Equity-efficiency bicriteria location with squared Euclidean distances

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April 16, 2008

Abstract

A facility has to be located within a given region taking two criteria of equity and efficiency into account. Equity is sought by minimizing the inequality in the inhabitant-facility distances, as measured by the sum of the absolute differences between all pairs of squared Euclidean distances from inhabitants to the facility. This measure meets the Pigou-Dalton condition of transfers, and can easily be minimized. Efficiency is measured through optimizing the sum of squared inhabitant-facility distances, either to be minimized or maximized for an attracting or repellent facility respectively. Geometric localization results are obtained for the whole set of Pareto optimal solutions for each of the two resulting bicriteria problems within a convex polygonal region. A polynomial procedure is developed to obtain the full bicriteria plot, both trade-off curves and the corresponding efficient sets.

Subject classification: Facilities planning: equity continuous location. Programming: bicriteria

Area of review: Optimization

1 Introduction

Over the last decade, many studies in location science have been made concerning equity in a geographical setting. In order to quantify the inequalities based on facility-inhabitant distances, many equity measures, for example, *range*, *variance*, *mean absolute deviation*, *sum of absolute deviations*, *Gini coefficient* have been incorporated into location models as objective functions. Comprehensive reviews of such equity studies can be found in Erkut(1993), Marsh and Schilling(1994), Eiselt and Laporte(1995).

This study formulates equity location models using the *sum of absolute deviations*, one of the simplest measures discussed, e.g. by Ogryczak(2000). That is, it seeks facility locations where the sum of absolute differences between all pairs of facility-inhabitant squared distances is minimized. This measure is strictly improved if a unit distance from a longer facility-inhabitant squared distance is transferred to a shorter facility-inhabitant squared distance. This *Pigou-Dalton condition of transfers* is regarded by economists as a mandatory requirement for adequate measures of equity, see Sen(1973), Marshall and Olkin(1979), a point of view we follow here. This equity measure is also the numerator of the well known Gini index, so it is intimately related to the Lorenz curve which is frequently used to measure equity of income in economics: see, e.g. Sen(1973), Stiglitz(1986) and Krugman(1993).

As efficiency measure we consider two cases, depending on the type of facility. For locating an attractive facility we use a Weber problem, for an obnoxious facility an anti-Weber problem, i.e. minimizing (resp. maximizing) the sum of some function of distances from the inhabitants to the facility. For a survey of the popular Weber problem and its extensions, see Drezner et al.(2004). Anti-Weber problems were initiated in Hansen et al.(1981a) and recently surveyed by Lozano and Mesa (2000).

Combining the equity measure with each efficiency measure, two bicriteria models are obtained and studied here. Thus, we consider two bicriteria formulations, respectively applicable to attractive facilities and to obnoxious facilities.

This work differs from existing equity location studies in several respects.

First, most existing models, for example Mandell(1991), consider a discrete framework in which a finite candidate set of potential facility sites is given. They are site-selection models, solvable by standard combinatorial optimization methods, as pointed out by Love et al.(1988). This paper is an exploratory site-generation model, intended to indicate where to look for the more interesting sites. It therefore works in a continuous plane. Equity location models in such a setting are rather rare. One such study is Drezner et al.(1986), who search for the location in the plane which minimizes the range, i.e., the difference between the maximum and minimum facility-inhabitant distances. But the range does not satisfy the Pigou-Dalton condition of transfers. Another one is Carrizosa (1999), who minimizes the variance of all Euclidean facility-inhabitant distances. It is not clear, however, how to handle a combination of this objective with an efficiency criterion as we do.

Second, like in Ohsawa(1999) and partly in Drezner and Wesolowsky(1991), the *squared* Euclidean distances between inhabitants and the facility are used rather than the more popular Euclidean distances. Such a quadratic formulation is appropriate in the location of fire stations, particularly in environments where fire spreads easily in all directions, such as forests or urban areas with wood-based infrastructure. Indeed, in case of a fire the affected area, and therefore the amount of damage, will grow quadratically with the amount of time before the fire fighting starts, while this time is approximately linearly related to the travel-distance to the site. Similarly, in most other emergency services, conditions at a site where an emergency arises deteriorate rapidly before the intervention, leading to dis-economies of scale in the damage as a function of intervention time; in order to obtain a good approximation of this convex nonlinear effect one will have to use at least a second-order Taylor expansion, i.e. one better uses a quadratic function of distance rather than a linear one. We will demonstrate in this paper that this squared distance view has the important advantage to allow exact analysis of all single criteria and both bicriteria problems, leading to full solution by geometrical means.

Third, we consider a locational constraint, as defined by a bounded feasible region. In practice such a constraint is always present. In the obnoxious facility case this is also necessary in theory to ensure existence of optimal solutions for the efficiency measure. For technical reasons we restrict ourselves to a convex polygonal region. Polygons are typical in GIS-based spatial data. Alleviating the convexity assumption will be briefly discussed in the concluding section.

Finally, we examine the conflict between equity and efficiency through two bicriteria models, one for attractive facilities, and one for obnoxious facilities. The set of Pareto-optimal solutions associated with each of these problems are then the most interesting sites. To the best knowledge of the authors, this is the first work on the equity-efficiency trade-off attempting such an approach. We thereby obtain the full tradeoff curves between the two objectives, which give all necessary value-information for the comparison of the Pareto-optimal solutions. In fact we obtain a full picture of all pairs of values obtainable within the feasible region. This bicriteria plot may be used further to evaluate visually for *any* feasible site how much improvement is obtainable in each objective without loss, or with some allowance, on the other objective.

As pointed out by Carrizosa and Plastria(1999), determining the Pareto-set in a continuous plane is usually a rather difficult problem because standard procedures of convex analysis are not always directly helpful. However, Ohsawa(2000) gave a polynomial algorithm to compute the

Pareto-set when maximin and minimax Euclidean distances are used as push and pull objectives. Ohsawa and Tamura(2003) extended this work to bicriteria models combining on the one hand maximin elliptic and minisum rectilinear distances, and maximin or minimax rectilinear distances on the other hand. Ohsawa et al.(2006) introduced partial covering in the bicriteria model of Ohsawa (2000). Melachrinoudis and Xanthopulos(2003) proposed a numerical approach for a bicriteria problem with maximin and minisum Euclidean distances, based on Voronoi diagrams and optimality conditions in nonlinear programming. Our general localization results are somewhat comparable to this latter work, but from the algorithmic point of view, our solution method based on computational geometry is rather similar to the one by Ohsawa(2000). We present a polynomial algorithm using the line tessellation generated by the perpendicular bisectors of all pairs of residence locations, and show that this allows the construction of the bicriteria plot. Its envelope then determines the two types of Pareto-sets. This shows that the framework developed by Ohsawa(2000) may be extended to other bicriteria location models.

The remainder of this paper is structured as follows. Section 2 explores the single-objective location models, first the equity objective, then the efficiency ones. Section 3 discusses the bicriteria models combining the equity model with either Weber or anti-Weber models, giving proofs of the localization theorems for the Pareto-sets, then describing polynomial algorithms to construct the bicriteria plot and the Pareto-sets. Section 4 contains a number of concluding remarks on possible extensions and further research.

2 Single-Objective Problems

2.1 Equity model

We propose to use the sum of absolute differences of squared distances as an equity criterion to be minimized. We start by showing, contrary to what was stated by Eiselt and Laporte (1995), that this satisfies the Pigou-Dalton condition under the following form given by Erkut (1993): when changing the distance distribution $S : d_1 \leq d_2 \leq \ldots \leq d_k \leq \ldots \leq d_l \leq \ldots \leq d_n$ into the distribution $S' : d_1 \leq d_2 \leq \ldots \leq d_k + \epsilon \leq \ldots \leq d_l - \epsilon \leq \ldots \leq d_n$, in which a small enough amount ϵ is transferred from the larger d_l to the lower d_k, so as not to change the order, the equity value should have decreased.

Proposition 1 *The sum of absolute differences*

$$PD(S) \equiv \sum_{i=1}^{n} \sum_{j=1}^{n} |d_i - d_j|$$

satisfies the Pigou-Dalton condition of transfers.

Proof It is easy to check that for $d_1 \leq d_2 \leq \ldots \leq d_i \leq \ldots \leq d_j \leq \ldots \leq d_n$

$$\begin{aligned} PD(S) &= 2\sum_{i=1}^{n}\sum_{j>i}^{n}(d_j - d_i) \\ &= 2\{(1-n)d_1 + (3-n)d_2 + \ldots + (n-3)d_{n-1} + (n-1)d_n\} \\ &= 2\sum_{i=1}^{n}(2i-1-n)d_i. \end{aligned} \tag{1}$$

Since $k < l$, one then obtains

$$PD(S') - PD(S) = 2\{(2k-1-n)\epsilon - (2l-1-n)\epsilon\} = 4(k-l)\epsilon < 0,$$

as required. ∎

In our location context a number of residences are given on an Euclidean plane. Let I and $\{\mathbf{p}_1, \mathbf{p}_2, \cdots, \mathbf{p}_{|I|}\}$ be their index and location sets. Let $\{\omega_1, \omega_2, \cdots, \omega_{|I|}\}$ be their number of inhabitants, expressed here as positive weights, i.e., $\omega_i > 0$. To simplify notations, and without loss of generality, we assume that $\sum_{i\in I} \omega_i = 1$. Suppose that a facility can be built within the convex polygon Ω. Let $\partial\Omega$ and $|\partial\Omega|$ be the boundary and the number of sides of Ω, respectively. We may allow that some $\mathbf{p}_i \notin \Omega$, i.e., some residence points may be situated outside the feasible region. To avoid unnecessary complications we assume that $|I| \geq 3$, and that all $\mathbf{p}_i$ and the vertices of $\partial\Omega$ are distinct.

Our equity problem seeks a site $\mathbf{x}$ in Ω minimizing the sum of absolute differences between all pairs of facility-inhabitant squared Euclidean distances. Its formulation is therefore (up to a positive factor)

$$\min_{\mathbf{x}\in\Omega} \qquad F(\mathbf{x}) \equiv \sum_{i\in I}\sum_{j\in I} \omega_i\omega_j \left| \|\mathbf{x}-\mathbf{p}_i\|^2 - \|\mathbf{x}-\mathbf{p}_j\|^2 \right|. \tag{2}$$

Let $i_1, \cdots, i_{|I|}$ be an ordering of the residence set I. Define $V_{i_1,i_2,\cdots,i_{|I|}}$ to be the ordered order-$|I|$ Voronoi polygon associated with $\mathbf{p}_{i_1}, \mathbf{p}_{i_2}, \cdots, \mathbf{p}_{i_{|I|}}$: see Okabe et al.(1999), as

$$V_{i_1,i_2,\cdots,i_{|I|}} \equiv \{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x}-\mathbf{p}_{i_1}\| \leq \|\mathbf{x}-\mathbf{p}_{i_2}\| \leq \cdots \leq \|\mathbf{x}-\mathbf{p}_{i_{|I|}}\|\}.$$

This set will be empty for many orderings $i_1, i_2, \cdots, i_{|I|}$ of I. The collection of all $V_{i_1,i_2,\cdots,i_{|I|}}$'s with non-empty interior (the *faces*) is called the *ordered order-$|I|$ Voronoi diagram.* Observe that the boundary of the cells of this diagram coincides with the arrangement of the perpendicular bisectors l_{ij}'s of all pairs $\mathbf{p}_i, \mathbf{p}_j$'s. Arrangements of lines occur in many different applications such as computer graphics and robotics, as pointed out by Boissonnat and Yvinec(1998). The number of bisectors is at most $\frac{|I|(|I|-1)}{2}$, i.e., $O(|I|^2)$. Since each bisector intersects each of the others at most once, or coincides with it, the number of edges of the Voronoi diagram is $O(|I|^4)$. Hence, this diagram is a planar graph and has $O(|I|^4)$ vertices, edges and faces. Let ∂V be the collection of the boundaries of $V_{i_1,i_2,\cdots,i_{|I|}}$'s within Ω. The line tessellation ∂V and $\partial\Omega$ is illustrated in Figure 1, where five inhabitants $\mathbf{p}_1, \cdots, \mathbf{p}_5$ are indicated by filled circles, and ∂V and $\partial\Omega$ are shown as thin and thick lines, respectively. The nearest point of any position within $V_{3,2,4,1,5}$ is $\mathbf{p}_3$, its second one is $\mathbf{p}_2$, its third one is $\mathbf{p}_4$ and so on.

Similarly as the derivation of (1), and using $\sum_{i\in I} w_i = 1$, within each face $\mathbf{x} \in V_{i_1,i_2,\cdots,i_{|I|}}$, $F(\mathbf{x})$ in (2) can be rewritten as

$$F(\mathbf{x}) = 2\left(u_{i_1}\|\mathbf{x}-\mathbf{p}_{i_1}\|^2 + u_{i_2}\|\mathbf{x}-\mathbf{p}_{i_2}\|^2 + \cdots + u_{i_s}\|\mathbf{x}-\mathbf{p}_{i_s}\|^2 + \cdots + u_{i_{|I|}}\|\mathbf{x}-\mathbf{p}_{i_{|I|}}\|^2\right), \tag{3}$$

where $u_{i_1} \equiv \omega_{i_1}(\omega_{i_1}-1)$, $u_{i_2} \equiv \omega_{i_2}(2\omega_{i_1}+\omega_{i_2}-1)$, $u_{i_s} \equiv \omega_{i_s}(\sum_{k=1}^{s}\omega_{i_k} - \sum_{k=s}^{|I|}\omega_{i_k})$ and $u_{i_{|I|}} \equiv \omega_{i_{|I|}}(1-\omega_{i_{|I|}})$.

Proposition 2 *The function $F(\mathbf{x})$ is convex on Ω, and piecewise linear with pieces the Voronoi polygons $V_{i_1,i_2,\cdots,i_{|I|}}$.*

Proof For two points $\mathbf{p}_i$ and $\mathbf{p}_j$ $(i \neq j)$, and writing scalar product as $\langle \cdot\,;\,\cdot \rangle$, we have

$$\begin{aligned} \|\mathbf{x}-\mathbf{p}_i\|^2 - \|\mathbf{x}-\mathbf{p}_j\|^2 &= \langle\, \mathbf{x}-\mathbf{p}_i\,;\,\mathbf{x}-\mathbf{p}_i \,\rangle - \langle\, \mathbf{x}-\mathbf{p}_j\,;\,\mathbf{x}-\mathbf{p}_j \,\rangle \\ &= 2\langle\, \frac{\mathbf{p}_i+\mathbf{p}_j}{2} - \mathbf{x}\,;\,\mathbf{p}_i-\mathbf{p}_j \,\rangle. \end{aligned} \tag{4}$$

This shows that this difference of squared distances is an affine function taking, of course, value zero along the bisector of $\mathbf{p}_i$ and $\mathbf{p}_j$, i.e. the line l_{ij} of equation $\langle\, \frac{\mathbf{p}_i+\mathbf{p}_j}{2} - \mathbf{x}\,;\,\mathbf{p}_i-\mathbf{p}_j \,\rangle = 0$. Its absolute value is therefore convex and piecewise linear, with pieces the two halfplanes bounded by l_{ij}.

$F(\mathbf{x})$ is the sum of such functions over all $i, j \in I$, and therefore also convex and piecewise linear, with pieces the intersections of halfplanes bounded by the l_{ij}'s, which are exactly the Voronoi faces $V_{i_1,i_2,\cdots,i_{|I|}}$. ∎

The piecewise linearity of $F(\mathbf{x})$ leads to the important fact that the level curves of $F(\mathbf{x})$ within a Voronoi polygon consist of parallel lines. Some resulting level curves of $F(\mathbf{x})$ for our problem are shown in Figure 2. In addition, the minimum of $F(\mathbf{x})$ within each Voronoi polygon restricted to Ω is reached at some vertex. Thus, we obtain

Proposition 3 *An optimal solution $\mathbf{d}^*$ of (2) exists at some vertex of the planar graph $\partial V \cup \partial\Omega$.*

Although $\mathbf{d}^*$ may not be unique, the whole solution set is given by the convex hull of all the nodes of the graph $\partial V \cup \partial\Omega$ which minimize $F(\mathbf{x})$ because the set must be convex by convexity of $F(\mathbf{x})$. Because of the piecewise linearity this set will also be a union of faces. Thus, we have the following algorithm to identify the whole set of optimal solutions:

Algorithm 1

Step 1. Define the planar graph $\partial V \cup \partial\Omega$.

Step 2. Find the minimum solution of $F(\mathbf{x})$ from the nodes of the graph $\partial V \cup \partial\Omega$.

Step 3. Detect the whole set of minimum solutions of $F(\mathbf{x})$, unless a unique solution exists.

Using this algorithm we obtain

Proposition 4 *A solution $\mathbf{d}^*$ can be found in $O(|I|^5 + |I||\partial\Omega|)$ time.*

Proof Since the number of the bisectors l_{ij} is $O(|I|^2)$, we can compute the arrangement of these bisectors in $O(|I|^4)$ time using an incremental algorithm, as shown in Edelsbrunner et al.(1986). Since $\partial\Omega$ is a closed polygonal line and the arrangement consists of convex regions, the graph $\partial V \cup \partial\Omega$ can be constructed in $O(|I|^4 + |\partial\Omega|)$ via the method by O'Rourke et al.(1982). Hence, Step 1 takes $O(|I|^4 + |\partial\Omega|)$ time. Note that the graph $\partial V \cup \partial\Omega$ has $O(|I|^4 + |\partial\Omega|)$ vertices. To compute $F(\mathbf{x})$ according to (3), which is a less demanding expression than (2), the $\mathbf{p}_i$'s are arranged according to the facility-inhabitant distances. Once the facility-inhabitant distances within one Voronoi polygon have been sorted in $O(|I|\log|I|)$, we can compute $F(\mathbf{x})$ not only at any vertex of the polygon but also at any vertex of its *surrounding* polygons in $O(|I|)$ using (3). This is because moving from one polygon to an adjacent polygon across some Voronoi edge interchanges only the order of two residence points: see, e.g. $V_{2,3,4,1,5}$ and $V_{3,2,4,1,5}$ in Figure 1. Since the graph $\partial V \cup \partial\Omega$ has $O(|I|^4 + |\partial\Omega|)$ edges, Step 2 runs in $O(|I|^5 + |I||\partial\Omega|)$ time by traversing at most twice each edge of the graph $\partial V \cup \partial\Omega$. Step 3 can be performed in $O(|I|^4 + |\partial\Omega|)$ time by comparing $F(\mathbf{x})$ evaluated at all vertices within each edge and each face of $\partial V \cup \partial\Omega$. ∎

Alternatively, Step 2 may also be done by grouping the calculations along each bisector as follows. Consider a bisector $l_{i_0 j_0}$ with parametric equation $\mathbf{x} = \mathbf{c} + t\mathbf{q}$ where $\mathbf{c} = (\mathbf{p}_{i_0} + \mathbf{p}_{j_0})/2$ and $\mathbf{q}$ is orthogonal to $\mathbf{p}_{i_0} - \mathbf{p}_{j_0}$. $l_{i_0 j_0}$ either contains no points of $\partial\Omega$, or just 1, or exactly 2, or, exceptionally a full line segment, side of Ω, which may in this first stage be reduced to its two endpoints. In the two last cases we have to search further along $l_{i_0 j_0}$. Using expression (4) one easily obtains the following new expression for $F(\mathbf{x})$ along $l_{i_0 j_0}$

$$F(\mathbf{x}) = F(\mathbf{c}+t\mathbf{q}) = \sum_{i \neq j \in I\setminus\{i_0, j_0\}} \omega_i \omega_j \alpha_{ij} |t_{ij} - t| + \sum_{i \in I\setminus\{i_0\}} \omega_{i_0} \omega_j \alpha_{i_0 j} |t_{i_0 j} - t| + \sum_{j \in I\setminus\{j_0\}} \omega_i \omega_{j_0} \alpha_{i j_0} |t_{i j_0} - t|, \tag{5}$$

where

$$\alpha_{ij} \equiv |\langle\, 2\mathbf{q} \,;\, \mathbf{p}_i - \mathbf{p}_j \,\rangle|, \qquad t_{ij} \equiv 2\langle\, \frac{\mathbf{p}_i + \mathbf{p}_j}{2} - \mathbf{c} \,;\, \mathbf{p}_i - \mathbf{p}_j \,\rangle / \alpha_{ij}.$$

Note that there are two types of t_{ij} corresponding to two different types of Voronoi vertices on $l_{i_0 j_0}$: the points of form $\mathbf{c} + t_{ij}\mathbf{q}$ of order 4, which are intersections of type $l_{i_0 j_0} \cap l_{ij}$ with $|\{i_0, j_0, i, j\}| = 4$, and those of form $\mathbf{c} + t_{i_0 j}\mathbf{q}$ or $\mathbf{c} + t_{i j_0}\mathbf{q}$ of order 6, which are those of type $l_{i_0 j_0} \cap l_{i_0 i} \cap l_{i j_0}$ with $|\{i_0, j_0, i\}| = 3$.

Expression (5) shows that $F(\mathbf{c}+t\mathbf{q})$ as a function of t along $l_{i_0j_0}$ is a weighted sum of distances to the points t_{ij}, and its minimization is a one-dimensional Weber problem, well known to be solved at a (weighted) median point: see, e.g. Francis et al.(1992). The minimizer can be obtained in time linear in the number of points by the procedure of Balas and Zemel (1980). Thus we can obtain the unconstrained minimum point of $F(\mathbf{x})$ along each bisector in $O(|I|^2)$, and then in constant time its constrained minimum by simple comparison with the boundary vertices.

In fact a descent procedure, which first finds the median along one bisector, then iteratively moves to another bisector which passes at this point and along which there is a decrease direction, until no such direction exists, will work much quicker in practice. (Note that the median finding technique applies along any line, so may also be used, if necessary, along Ω's boundary.) It is not clear, however, if a better worst case complexity may be shown for this method.

The solution $\mathbf{d}^*$ for our sample problem with $\omega_1 = \omega_2 = \ldots = \omega_5$ lies on a vertex of ∂V, as shown in Figure 2.

2.2 Efficiency models

As efficiency criterion, we consider the following typical single-objective function:

$$G(\mathbf{x}) \equiv \sum_{i \in I} \omega_i \|\mathbf{x} - \mathbf{p}_i\|^2. \tag{6}$$

The problems to either minimize or maximize $G(\mathbf{x})$ are called (quadratic distance) *Weber* or *anti-Weber problems*, respectively, as in Hansen et al.(1981a). The minimization of $G(\mathbf{x})$ may be used for locating a purely attractive facility, whereas the maximization of $G(\mathbf{x})$ may be applicable to a purely repellent facility in the sense that the total distance to inhabitants is maximized.

As shown in Francis et al.(1992), it is straightforward to check that

$$G(\mathbf{x}) = \|\mathbf{x} - \overline{\mathbf{p}}\|^2 + \sum_{i \in I} \omega_i \|\mathbf{p}_i\|^2 - \|\overline{\mathbf{p}}\|^2 \tag{7}$$

where $\overline{\mathbf{p}} \equiv \sum_{i \in I} \omega_i \mathbf{p}_i$ is the center of gravity (centroid) of the points $\mathbf{p}_i$. This shows that $G(\mathbf{x})$ is a strictly convex function of $\mathbf{x}$.

It also immediately leads to the well-known result that the unconstrained minimum of $G(\mathbf{x})$ is given by $\overline{\mathbf{p}}$ and that the level sets of $G(\mathbf{x})$ consist of circular disks with center $\overline{\mathbf{p}}$. The constrained minimum $\mathbf{m}^*$ of $G(\mathbf{x})$ on Ω is unique, due to strict convexity, and is evidently the projection of $\overline{\mathbf{p}}$ on Ω, i.e. the point of Ω closest to $\overline{\mathbf{p}}$.

The strict convexity of $G(\mathbf{x})$ also implies that $G(\mathbf{x})$ can be maximum on Ω only at a vertex of Ω which we will denote by $\mathbf{a}^*$, as discussed in Hansen et al.(1981a).

The solutions $\mathbf{m}^*$ and $\mathbf{a}^*$ for the sample problem are also shown in Figure 2. Note that in this example $\mathbf{m}^* = \overline{\mathbf{p}}$, because $\overline{\mathbf{p}} \in \Omega$.

3 Biobjective Problems

3.1 Formulation

In order to examine the tradeoff between equity and efficiency, we formulate the following two bicriteria problems combining the equity model (2) with either minimizing or maximizing $G(\mathbf{x})$ in (6):

$$\min_{\mathbf{x} \in \Omega} \{F(\mathbf{x}), G(\mathbf{x})\}, \tag{8}$$

$$\min_{\mathbf{x} \in \Omega} \{F(\mathbf{x}), -G(\mathbf{x})\}. \tag{9}$$

As usual, the *weakly Pareto-optimal solutions* are feasible locations simultaneously at least as good for both objectives than any other feasible location, and they are *Pareto-optimal* when always strictly better for at least one objective. Let E_+^* and E_-^* be the Pareto-set associated with the problems (8) and (9), respectively. We call *bicriteria plot* the set of pairs $(F(\mathbf{x}), G(\mathbf{x}))$ in the objective space for all $\mathbf{x}$ in the feasible region, and *tradeoff curve* that for the Pareto-set only. Writing for any subset S of the plane, $(F, G)(S) \equiv \{(F(\mathbf{x}), G(\mathbf{x}))|\mathbf{x} \in S\}$, the bicriteria plot is given by $(F, G)(\Omega)$, and the tradeoff curves associated with the problems (8) and (9) are the sets $(F, G)(E_+^*)$ and $(F, G)(E_-^*)$, respectively. In what follows we always consider the objective space with the horizontal (vertical) axis measuring the values of $F(\mathbf{x})$ ($G(\mathbf{x})$). For the attractive facility location (8), since the left and lower directions on the objective space are better in terms of $F(\mathbf{x})$ and $G(\mathbf{x})$, respectively, a Pareto-optimal solution has no alternative in any southwesternly quadrant direction. Graphically, therefore, the Pareto-set E_+^* is given by the set of locations corresponding to the lower-left envelope of the bicriteria plot $(F, G)(\Omega)$. In the same way, the Pareto-set E_-^* is given by the set of locations corresponding to the upper-left envelope of the bicriteria plot $(F, G)(\Omega)$.

3.2 Localization results

3.2.1 Obnoxious facility

Within each constrained ordered Voronoi cell $C = V_{i_1,i_2,\cdots,i_{|I|}} \cap \Omega$, $-F(\mathbf{x})$ is a linear function, and $G(\mathbf{x})$ is a strictly convex function. Thus problem (9) is a bicriteria convex maximization problem on a convex polygonal region. It was shown in general by Carrizosa and Plastria (2000) (proposition 19 p53) that when maximizing k quasiconvex criteria on a polytope P any point of P is dominated by some point on a $(k-1)$-dimensional face of P. In our situation $k = 2$, so any point of C is dominated by some point of C's boundary. This result may be slightly strengthened in our case. Compare also with a similar result in Ohsawa(2000).

Proposition 5 *The Pareto-set E_-^* is a subset of $\partial V \cup \partial\Omega$.*

Proof Consider any face $C = V_{i_1,i_2,\cdots,i_{|I|}} \cap \Omega$ and any point c in the interior of C. Since $F(\mathbf{x})$ is linear on C, the set of points within C with same F-value as c is either a line segment, or the whole cell C, so in any case it is a polytope with relative extreme points on C's boundary, while c lies in its relative interior. By strict convexity of $G(\mathbf{x})$, any maximum of $G(\mathbf{x})$ on this set will be found at such an extreme point, so cannot be c itself. Therefore interior points of cells cannot belong to E_-^*. ∎

3.2.2 Attractive facility

Both $F(\mathbf{x})$ and $G(\mathbf{x})$ are convex functions all over the plane, and thus problem (8) is a bicriteria convex minimization problem constrained to a convex polygonal region. For this type of problems Plastria and Carrizosa (1996) derived the following weakly Pareto-optimality condition at a point $\mathbf{x}$: the convex hull of the subdifferentials of both objectives at $\mathbf{x}$ must intersect the opposite of the normal cone of the constraint region at $\mathbf{x}$. When $\mathbf{x}$ is an interior point of some constrained ordered Voronoi cell $V_{i_1,i_2,\cdots,i_{|I|}} \cap \Omega$, this condition is very easily checked since both objectives are differentiable, so their subdifferentials consist of their gradient only, and in the interior of the constraint set the normal cone is reduced to the zero vector. The gradient of $F(\mathbf{x})$ at $\mathbf{x}$ is orthogonal to $F(\mathbf{x})$'s level curves, while the gradient of $G(\mathbf{x})$ at $\mathbf{x}$ is given by $\mathbf{x} - \overline{\mathbf{p}}$. Thus such an interior point $\mathbf{x}$ can only be (weakly) Pareto-optimal if these two gradients are parallel and of opposite sign, i.e. $\mathbf{x} - \overline{\mathbf{p}}$ is orthogonal to $F(\mathbf{x})$'s level curves in the direction of decrease of $F(\mathbf{x})$, and, by (3), $\mathbf{x} - \overline{\mathbf{p}}$ must be of the form $-\sum_{k=1}^{|I|} u_{i_k}\mathbf{p}_{i_k}$ when $\mathbf{x}$ is in $V_{i_1,i_2,\cdots,i_{|I|}}$. Define therefore $L_{i_1,i_2,\cdots,i_{|I|}}$ as the intersection of $V_{i_1,i_2,\cdots,i_{|I|}} \cap \Omega$ with the halfline issued from $\overline{\mathbf{p}}$ in the direction of $-\sum_{k=1}^{|I|} u_{i_k}\mathbf{p}_{i_k}$. Note that many of these intersections will be empty, and let L be the collection of nonempty $L_{i_1,i_2,\cdots,i_{|I|}}$'s. This set is illustrated in Figure 3, where L consists of eleven broken line-segments (recall that here $\mathbf{m}^* = \overline{\mathbf{p}}$).

It now follows immediately:

Proposition 6 *The Pareto-set E_+^* is a subset of $\partial V \cup \partial\Omega \cup L$.*

The effects of Propositions 5 and 6 are that we may restrict our search for Pareto solutions to edges of the graphs $\partial V \cup \partial\Omega$ and $\partial V \cup \partial\Omega \cup L$, respectively.

3.3 Constructing the bicriteria plot

Consider now any region W of the line tessellation defined by $\partial V \cup \partial\Omega \cup L$. Any such region W is simply-connected. By continuity of $F(\mathbf{x})$ and $G(\mathbf{x})$, this means that the set of the plots $(F, G)(W)$ in the objective space is also simply-connected. Along any line, $F(\mathbf{x})$ is a linear function and $G(\mathbf{x})$ is a quadratic function, so we can express $G(\mathbf{x})$ as a quadratic function of $F(\mathbf{x})$, the exact form of which is easily obtained from (7), suitably adapted to the line under consideration. Furthermore, this relationship between $F(\mathbf{x})$ and $G(\mathbf{x})$ is monotonic because within W the scalar product of the gradients of both functions has constant sign. This implies that the set $(F, G)(W)$ is the bounded domain surrounded by the plots of the line-segments of the boundary of W, and each of these latter is a piece of an upright parabola.

Therefore we have:

Proposition 7 *The bicriteria plot $(F, G)(\Omega)$ is given by the union of the domains bounded by the plots of ∂V, $\partial\Omega$ and L.*

This bicriteria plot is therefore obtained by following algorithm, whose validity follows from Proposition 7.

Algorithm 2.

Step 1. Construct the planar graph $\partial V \cup \partial\Omega \cup L$.

Step 2. For each face C of the graph, hatch the region bounded by the loci $(F, G)((\partial V \cup L) \cap C)$ in the objective space.

Using this algorithm we obtain

Proposition 8 *The bicriteria plot $(F, G)(\Omega)$ can be drawn in $O(|I|^5 + |I||\partial\Omega|)$ time.*

Proof The planar graph $\partial V \cup \partial\Omega$ can be constructed in $O(|I|^4 + |\partial\Omega|)$ time, as we have seen in Step 1 of Algorithm 1. Analogous to Step 2 of Algorithm 1, once the facility-inhabitant distances within one Voronoi polygon $V_{i_1, i_2, \cdots, i_{|I|}}$ are sorted in $O(|I| \log |I|)$, not only L within the Voronoi polygon but also L within any of its surrounding polygons can be identified in $O(|I|)$ time, respectively. Since the graph $\partial V \cup \partial\Omega$ has $O(|I|^4 + |\partial\Omega|)$ edges, the planar graph $\partial V \cup \partial\Omega \cup L$ can be constructed in $O(|I|^5 + |I||\partial\Omega|)$ time. Thus, the complexity of Step 1 is $O(|I|^5 + |I||\partial\Omega|)$. Note that the graph $\partial V \cup \partial\Omega \cup L$ has $O(|I|^4 + |\partial\Omega|)$ edges. Similarly as for Step 1, by traversing at most twice each edge of the graph, plots corresponding to the boundaries of all cells of the graph can be constructed in $O(|I|^5 + |I||\partial\Omega|)$ time. This leads to an $O(|I|^5 + |I||\partial\Omega|)$ time for Step 2. ∎

The bicriteria plot $(F, G)(\Omega)$ corresponding to our sample problem is indicated by the gray region in Figure 4, where the plots corresponding to ∂V, $\partial\Omega$ and L are also indicated by thin, thick and broken lines, respectively. One recognizes the image of $\mathbf{m}^*$ as the lowest point, and that of $\mathbf{d}^*$ as the left-most point. The corresponding solution points are quite central within Ω, as can be seen in Figure 3. The two halves of Ω above and below these points are 'folded' on top of each other by the mapping (F, G), yielding the two superposed 'wings' towards the right on the bicriteria plot, with extreme points $(F, G)(\mathbf{s}_3)$ and $(F, G)(\mathbf{a}^*)$ at the top.

Figure 4 shows that, roughly speaking, the relationship between the equity and efficiency criteria slope upward to the right, and that the bicriteria plot has a quite complicated shape. Plotting the image of any proposed location on this bicriteria plot provides an explicit representation of the simultaneous gains and losses that may be obtained for both objectives within the feasible region.

3.4 Determination of the Pareto-sets

3.4.1 Attractive facility

Consider the following single objective minimization problem for any given $0 \leq \theta \leq 1$:

$$\min_{\mathbf{x} \in \Omega} \theta F(\mathbf{x}) + (1-\theta) G(\mathbf{x}). \tag{10}$$

As we have shown, $F(\mathbf{x})$ is convex and $G(\mathbf{x})$ is strictly convex, so the convex minimization problem (10) possesses a unique solution for any $0 \leq \theta < 1$. The uniqueness implies that this solution is Pareto optimal: see, e.g. Miettinen(1999) p79. For $\theta = 1$, problem (10) may have a set of minimizers for $F(\mathbf{x})$, but only one of these will be a Pareto solution: the point, denoted by $\bar{\mathbf{d}}^*$, of this set nearest to $\mathbf{m}^*$.

Therefore, the Pareto set E^*_+ is given by a simple continuous curve connecting $\bar{\mathbf{d}}^*$ and $\mathbf{m}^*$, as discussed in Hansen et al.(1981b) and Ohsawa(1999). Combining this result with Proposition 6 leads to the following algorithm

Algorithm 3

Step 1. Construct the planar graph $\partial V \cup \partial\Omega \cup L$.

Step 2. Identify $\mathbf{m}^*$.

Step 3. Find all the nodes of the graph $\partial V \cup \partial\Omega$ minimizing $F(\mathbf{x})$.
Among these nodes determine $\bar{\mathbf{d}}^*$ as the one closest to $\mathbf{m}^*$.

Step 4. Find the steepest descent path of $F(\mathbf{x})$ from $\mathbf{m}^*$ to $\bar{\mathbf{d}}^*$ on the planar graph.

It follows

Proposition 9 *The Pareto-set* E^*_+ *can be found in* $O((|I|^5 + |I||\partial\Omega|)$ *time.*

Proof As we have seen in Step 1 of Algorithm 2, Step 1 takes $O(|I|^5 + |I||\partial\Omega|)$ time. If the center of gravity $\overline{\mathbf{p}}$, found in $O(|I|)$ is outside Ω, then $\mathbf{m}^*$ is the point of Ω closest to $\overline{\mathbf{p}}$, which is found in $O(|\partial\Omega|)$. Hence, Step 2 takes at most $O(|I| + |\partial\Omega|)$ time. Since the graph $\partial V \cup \partial\Omega$ has $O(|I|^4 + |\partial\Omega|)$ vertices, Step 3 can be performed in $O(|I|^5 + |I||\partial\Omega|)$ time, as in Algorithm 1. Since the slope of $F(\mathbf{x})$ on any edge of the graph can be determined in $O(|I|)$ via (3), and the graph $\partial V \cup \partial\Omega \cup L$ has $O(|I|^4 + |\partial\Omega|)$ edges, the complexity of Step 4 is at most $O(|I|^5 + |I||\partial\Omega|)$. ∎

3.4.2 Obnoxious facility

It follows from Proposition 5 that the tradeoff curve of the obnoxious facility problem coincides with the upper-left envelope of $(F, G)(\partial V \cup \partial\Omega)$. Hence, the Pareto-set E^*_- can be identified by the following algorithm, which generalizes slightly the method by Ohsawa(2000).

Algorithm 4

Step 1. Build the planar graph $\partial V \cup \partial\Omega$.

Step 2. Plot the parabolic loci of $(F, G)(\partial V \cup \partial\Omega)$ in objective space.

Step 3. Find the upper-left envelope of the loci.

Step 4. Determine the subedges corresponding to the envelope in the geographical space.

Hence we have

Proposition 10 *The Pareto-set E^*_- can be determined in $O((|I|^4 + |\partial\Omega|)(|I| + \log|\partial\Omega|))$ time.*

Proof As we have shown in Algorithm 1, Step 1 has a time complexity of $O(|I|^4 + |\partial\Omega|)$. Step 2 takes $O(|I|^5 + |I||\partial\Omega|)$ time using (3). One parabolic locus corresponding to one edge within the planar graph intersects any other parabolic locus in at most two points. Since the graph has $O(|I|^4 + |\partial\Omega|)$ edges, the envelope can be defined in $O((|I|^4 + |\partial\Omega|)\log(|I|^4 + |\partial\Omega|))$, as shown in Boissonnat and Yvinec(1998). Thus, Step 3 requires $O((|I|^4 + |\partial\Omega|)\log(|I|^4 + |\partial\Omega|))$. Step 4 can be accomplished in $O(|I|^4 + |\partial\Omega|)$. Finally $O(|I|^5 + |I||\partial\Omega|) + O((|I|^4 + |\partial\Omega|)\log(|I|^4 + |\partial\Omega|)) = O((|I|^4 + |\partial\Omega|)(|I| + \log|\partial\Omega|))$. ∎

In a similar way, but based on Proposition 6, the Pareto-set E^*_+ can also be identified by tracing out the lower-left envelope of $(F,G)(\partial V \cup \partial\Omega \cup L)$ in $O((|I|^4 + |\partial\Omega|)(|I| + \log|\partial\Omega|))$ time, which is larger than $O((|I|^5 + |I||\partial\Omega|)$ in Proposition 9.

The tradeoff curves $(F,G)(E^*_+)$ and $(F,G)(E^*_-)$ are visualized collectively in Figure 4 by thick curves. The tradeoff curve $(F,G)(E^*_+)$ consists of a continuous portion, but the other tradeoff curve $(F,G)(E^*_-)$ has three connected components.

Figure 4 enables us to perceive clearly and quickly to what extent the Pareto-solutions are really better than other candidates within Ω by comparing the bicriteria plot $(F,G)(\Omega)$ corresponding to all alternatives. Thus, our solution supports well-informed decision making.

The Pareto-sets E^*_+ and E^*_- are traced out in Figure 3 by using the corresponding lines in Figure 4, respectively. The Pareto-set E^*_+ is a connected piecewise linear curve from $\mathbf{m}^*$ to $\bar{\mathbf{d}}^*$, as was to be expected. The other Pareto-set E^*_- is composed of a piecewise linear curve joining $\bar{\mathbf{d}}^*$ with $\mathbf{s}_1$, a second connecting $\mathbf{s}_2$ with $\mathbf{s}_3$ and a third connecting $\mathbf{s}_4$ with $\mathbf{a}^*$. Thus, the Pareto-optimal solutions for a repellent facility E^*_-, are more spread out over the feasible region than for an attractive facility E^*_+. This result is intuitively reasonable, because the Pareto-optimal locations for a repellent facility directly depend on the shape of the feasible region Ω.

4 Concluding remarks

This paper has formulated a new location problem using an equity measure of facility-inhabitant distances which satisfies the Pigou-Dalton condition of transfers. On the one hand, we have characterized the optimal solutions for the single-objective equity problem, the Pareto-sets and the bicriteria plot associated with the problem which combines this equity objective with Weber-type objectives. On the other hand, we have presented polynomial-time graphical methods to trace out the solutions, the Pareto-sets, and the bicriteria plot by using line tessellations.

Some further remarks on extensions of our formulation and results may be made.

Let us first address the shape of the feasible region Ω, and see if the assumption of being a bounded convex polygonal region may be lifted. It is well known that any finite union of non-convex polygons can be partitioned into convex subareas in polynomial time: see, e.g. Hertel and Mehlhorn (1983). Therefore, the solution, the bicriteria plot, and the Pareto-sets can be obtained in polynomial time by separate treatment of each convex subpolygon and merging of the results. Lifting the polygonality assumption is not directly possible, since this assumption was extensively used in our arguments, in particular when the precise shape of the bicriteria plot of line segments is needed in the final construction. However, with the advent of GIS-based spatial data, polygonality may often be considered as given.

Another point arises in our choice of equity and efficiency objectives. Evidently any of these may be replaced by any increasing function of the same objectives, without affecting the Pareto-sets we constructed here. It should be possible to adapt the efficiency objective in other ways too. The arguments used in Propositions 5 and 6 are sufficiently general to remain applicable for any other *convex* efficiency objective, like minimizing the sum of distances, or minimizing the maximum distance, etc. Similar localization results will then still hold. But they will again be harder to exploit algorithmically, since the nice parabolic shapes we have been using here will be lost.

Replacing the equity objective (sum of pairwise absolute deviations) used here by some other one, like the variance or the Gini coefficient and/or replacing squared Euclidean distances by other distance measures, will again invalidate many of our results, in which the linearity of the equity objective was fundamental. How to handle such cases remains open for further research.

Acknowledgment

This research was partially funded by the Kajima Foundation's Research Grant (2003). The authors are grateful to Masahiro Hachimori, Atsushi Masuyama and anonymous reviewers.

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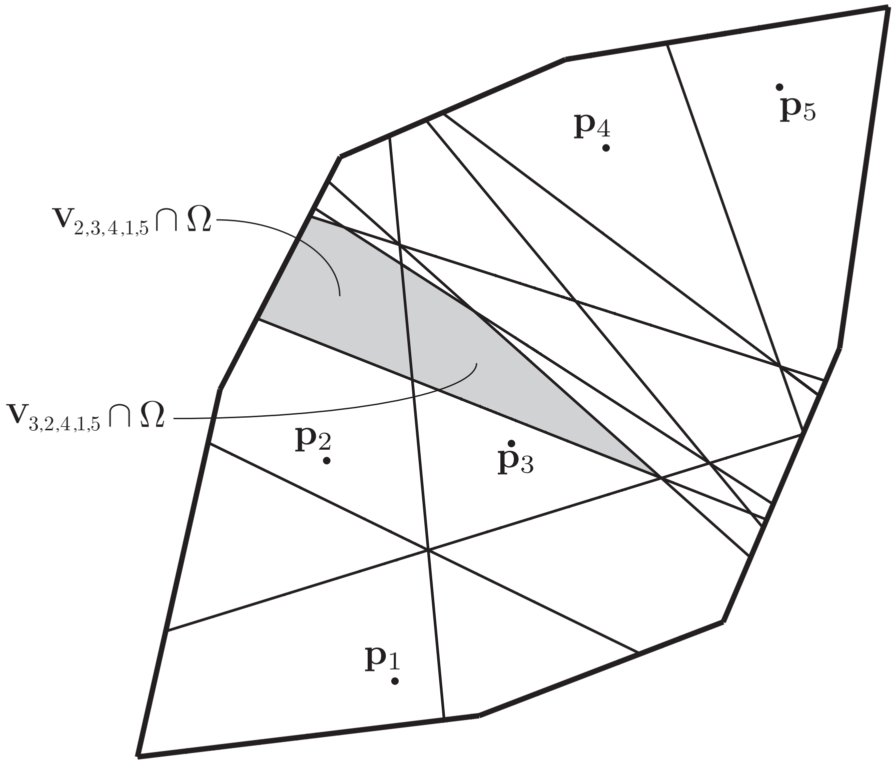

Figure 1: Line tessellation

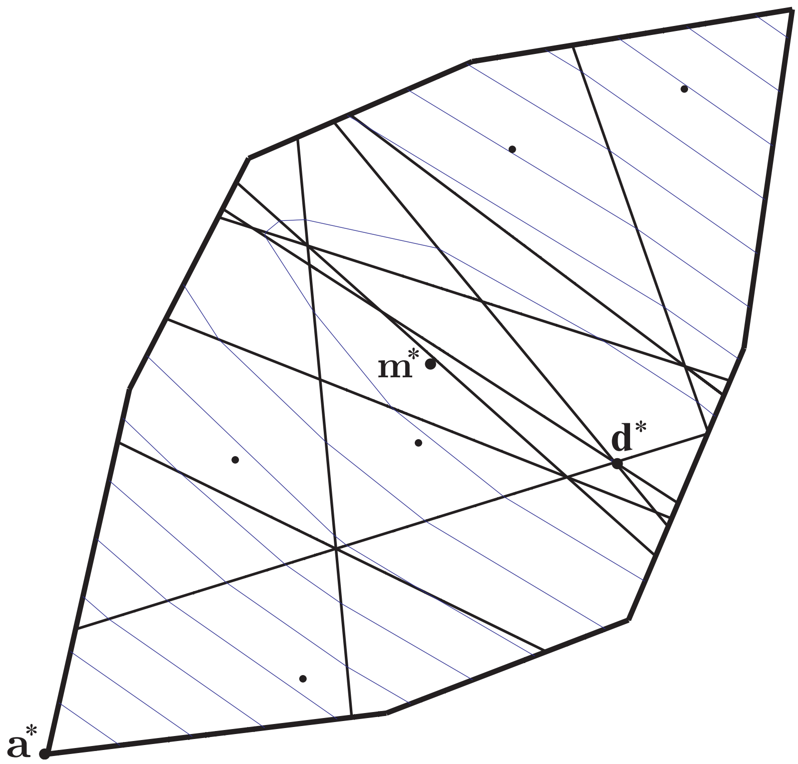

Figure 2: Level sets of $F(\mathbf{x})$ and solutions $\mathbf{a}^*$, $\mathbf{m}^*$, $\mathbf{d}^*$

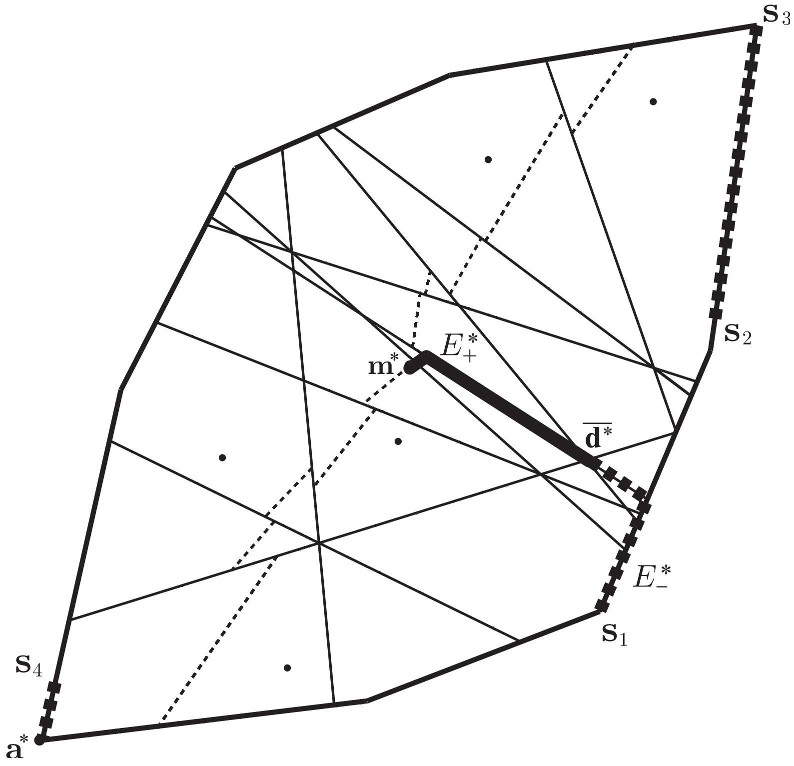

Figure 3: Pareto sets E_-^* and E_+^*

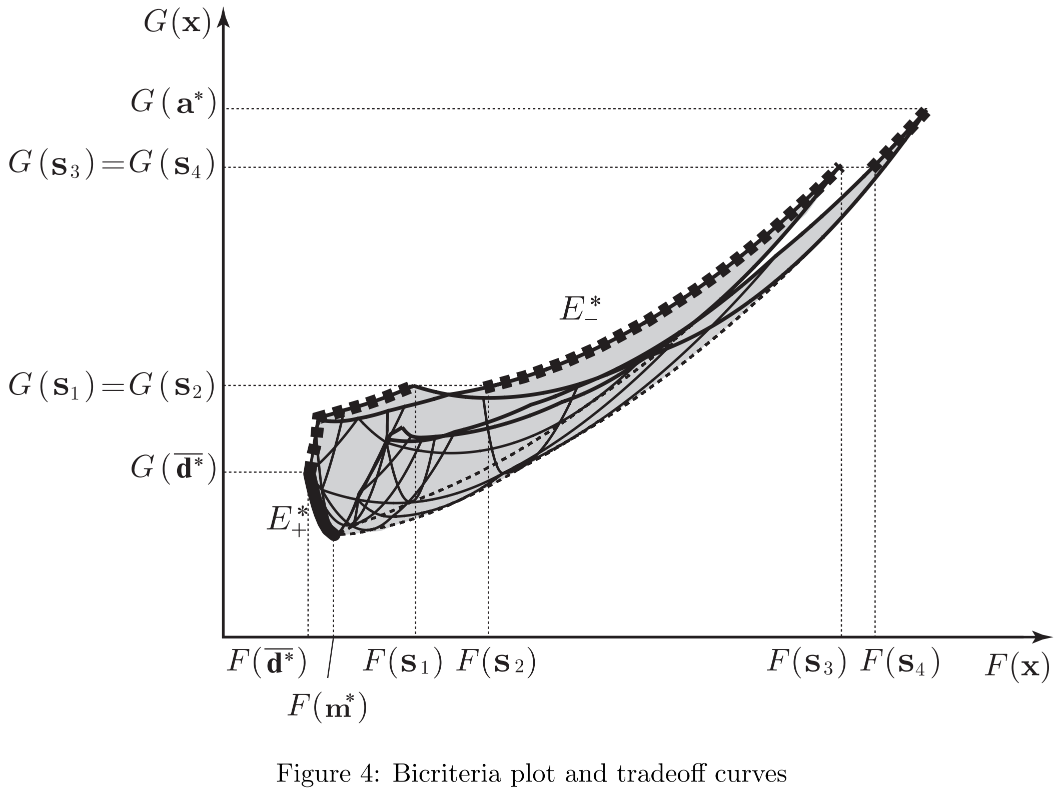

Figure 4: Bicriteria plot and tradeoff curves