mathcal{Q} \).

(6.1) **Proposition.** Let \( f: X \to Y \) be a map and \( Y \) a normal space. Then \( f \) is approximatively \((-1)\)-connected iff \( f(X) = Y \).

**Proof.** First we assume that \( f \) is approximatively \((-1)\)-connected. Take any \( y \in Y \) and any open neighborhood \( V \) of \( y \) in \( Y \). Since \( Y \) is normal, \( c\mathcal{N} = \{ V, Y - \{ y \} \} \subset \mathcal{C}_{ov}(Y) \). By (AI3) there exists \( b \in B \) such that \( c\mathcal{N} > q_b \cdot st \in c\mathcal{N}_b \). Since \( f \) is an approximative resolution of \( f \),

(1) \[
(q_b f, f_b p_{f(b)}) < c\mathcal{N}_b.
\]

By (AR2) there exists \( a > f(b) \) such that \( st(p_{f(b)}(X), \mathcal{U}_{f(b)}) \supset p_{a \cdot f(b)}(X_a) \). Since \( f \) is approximatively \((-1)\)-connected, there exists an admissible pair \( (a_1, b_1) > (a, b) \) satisfying (ALP) with respect to \( \{ (\mathcal{A}, \mathcal{F}) \} \) for \( (a, b) \).

Take a map \( g: \mathcal{A} \to Y_{b_1} \) such that \( g(\mathcal{A}) = q_{b_1}(y) \). By the choice of \( (a_1, b_1) \) there exists a map \( H: \mathcal{A} \to X_a \) such that

(2) \[
(f_b p_{a \cdot f(b)} H, q_{b_1} \cdot bG) < st c\mathcal{N}_b.
\]

By the choice of \( a \), \( p_{a \cdot f(b)} H(\mathcal{A}) \subset st(p_{f(b)}(X), \mathcal{U}_{f(b)}) \) and then there exist \( x \in X \) and \( U \subset \mathcal{U}_{f(b)} \) such that \( p_{f(b)}(x), p_{a \cdot f(b)} H(\mathcal{A}) \subset U \). By (AM1) there exists \( V_1 \subset c\mathcal{N}_b \) such that

(3) \[
f_b p_{f(b)}(x), f_b p_{a \cdot f(b)} H(\mathcal{A}) \subset V_1.
\]

By (1) and (2) there exist \( V_2, V_3 \subset c\mathcal{N}_b \) such that

(4) \[
q_b f(x), f_b p_{f(b)}(x) \subset V_2, \text{ and}
\]

(5) \[
f_b p_{a \cdot f(b)} H(\mathcal{A}), q_{b_1} \cdot bG(\mathcal{A}) \subset st(V_3, c\mathcal{N}_b).
\]
Since \( q_{b_1, b_2}(\mathcal{X}) = q_b(y) \), by (3)-(5), \( q_b(x), q_b(y) \in \text{st}(\text{st}(V_3, \mathcal{C}V_\delta), \mathcal{C}V_\delta) \). By the choice of \( b \) and the definition of \( V \), \( f(x), y \in V \) and then \( V \cap f(X) \neq \emptyset \). Hence \( f(X) \) is dense in \( Y \).

Next we assume that \( f(X) \) is dense in \( Y \). Take any admissible pair \( (a, b) \) of \( f \). By (B4) there exists \( b_i > b \) such that \( \text{st}(q_b(Y), \mathcal{C}V_\delta) \supset q_{b_1, b}(Y_{b_1}) \). Take any \( a_i > a, f(b_i) \). We will show that the admissible pair \( (a_i, b_i) \) has the required property. Take any map \( g : \mathcal{X} \to Y_{b_1} \). By the choice of \( b_1 \) there exist \( y \in Y \) and \( V \in \mathcal{C}V_\delta \) such that

\[
q_{b_1, b}(y), q_{b_1, b}(\mathcal{X}) \in V_i.
\]

Since \( f(X) \) is dense in \( Y \), there exists \( x \in X \) such that

\[
f(x) \in q_{b_1, b}^{-1} V_i.
\]

We define a map \( H : \mathcal{X} \to X \) by \( H(\mathcal{X}) = p_a(x) \). By (1) there exists \( V \in \mathcal{C}V_\delta \) such that

\[
q_{b_1, b}(x), q_{b_1, b}(\mathcal{X}) \in V_i.
\]

Since \( f(X) \) is dense in \( Y \), there exists \( x \in X \) such that

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We define a map \( H : \mathcal{X} \to X \) by \( H(\mathcal{X}) = p_a(x) \). By (1) there exists \( V \in \mathcal{C}V_\delta \) such that

\[
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\]

Since \( f(X) \) is dense in \( Y \), there exists \( x \in X \) such that

\[
f(x) \in q_{b_1, b}^{-1} V_i.
\]

We define a map \( H : \mathcal{X} \to X \) by \( H(\mathcal{X}) = p_a(x) \). By (1) there exists \( V \in \mathcal{C}V_\delta \) such that

\[
q_{b_1, b}(x), q_{b_1, b}(\mathcal{X}) \in V_i.
\]

Thus \( f(X) \) is approximatively \((-1)\)-connected.

(6.2) Corollary. Let \( f : X \to Y \) be a closed map and \( Y \) a normal space. Then \( f \) is approximatively \((-1)\)-connected iff \( f(X) = Y \).

(6.3) Theorem. Let \( Y \) be a paracompact space and \( f : X \to Y \) a closed map with (\#):

(\#) \( f^{-1}(Y_0) \) is \( P \)-embedded in \( X \) for any closed subset \( Y_0 \) of \( Y \).

For each integer \( n \geq -1 \) the following statements are equivalent:

(i) \( f \) is onto and \( f^{-1}(y) \) is approximatively \( n \)-connected for any \( y \in Y \).

(ii) \( f \) is onto and the decomposition \( D = \{ f^{-1}(y) : y \in Y \} \) has AFEP with respect to \( n^{+}\text{SC}_{\text{pairs}} \).

(iii) \( f \) is approximatively \( n \)-connected.

Proof. When \( n = -1 \) (6.3) follows from (6.2). We assume \( n \geq 0 \). We show (iii) \( \Rightarrow \) (i). By (6.2) \( f \) is onto. By (3.3) \( f \) has ALP with respect to \( n^{+}\text{POL}_{\text{pairs}} \) and then by (2.7) and (\#) so does \( f|f^{-1}(y) : f^{-1}(y) \to \{ y \} \) for each \( y \in Y \). By (3.4) \( \text{pro-} \pi_k(f|f^{-1}(y)) : \text{pro-} \pi_k(f^{-1}(y), y) \to \text{pro-} \pi_k(\{ y \}, y) = 0 \) is an isomorphism for \( 0 \leq k \leq n \). Then \( \text{pro-} \pi_k(f^{-1}(y), y) = 0 \) for \( 0 \leq k \leq n \). By (4.14) \( f^{-1}(y) \) is approximatively \( n \)-connected.

(5.14) means that (i) \( \Rightarrow \) (ii). We show (ii) \( \Rightarrow \) (iii). By (1.4.9) there exist
approximative ANR-resolutions \( p : X \to (X, \mathcal{U}) \), \( q : Y \to (Q, \mathcal{V}) \) and an approximative resolution \( f : (X, \mathcal{U}) \to (Q, \mathcal{V}) \) of \( f \) with respect to \( p \) and \( q \) satisfying (RM1) and (RM2). By (RM2) and (2.1) we may assume that \( st(p_a(X), \mathcal{U}_a) = X_a \) and \( st(q_b(Y), \mathcal{V}_b) = Y_b \) for all \( a \in A \) and all \( b \in B \).

**Claim 1.** \( f \) has (ALP**) with respect to \((D, S_n)\).

Take any admissible pair \((a, b)\) of \( f \) and then there exists \( b_1 > b \) such that \( q_b^{-1} \circ \mathcal{V}_b > st \mathcal{V}_b \). By (RM1) for \( f \) there exists \( a_1 > a \), \( f(b_1) \) such that \( f_b \circ p_{a_1, f(b_1)} = q_{b_1} \circ f_{a_1, f(b_1)} \). By the assumption and (5.2) there exists \( a_2 > a_1 \) satisfying (FEP) with respect to \( ^*+^* \mathcal{SC} \) for \( a_1 \). By (5.7) there exists \( a_3 > a_2 \) such that \( p_{a_2, a_3} \circ \delta_{a_3} > s f \delta_{a_3} \). Here \( D(y, a) = st(p_a(D_y), \mathcal{U}_a) \) and \( \delta_a = (D(y, a) : y \in Y) \) for \( a \in A \) and \( y \in Y \). Since \( f \) is closed, there exists an open neighborhood \( V_y \) of \( y \) in \( Y \) such that

\[
D_y \subseteq f^{-1}(V_y) \subseteq p_{a_2}^{-1}(st(p_{a_3}(D_y), \mathcal{U}_{a_3})) \quad \text{for each} \quad y \in Y.
\]

Since \( Y \) is paracompact, \( \mathcal{V} = \{ V_y : y \in Y \} \subseteq C_{ov}(Y) \) and then there exists \( b_1 > b \) such that \( \mathcal{V} \supseteq q_{b_1}^{-1} \circ \mathcal{V}_b \).

Let \( \mathcal{V}_{b_1} = \{ V_m : m \in M \} \) and \( \mathcal{M}' = \{ m \in M : V_m \cap q_{b_1}(Y) \neq \emptyset \} \). By Theorem 2 of Kuratowski [17, p. 226] there exist open sets \( G(V_m) \) in \( Y_{b_1} \) for \( m \in \mathcal{M}' \) such that

\[
\begin{align*}
&\text{(2)} & V_m \cap q_{b_1}(Y) &= G(V_m) \cap \overline{q_{b_1}(Y)} \quad \text{and} \quad G(V_m) \subseteq V_m \quad \text{for} \ m \in \mathcal{M}', \\
&\text{(3)} & \text{for each finite subset} \ \{ m_1, m_2, \ldots, m_s \} \ \text{of} \ \mathcal{M}', \ V_m_1 \cap V_m_2 \cap \cdots \cap V_m_s \cap \overline{q_{b_1}(Y)} = \emptyset \ \text{implies} \ G(V_m_1) \cap G(V_m_2) \cap \cdots \cap G(V_m_s) = \emptyset.
\end{align*}
\]

Since \( q_{b_1}(Y) \subseteq \bigcup \{ G(V_m) : m \in \mathcal{M}' \} \) by (2), by (B3) there exists \( b_1 > b_1 \) such that \( q_{b_1, b_2}(Y_{b_1}) \subseteq \bigcup \{ G(V_m) : m \in \mathcal{M}' \} \). Since \( \{ q_{b_1, b_2}G(V_m) : m \in \mathcal{M}' \} \subseteq C_{ov}(Y_{b_2}) \), by (A13) there exists \( b_2 > b_1 \) such that \( q_{b_1, b_2}G(V_m) : m \in \mathcal{M}' \}\supset q_{b_1, b_2}G(V_m) \). By (RM1) for \( f \) there exists \( a_4 > a_3 \), \( f(b_2) \) such that \( f_b \circ p_{a_4, f(b_2)} = q_{b_2} \circ f_{a_4, f(b_2)} \). Let \( \mathcal{U}_{a_4} = \{ U_e : e \in E \} \) and \( E' = \{ e \in E : U_e \cap p_{a_4}(X) \neq \emptyset \} \). Since \( p_{a_4}(X) \subseteq \bigcup \{ U_e : e \in E' \} = st(p_{a_4}(X), \mathcal{U}_{a_4}) \), by (B3) there exists \( a_5 > a_4 \) such that \( \mathcal{V}_{b_1, b_2} \subseteq \bigcup \{ U_e : e \in E' \} \). There exists \( a_5 > a_5 \) such that \( \mathcal{V}_{b_1, b_2} \subseteq \bigcup \{ U_e : e \in E' \} \). We will show that the admissible pair \((a_5, b_2)\) has the required property. To do so we take any \((D^{k+1}, \partial D^{k+1})\) in \((D, S_n)\) and any maps \( g : D^{k+1} \to Y_{b_2}, h : \partial D^{k+1} \to X_{a_5} \) such that \( g | \partial D^{k+1} = f_b \times p_{a_4, f(b_2)}h \). Take \( W \in C_{ov}(D^{k+1}) \) such that \( W < g^{-1}(\mathcal{V}_{b_2}) \) and \( W | \partial D^{k+1} < h^{-1} \mathcal{U}_{a_5} \). There exists a simplicial complex \( K \) and a subcomplex \( L \) of \( K \) such that \( |K| = D^{k+1}, |L| = \partial D^{k+1}, L \) is a full subcomplex of \( K \) and for each simplex \( s \) of \( K \) there exists \( W_s \subseteq W \) such that \( s \subseteq W_s \). Take any simplex \( s \) of \( K \). By the choices of \( b_1 \) and \( W_s \) there exist \( V_s \subseteq C_{ov} \) and
m(s) \in M'$ such that $g(W_s) \subset V_s$ and $q_{b_1,b_2}(V_s) \subset G(V_m(z))$. Thus

\[ q_{b_1,b_2}g(|s|) \subset G(V_m(z)) \text{ for each simplex } s \text{ of } K. \]

Since $V_m(z) \cap q_{b_2}(Y) \neq \emptyset$, by the choice of $b_2$ there exists $y(s) \in Y$ such that $V_{y(z)} \supseteq q_{b_2}V_m(z)$, and then there exists $x(s) \in X$ such that $x(s) \in f^{-1}q_{b_2}V_m(z) \subset f^{-1}V_{y(z)}$. Then by (1)

\[ p_{a_3}(x(s)) \in \text{st}(p_{a_3}(D_{y(z)}), U_{a_3}) \text{ for each simplex } s \text{ of } K. \]

Now we define a map $g^k: K \to X_{a_3}$ as follows: $g^k(z) = p_{a_3,a_3}h(z)$ for $z \in L$ and $g^k(z) = p_{a_3}(x(v))$ for $z = v \in K^o - L$. It is easy to show that $g^k$ is well-defined and continuous.

**Claim 2.** For each simplex $s$ of $K$ there exists $y \in Y$ such that $g^k(s \cap (K^o \cup L)) \subset \text{st}(p_{a_3}(D_y), U_{a_3}).$

To prove Claim 2 we take any simplex $s = \{v_0, v_1, \ldots, v_p\}$ of $K$. By (4) for each $i$, $0 \leq i \leq p$, $q_{b_1,b_2}(v_i) \in G(V_m(z)) \cap G(V_m(z)) \neq \emptyset$ and then by (2) $V_m(z) \cap V_m(z) \cap q_{b_2}(Y) \neq \emptyset$. Since $V_m(z) \cap V_m(z) \cap q_{b_2}(Y) \neq \emptyset$, $V_{y(z)} \cap V_{y(z)} \supseteq q_{b_2}(V_m(z) \cap V_m(z)) \neq \emptyset$ and then by (1) $\text{st}(p_{a_3}(D_{y(z)}), U_{a_3}) \cap \text{st}(p_{a_3}(D_{y(z)}), U_{a_3}) = p_{a_3}(f^{-1}(V_{y(z)} \cap V_{y(z)})) \neq \emptyset$, that is,

\[ \text{st}(p_{a_3}(D_{y(z)}), U_{a_3}) \cap \text{st}(p_{a_3}(D_{y(z)}), U_{a_3}) \neq \emptyset \text{ for } 0 \leq i \leq p. \]

We consider the following cases: (i) All vertexes of $s$ are in $K^o - L$. (ii) All vertexes of $s$ are in $L$. (iii) Some vertexes of $s$ are in $K^o - L$ and some vertexes of $s$ are in $L$.

We consider the case (i). In this case $s \cap (K^o \cup L) = \{v_0, v_1, \ldots, v_p\}$ and then $g^k(s \cap (K^o \cup L)) = \{p_{a_3}(x(v_0)), p_{a_3}(x(v_1)), \ldots, p_{a_3}(x(v_p))\}$. Since $p_{a_3}(x(v_i)) \in \text{st}(p_{a_3}(D_{y(z)}), U_{a_3})$ for $i = 0, 1, \ldots, p$ by (5), then by (6) $\text{st}(p_{a_3}(D_{y(z)}), U_{a_3}) \cap \text{st}(p_{a_3}(D_{y(z)}), U_{a_3}) \subset \text{st}(p_{a_3}(D_{y(z)}), U_{a_3})$. Hence $g^k(s \cap (K^o \cup L)) \subset \text{st}(p_{a_3}(D_{y(z)}), U_{a_3})$.

Next we consider the case (ii). Since $L$ is a full subcomplex of $K$, $s \subset L$ and then $g^k(s \cap (K^o \cup L)) = p_{a_3,a_3}h(s)$. By the choice of $W_s$ there exists $U_s \subset U_{a_3}$ such that $h(s) \subset h(W_s \cap L) \subset U_s$. By the choice of $a_3$ there exists $e(s) \in E'$ such that $p_{a_3,a_3}(U_s) \subset U_{e(s)}$. Since $U_{e(s)} \cap p_{a_3}(X) \neq \emptyset$, there exists $y(s) \in X$ such that $p_{a_3}(x(s)) \in U_{e(s)}$. Put $y(s) = f(s)' \in Y$ and then $x(s) \in D_{y(s)}'$. Then we have

\[ p_{a_3,a_3}h(s) \subset \text{st}(p_{a_3}(x(s)), U_{a_3}) \subset \text{st}(p_{a_3}(D_{y(s)}'), U_{a_3}) \]

and then

\[ g^k(s \cap (K^o \cup L)) = p_{a_3,a_3}h(s) \subset p_{a_3,a_3}\text{st}(p_{a_3}(D_{y(s)}'), U_{a_3}) \subset \text{st}(p_{a_3}(D_{y(s)}'), U_{a_3}). \]
Finally we consider the case (iii). In this case we may assume that \( v_0, v_1, \ldots, v_u \) are vertexes in \( K^*-L \) and \( v_{u+1}, v_{u+2}, \ldots, v_p \) are vertexes in \( L \) for some \( u, 0 \leq u \leq p \). Since \( L \) is full, \( t=[v_{u+1}, v_{u+2}, \ldots, v_p] \in L \) and \( s \cap (K^*:L)=\{v_0, v_1, \ldots, v_u\} \cup t \). Since \( t \in L \), by the case (ii) \( p_{a_0.a_1}h(t) \subseteq U_{\alpha(t)} \) and \( p_{a_k}x'(t) \subseteq U_{\alpha(t)} \). Since \( f_{b_1}p_{a_k}x'(t) \subseteq U_{\alpha(t)} \), there exists \( V'_i \subseteq V_{b_1} \) such that \( f_{b_i}p_{a_k}x'(U_{\alpha(t)}) \subseteq V'_i \). Since \( gL=f_{b_i}p_{a_k}x'(U_{\alpha(t)}) \subseteq H^L, f_{b_i}p_{a_k}x'(t) \subseteq V'_i \) and \( g(t)=f_{b_i}p_{a_k}x'(t) \subseteq V'_i \). Since \( g(t) \subseteq g(s) \subseteq g(W_t) \subseteq V_{b_1} \), \( V_{b_1} \cap V'_i \cap g(t) \cap \emptyset \). By the choice of \( b_1 \) there exists \( m_0(t) \subseteq \emptyset ^* \) such that \( q_{b_1,b_2}V'_i \subseteq G(V_{m_0(t)}) \). Since \( q_{b_1,b_2}V_s \subseteq G(V_{m_0(t)}) \), \( G(V_{m_0(t)}) \cap G(V_{m_0(t)}) \supseteq q_{b_1,b_2}(V_s \cap V'_i) \neq \emptyset \). By (3), \( V_{m_0(t)} \cap V_{m_0(t)} \cap q_{b_1}(Y) \neq \emptyset \). By the choice of \( b_1 \) there exists \( y_0(t) \subseteq Y \) such that \( V_{y_0(t)} \supseteq q_{b_1}(V_{m_0(t)}) \). Since \( V_{y_0(t)} \cap V_{y_0(t)} \supseteq q_{b_1}(V_{m_0(t)}) \neq \emptyset \), by (1) \( st(p_{a_3}(D_{y_0(t)}), U_{a_3}) \cap st(p_{a_3}(D_{y_0(t)}), U_{a_3}) \supseteq \emptyset \). This means that

\[
\text{(8)} \quad st(p_{a_3}(D_{y_0(t)}), U_{a_3}) \cap st(p_{a_3}(D_{y_0(t)}), U_{a_3}) \supseteq \emptyset.
\]

Since \( f \) satisfies (RM2), \( q_{b_1}f=f_{b_1}p_{a_1}f_{a_2} \) and then by (8) \( q_{b_1}f(x'(t)) \subseteq V'_i \). By the choice of \( m_0(t) \), \( q_{b_1}f(x'(t)) \subseteq q_{b_1}(V_{m_0(t)}) \) and then by (2) \( q_{b_1}f(x'(t)) \subseteq q_{b_1}(V_{m_0(t)}) \supseteq q_{b_1}(V_{y_0(t)}) \supseteq V_{y_0(t)} \supseteq V_{y_0(t)} \supseteq q_{b_1}(Y) \neq \emptyset \). By the choice of \( b_1 \) there exists \( y_0(t) \subseteq Y \) such that \( V_{y_0(t)} \supseteq q_{b_1}(V_{m_0(t)}) \). Since \( V_{y_0(t)} \cap V_{y_0(t)} \supseteq q_{b_1}(V_{m_0(t)}) \neq \emptyset \), by (1) \( p_{a_3}(D_{y_0(t)}), U_{a_3}) \cap st(p_{a_3}(D_{y_0(t)}), U_{a_3}) \supseteq \emptyset \). This means that

\[
\text{(9)} \quad st(p_{a_3}(D_{y_0(t)}), U_{a_3}) \cap st(p_{a_3}(D_{y_0(t)}), U_{a_3}) \supseteq \emptyset.
\]

Since \( t \subseteq L \), by (5)-(9) \( p_{a_3}x(v_i) : i=u+1, \ldots, p \} \cap p_{a_3}x(h(t)) \subseteq \bigcup \{st(p_{a_3}(D_{y_0(t)}), U_{a_3}) : i=u+1, \ldots, p \} \cap st(p_{a_3}(D_{y_0(t)}), U_{a_3}) \cap st(p_{a_3}(D_{y_0(t)}), U_{a_3}) \cap st(p_{a_3}(D_{y_0(t)}), U_{a_3}) \cap st(p_{a_3}(D_{y_0(t)}), U_{a_3}) \cap st(p_{a_3}(D_{y_0(t)}), U_{a_3}) \cap st(p_{a_3}(D_{y_0(t)}), U_{a_3}) \). Thus \( g' \) satisfies (RM2) and \( q_{b_1}f(x'(t)) \subseteq V'_i \). By the choice of \( a_3 \) there exists \( y \subseteq Y \) such that \( p_{a_3}x(v_i) : i=0, 1, \ldots, u \} \cap p_{a_3}x(h(t)) \subseteq st(p_{a_3}(D_{y_0(t)}), U_{a_3}) \). Thus we have completed the proof of Claim 2.

Since \( (K, K^*:L) \subseteq \mathbf{P}OL \) pairs, by Claim 2 and the choice of \( a_3 \) there exists a map \( g^1 : K \to X_{a_1} \) satisfying the following conditions:

\[
\text{(10)} \quad g^1 \left| K^*:L = p_{a_3,a_1}g^\circ \right.
\]

and

\[
\text{(11)} \quad \text{for each simplex } s \text{ of } K \text{ there exists } y \subseteq Y \text{ such that } g^1(s) \subseteq st(p_{a_3}(D_{y_0}), U_{a_3}).
\]

Claim 3. \( (f_{b_1}p_{a_1}, f_{b_1})g^1, q_{b_1,b_1}g) < st^L \cap U_{b_1} \).

Take any \( z \subseteq K \) and then there exists a simplex \( s \) of \( K \) with \( z \subseteq s \). First we assume that \( s \subseteq L \). In this case by the choices of \( a_4 \), \( g \) and (10), \( f_{b_1}p_{a_1}, f_{b_1})g^1 | s = f_{b_1}p_{a_1}, f_{b_1})p_{a_2}, a_1 \circ g \} | s = f_{b_1}p_{a_1}, f_{b_1})h | s = q_{b_1,b_1}f_{b_1}p_{a_1}, f_{b_1})h | s = q_{b_1,b_1}g | s \). Thus Claim 3 holds in this case.

Next we assume that \( s \subseteq L \). We put \( s=[v_0, v_1, \ldots, v_p] \). Since \( L \) is full,
we may assume that \( v_0 \in K^* - L \). By (11) there exists \( y \in Y \) such that \( g^l(s) \subset \text{st}(p_{a_1}(D_y), \mathcal{V}_{a_1}) \). By (RM2) for \( f \)

\[
q_b f = f_b p_{f(b)}.
\]

Since \( f_b p_{a_1} \circ f(b) = q_{b_1}(y) \) by (12) and \( (f_b p_{a_1} \circ f(b))^{-1} \subset \mathcal{V}_{b_1} \), \( f_b p_{a_1} \circ f(b) g^l(s) \subset f_b p_{a_1} \circ f(b)(\text{st}(p_{a_1}(D_y), \mathcal{V}_{a_1})) \subset \text{st}(q_{b_1}(y), \mathcal{V}_{b_1}) \). Since \( z, v_0 \in s \),

\[
f_b p_{a_1} \circ f(b) g^l(z), f_b p_{a_1} \circ f(b) g^l(v_0) \in \text{st}(q_{b_1}(y), \mathcal{V}_{b_1}).
\]

By (10) and (12), \( f_b p_{a_1} \circ f(b) g^l(v_0) = q_{b_1}(x(v_0)) \). Since \( V_{m(t_s)}, V_{m(<t_0)} \subset \mathcal{V}_{b_2} \), there exist \( V_1, V_2 \subset \mathcal{V}_{b_1} \) such that \( q_{b_2, b_1}(V_{m(t_s)}) \subset V_1 \) and \( q_{b_2, b_1}(V_{m(<t_0)}) \subset V_2 \). By the choice of \( x(v_0) \), \( q_{b_1}(x(v_0)) \in V_{m(<t_0)} \) and then

\[
f_b p_{a_1} \circ f(b) g^l(v_0) \in V_2.
\]

By (2), (4) and the choice of \( V_1 \), \( q_{b_2, b_1} q_{b_2, b_1} g(s) \subset q_{b_2, b_1} G(V_{m(t_s)}) \subset q_{b_2, b_1} V_{m(t_s)} \subset V_1 \)

and then

\[
q_{b_1, b_1} g(x), q_{b_1, b_1} g(y) \in V_1.
\]

By (2), (4) and the choice of \( V_2 \), \( q_{b_2, b_1} g(v_0) \in q_{b_2, b_1} G(V_{m(<t_0)}) \subset q_{b_2, b_1} V_{m(<t_0)} \subset V_2 \),

that is,

\[
q_{b_2, b_1} g(v_0) \in V_2.
\]

From (13)-(16), \( f_b p_{a_1} \circ f(b) g^l(x), q_{b_1, b_1} g(x) \in \text{st}(\text{st}(V_2, \mathcal{V}_{b_1}), \mathcal{V}_{b_1}) \). Thus in this case we have the required condition and hence we have Claim 3.

We put \( G = p_{a_1, a} g^l : A^{k+1} \to X_\alpha \). By the choices of \( a_1, b_1 \), (10) and Claim 3, \( (f_b p_{a_1} \circ f(b) G, q_{b_1, b_1} G) \subset \mathcal{V}_{b_1} \) and \( G|\partial A^{k+1} = p_{a_0, a} h \). Thus \( f \) satisfies (ALP**) with respect to \( (D, S) \), and hence we have Claim 1.

By Claim 1 and (3.1) \( f \) has ALP with respect to \( (D, S)_\alpha \). Thus \( f \) is approximatively \( n \)-connected. Hence we have completed the proof. \( \blacksquare \)

(6.4) COROLLARY. Let \( Y \) be a paracompact space and \( f : X \to Y \) a closed onto map with (\#). If \( f^{-1}(y) \) is approximatively \( n \)-connected for each \( y \in Y \), then we have the following:

1. \( \text{pro-}\pi_k(f) : \text{pro-}\pi_k(X, x) \to \text{pro-}\pi_k(Y, f(x)) \) is an isomorphism for \( 0 \leq k \leq n \) and epimorphism for \( k = n+1 \) in pro-groups for each \( x \in X \).
2. \( \pi_k(f) : \pi_k(X, x) \to \pi_k(Y, f(x)) \) is an isomorphism for \( 0 \leq k \leq n \) and for each \( x \in X \).
3. If \( n+1 = \text{Max}(sd X+1, sd Y) < \infty \), then \( f : (X, x) \to (Y, f(x)) \) induces a shape equivalence for each \( x \in X \).
(6.5) COROLLARY. Let \( f : X \rightarrow Y \) be a closed onto map from a paracompact space \( X \). If \( f^{-1}(y) \) is approximately \( n \)-connected for each \( y \in Y \), then (i)-(iii) in (6.4) hold.

(6.4) follows from (3.4), (3.6), (3.7) and (6.3). (6.5) follows from (6.5) and Michael's Theorem (see Engelking [9, p. 385]). □

We say that \( f : X \rightarrow Y \) is a cell-like map, in notation CE-map, provided that \( f^{-1}(y) \) has trivial shape for any \( y \in Y \).

(6.6) COROLLARY. Let \( f : X \rightarrow Y \) be a closed map from a paracompact space \( X \). If \( f \) is a CE-map, then we have the following:

(i) \( \text{pro-} \pi_k(f) : \text{pro-} \pi_k(X, x) \rightarrow \text{pro-} \pi_k(Y, f(x)) \) is an isomorphism in pro-groups for each \( k \) and each \( x \in X \).

(ii) \( \check{\pi}_k(f) : \check{\pi}_k(X, x) \rightarrow \check{\pi}_k(Y, f(x)) \) is an isomorphism for each \( k \) and each \( x \in X \).

(iii) If \( \text{sd} X, \text{sd} Y < \infty \), \( f : (X, x) \rightarrow (Y, f(x)) \) induces a shape equivalence for each \( x \in X \).

(6.6) follows from (6.5). □

(6.7) REMARK. Usually approximatively \( n \)-connected maps are called \( UV^n \)-maps (see Lacher [18]). Smale [30] and Kozlowski [13] studied these maps and showed special cases of theorems (6.4) and (6.3). Various Vietoris-Smale theorems in shape theory were studied by many authors, Bogatyi [2, 3], Dydak [4-7], Kodama [11, 12], Kuperberg [16], Kozlowski-Segal [15], Morita [27, 28]. Our results are the most general.

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