MAXIMAL FUNCTIONS OF PLURISUBHARMONIC FUNCTIONS

By

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Abstract. We show that for nonnegative plurisubharmonic functions on the unit ball of $\mathbb{C}^n$ the admissible maximal functions are dominated by the radial maximal functions in $L^p$-mean. This gives another characterization of the class $M^p$ of holomorphic functions and its invariance under the compositions by automorphisms of the unit ball. As a consequence of the invariance all onto endomorphisms of $M^1$ ($n=1$) are characterized.

1. Introduction.

Let $B$ be the unit ball of $\mathbb{C}^n$ and let $\sigma$ denote the Lebesgue measure on $S=\partial B$, normalized so that $\sigma(S)=1$. For a function $u:B \to \mathbb{C}$, the radial maximal function $\mathcal{M}u$ on $S$ is defined by

$$\mathcal{M}u(\eta)=\sup\{|u(r\eta)| : 0 \leq r < 1\}, \quad \eta \in S.$$  

For $\alpha>1$ and $\eta \in S$, we let

$$D_\alpha(\eta)=\{z \in B : |1-<z, \eta>|<\frac{\alpha}{2}(1-|z|^2)\}.$$  

The admissible maximal function $\mathcal{M}_\alpha u$ on $S$ is defined by

$$\mathcal{M}_\alpha u(\eta)=\sup\{|u(z)| : z \in D_\alpha(\eta)\}.$$  

We prove the following theorem.

Theorem I. For $0<p<\infty$, there is a positive constant $C=C(n, p, \alpha)$ such that if $u \geq 0$ is plurisubharmonic in $B$ then

$$\int_S \mathcal{M}_\alpha u(\eta)^p d\sigma(\eta) \leq C \int_S \mathcal{M}u(\eta)^p d\sigma(\eta).$$

For $n=1$, the corresponding theorem for harmonic functions on the upper half plane appears in [3, Theorem 3.6].

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For an application of Theorem I, we consider the class \( M^p(B)(0 < p < \infty) \) of holomorphic functions \( f \) on \( B \) for which
\[
\int_S (\log^+ Mf(\eta))^p d\sigma(\eta) < \infty.
\]
For \( n=1 \), these classes as topological algebras have been studied in [7, 10] for \( p>1 \) and in [2, 5, 6] for \( p=1 \). For \( n \geq 1 \), it is shown in [2] that
\[
\bigcup_{p>0} H^p \subseteq \bigcap_{p>1} M^p \subseteq M^1 \subseteq N^+,
\]
where \( H^p \) is the usual Hardy space and \( N^+ \) is the Smirnov class on \( B \). The main theorem of [2] concerns with the boundary behavior of functions in the class \( M^p(p \geq 1) \), with its application to outer factors of functions in \( M^1 \) when \( n=1 \).

If we take \( u=\log^+ |f| \) with holomorphic functions \( f \) on \( B \) in Theorem I, we get the following characterization of \( M^p \) immediately.

**Theorem II.** A holomorphic function \( f \) on \( B \) belongs to \( M^p \) if and only if
\[
\int_S (\log^+ Mf(\eta))^p d\sigma(\eta) < \infty.
\]
Since every automorphism of \( B \) maps any radius into a curve which approaches the boundary nontangentially, the following corollary is immediate.

**Corollary III.** The class \( M^p(0 < p < \infty) \) is invariant under the compositions of automorphisms of \( B \).

When \( p>1 \), this fact is not new because \( M^p(p>1) \) can be defined by means of boundary functions. See [2, 7]. As a consequence of this corollary we can characterize all onto algebra endomorphisms of \( M^1 \) for the case \( n=1 \). For the case \( p>1 \), see [7].

**Theorem IV.** Let \( n=1 \). Then \( \Gamma : M^1 \to M^1 \) is an onto algebra endomorphism if and only if
\[
\Gamma(f) = f \ast \varphi, \quad f \in M^1
\]
for some automorphism \( \varphi \) of the unit disc \( U \) of \( \mathbb{C}^1 \). In particular, \( \Gamma \) is invertible in this case and \( \Gamma^{-1}(f) = f \ast \varphi^{-1}, \ f \in M^1 \).

The proof will be given in the last section. The theorem might be true for \( n>1 \) but we do not have a proof.
2. An inequality of Hardy and Littlewood.

The following lemma is due to Hardy and Littlewood. It is stated in [3, 4] for $|u|$ with harmonic functions $u$ but the proof is exactly the same for non-negative subharmonic functions.

2.1. Lemma. If $u \geq 0$ is subharmonic on the disc $D(z_0, R)$ with center at $z_0$ and radius $R > 0$ in the complex plane $\mathbb{C}$ and if $0 < p < \infty$, then

$$u(z_0) \leq K \left( \frac{1}{\pi R^2} \int_{D(z_0, R)} u(x^p \, dx \, dy) \right)^{1/p},$$

where $K = K(p)$ is a positive constant independent of $u$.

The next lemma will be a polydisc version of the above inequality. Its statement is suitably adapted for the proof of Theorem I.

Let $z = r \zeta \in B$ and $R > 0$. Let $\zeta_1, \ldots, \zeta_n \in \mathcal{S}$ be such that $\zeta, \zeta_1, \ldots, \zeta_n$ form an orthonormal basis for $\mathbb{C}^n$. Define a polydisc $\Delta(z, R)$ with respect to the basis $\zeta, \zeta_1, \ldots, \zeta_n$ at $z$ as follows:

$$\Delta(z, R) = \Delta(z, R; \zeta, \zeta_1, \ldots, \zeta_n) = \{w = z + \lambda \zeta + \sum_{j=1}^{n} \lambda_j \zeta_j : |\lambda| < R, |\lambda_j| < R^{1/2}, 2 \leq j \leq n\}.$$

2.2. Lemma. Let $\Delta = \Delta(z, R) \subset B$. If $u \geq 0$ is plurisubharmonic in $B$ and $0 < p < \infty$, then

$$u(z)^p \leq K \frac{1}{m_n(\Delta)} \int_{\Delta} u(w)^p \, dm_n(w),$$

where $K = K(n, p)$ is a positive constant independent of $u$ and $dm_n$ is the Lebesgue measure on $\mathbb{C}^n$.

Proof. We define

$$v(\lambda, \lambda_1, \ldots, \lambda_n) = u(z_0 + \lambda \zeta + \lambda_1 \zeta_1 + \ldots + \lambda_n \zeta_n).$$

Since $u$ is plurisubharmonic in $B$, $v$ is an $n$-subharmonic function for $|\lambda| < R$, $|\lambda_j| < R^{1/2}(2 \leq j \leq n)$. We now apply Lemma 2.1 $n$ times to $v$. The positive constants $K$'s in the following are not the same in each occurrence but are independent of $v$.

$$v(0, \ldots, 0)^p \leq K \frac{1}{R} \int_{|\lambda_n| < R^{1/2}} v(0, \ldots, 0, \lambda_n) \, dm_1(\lambda_n) \leq \ldots \leq K \frac{1}{R^{n-1}} \int_{|\lambda_{j+1} < R^{1/2}| \ldots \zeta_j \leq \zeta_{n}} v(0, \lambda, \ldots, \lambda_n)^p \, dm_1(\lambda) \ldots dm_n(\lambda_n)$$
Therefore, we have
\[ u(z)^p \leq K \frac{1}{\gamma_n(\Delta)} \int u(w)^p \, dm_n(w). \tag{Q. E. D.} \]

3. Geometric lemmas.

3.1. Lemma. \( z = \zeta \in B \) and let \( \Delta(z, \varepsilon(1-r^*) \supset B \) for a choice of \( \zeta, \ldots, \zeta_n \in S \) and \( \varepsilon > 0 \). If \( r > 1/2 \) and \( w \in \Delta(z, \varepsilon(1-r^*)) \) then
\[ r - \delta(1-r^*) < |w| < r + \delta(1-r^*) \]
for some choice of a positive constant \( \delta = \delta(n, \varepsilon) \) independent of \( z \) and \( \zeta \)'s.

**Proof.** Suppose \( w = z + \lambda \zeta + \sum \lambda \zeta_j \in \Delta(z; \varepsilon(1-r^*)) \). Then
\[ |w|^2 = |r + \lambda|^2 + \sum_j |\lambda|^2 \leq r^2 + |\lambda|^2 + (n-1)\varepsilon(1-r^*) \]
\[ \leq r^2 + (n+2)\varepsilon(1-r^*). \]
Also,
\[ |w|^2 \geq (r - |\lambda|)^2 = r^2 - 2r|\lambda| + |\lambda|^2 \]
\[ \geq r^2 - 2|\lambda|^2 \geq r^2 - 2\varepsilon(1-r^*). \]
If \( r > 1/2 \) then
\[ |w| - r \leq 2|w|^* - r^* \leq 2(n+2)\varepsilon(1-r^*). \]
So we can take \( \delta = 2(n+2)\varepsilon. \tag{Q. E. D.} \)

The following lemma appears in [1] but its proof is included for the sake of completeness.

3.2. Lemma. \( \beta > \alpha > 1 \) and \( z = \zeta \in D_\alpha(\eta) \). Then there is a positive constant \( \varepsilon = \varepsilon(n, \alpha, \beta) \) such that
\[ \Delta(z, \varepsilon(1-r^*)) \subset D_\beta(\eta) \]
for any choice of \( \zeta, \ldots, \zeta_n \in S \).

**Proof.** Suppose \( w = z + \lambda \zeta + \sum \lambda \zeta_j \in \Delta(z; \varepsilon(1-r^*)) \). Then \( \lambda \leq \varepsilon(1-r^*) \) and \( |\lambda_j| \leq (\varepsilon(1-r^*))^{1/2} \). By the orthogonality of \( \zeta \) and \( \zeta_j \), the Schwarz lemma and the hypothesis \( z \in D_\alpha(\eta) \), we have
\[ \langle \zeta_j, \eta \rangle = |\zeta_j, \eta - r \zeta| \]
\[ \leq |\eta - r \zeta| \]
We compute
\[
|1 - \langle w, \eta \rangle| = \left| 1 - \left( r \langle \zeta, \eta \rangle + \lambda \langle \xi, \eta \rangle + \sum_{j=1}^{n} \lambda_j \langle \zeta_j, \eta \rangle \right) \right|
\]
\[
\leq \frac{\alpha}{2} (1-r^2) + \varepsilon (1-r^2) + \sum_{j=1}^{n} \varepsilon (1-r^2)^{1/2} |\langle \zeta_j, \eta \rangle|
\]
\[
\leq \left\{ \frac{\alpha}{2} + \varepsilon + (n-1)\varepsilon^{1/2} \alpha^{1/2} \right\} (1-r^2)
\]

On the other hand, from the proof of Lemma 3.1, we have
\[
1 - |w|^2 \geq (1-(n+2)\varepsilon)(1-r^2).
\]

Therefore we can choose \( \varepsilon = \varepsilon(n, \alpha, \beta) > 0 \) so small that
\[
|1 - \langle w, \eta \rangle| < \frac{\beta}{2} (1-|w|^2),
\]
for any \( w \in \Delta(z, \varepsilon(1-r^2)) \). Therefore \( \Delta(z, \varepsilon(1-r^2)) \subset D_{\beta}(\eta) \). Q.E.D.

We define the radial projection \( \pi \) from \( B \setminus \{0\} \) onto \( S \) as
\[
\pi(w) = w/|w|, \quad w \in B \setminus \{0\}.
\]
For \( \eta \in S \) and \( \delta > 0 \),
\[
Q(\eta, \delta) = \{ \zeta \in S : |1 - \langle \zeta, \eta \rangle| < \delta \}
\]
is the nonisotropic "ball" of radius \( \delta^{1/2} \) around \( \eta \). The volume \( \sigma(Q(\eta, \delta)) \) is roughly proportional to \( \delta^n \), i.e., \( \sigma(Q(\eta, \delta)) \approx \delta^n \). See [9, Proposition 5.1.4].

33. Lemma. Let \( z = r \zeta \in D_\alpha(\eta) \), \( r > 0 \) and \( \beta > \alpha > 1 \). Then there is a positive constant \( \varepsilon = \varepsilon(n, \alpha, \beta) \) so small that
\[
\pi(\Delta(z, \varepsilon(1-r^2))) \subset Q\left( \eta, \left( \frac{\beta}{2} + 1 \right)(1-r^2) \right)
\]
for any choice of \( \zeta, \ldots, \zeta_n \).

Proof. Choose \( \beta' \) so that \( \beta > \beta' > \alpha \). Let \( w = \rho w \in \Delta(z, \varepsilon(1-r^2)) \). Then
\[
|1 - \langle w, \eta \rangle| = |1 - \langle \rho w, \eta \rangle - (1-\rho)\langle w, \eta \rangle|
\]
\[
\leq |1 - \langle w, \eta \rangle| + (1-\rho^2).
\]
By Lemma 3.2, we can choose \( \varepsilon > 0 \) so small that
\[
|1 - \langle w, \eta \rangle| < \frac{\beta'}{2} (1-r^2).
\]
From the proof of Lemma 3.1, we have
\[ 1 - \rho^4 \leq (1 + 2s)(1 - r^4). \]
Therefore we have
\[ |1 - \langle \omega, \eta \rangle| < \left( \frac{\beta'}{2} + 1 + 2s \right)(1 - r^4). \]
If we choose \( \varepsilon = \varepsilon(n, \alpha, \beta) > 0 \) even smaller so that \( \beta' / 2 + 1 + 2s < \beta / 2 + 1 \), we have
\[ |1 - \langle \omega, \eta \rangle| < \left( \frac{\beta}{2} + 1 \right)(1 - r^4); \]
so that \( \omega \in Q(\eta, (\beta / 2 + 1)(1 - r^4)) \).

Q. E. D.

4. Proof of Theorem I.

It suffices to prove the theorem for a modified admissible maximal function (with the same notation) as
\[ \mathcal{M}_\alpha u(\eta) = \sup \left\{ |u(z)| : |\frac{z}{2}|, z \in D_\alpha(\eta) \right\}. \]
Let \( z = r \zeta \in D_\alpha(\eta), \ r \geq 1 / 2 \) and \( \beta > \alpha \). By Lemmas 3.1, 3.2 and 3.3, we can choose positive constants \( \varepsilon = \varepsilon(n, \alpha, \beta) \) and \( \delta = \delta(n, \varepsilon) = \delta(n, \alpha, \beta) \) so that
(i) \( \Delta = \Delta(z, s(1 - r^4)) \subseteq D_\beta(\eta) \) for a choice of \( \zeta_2, \cdots, \zeta_n \),
(ii) \( \pi(\Delta) \subset Q(\eta, (\beta / 2 + 1)(1 - r^4)) \),
(iii) \( r - \delta(1 - r^4) < |w| < r + \delta(1 - r^4) \) if \( w \in \Delta \).

Using Lemma 2.2, we have the following computation in which the constants \( K = K(n, \rho, \delta) \) are not the same in each occurrence, but are independent of \( u \).

\[
\begin{align*}
\mathcal{M}_\alpha u(z) &\leq K \frac{1}{(1 - r^4)^{n+1}} \int_\Delta u(w)^{p/2} d\sigma(w) \\
&\leq K \frac{1}{(1 - r^4)^{n+1}} \int_{r - \delta(1 - r^4)}^{r + \delta(1 - r^4)} \sigma^{n-1} \rho \int_{Q(\eta, (\beta / 2 + 1)(1 - r^4))} \mathcal{M}u(\omega)^{p/2} d\sigma(\omega) \\
&\leq K \frac{1}{(1 - r^4)^{n}} \int_{Q(\eta)} \mathcal{M}u(\omega)^{p/2} d\sigma(\omega) \\
&\leq K \left( \frac{1}{\sigma(Q)} \right) \int_{Q} \mathcal{M}u(\omega)^{p/2} d\sigma(\omega) \\
&\leq KM \{ \mathcal{M}u \}^{p/2}(\eta),
\end{align*}
\]
where \( M \) is the Hardy-Littlewood maximal function operator on \( S \). Therefore we have
\[ \{ \mathcal{M}_\alpha u(\eta) \}^{p/2} \leq KM \{ \mathcal{M}u \}^{p/2}(\eta). \]

We note that the constant \( K \) is eventually dependent on \( n, \rho, \alpha \) from the choice
of $\beta$ and $\delta$. By the Hardy-Littlewood maximal theorem [9, Theorem 5.2.6], we have
\[
\int_S M_n u(\eta)^p d\sigma(\eta) \leq C \int_S M u(\eta)^p d\sigma(\eta).
\]
for some positive constant $C = C(n, p, \alpha)$ independent of $u$. Q.E.D.

5. Proof of Theorem IV.

By corollary III, every automorphism $\varphi$ of $U$ defines an algebra isomorphism $\Gamma'(f) = f \cdot \varphi$, $\varphi \in M^1$. Conversely, let $\Gamma$ be any onto endomorphism of $M^1$. We will follow the corresponding proof for the case $N^+$ [8]. Let $\varphi = \Gamma(z)$ and let $\lambda = U$. ($z$ denotes the identity function on $U$.) Define $\gamma(f) = \Gamma'(f)(\lambda)$, $f \in M^1$. Since $\gamma$ is a multiplicative linear functional on $M^1$, $\gamma$ corresponds to the point evaluation at some $\beta \in U$ by Theorem 6.4 of [6]. Thus $\beta = \gamma(z) = \Gamma'(z)(\lambda) = \varphi(\lambda)$. Hence $\varphi(z) \in U$ for all $\lambda \in U$ and $\Gamma'(f)(\lambda) = f(\varphi(\lambda))$, $f \in M^1$, $\lambda \in U$. Since $\Gamma'$ is onto, $\varphi$ is not constant. Thus $\varphi(U)$ is open in $U$. Therefore $\Gamma'$ is one-to-one (and onto). Thus $\Gamma^{-1}$ is also an onto endomorphism, so $\Gamma^{-1}(f) = f \cdot \varphi$, $f \in M^1$, for some holomorphic self-map $\varphi$ of $U$. But then $z = \Gamma \Gamma^{-1}(z) = \Gamma(\varphi) = \varphi \cdot \varphi$ and $\varphi \cdot \varphi = z$. Therefore $\varphi$ is an automorphism of $U$. Q.E.D.

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References


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