Formal groups and unipotent affine groups in non-categorical symmetry ⋆

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Abstract

As is well known, in characteristic zero, the Lie algebra functor gives two category equivalences, one from the formal groups to the finite-dimensional Lie algebras, and the other from the unipotent algebraic affine groups to the finite-dimensional nilpotent Lie algebras. We prove these category equivalences in a quite generalized framework, proposed by Gurevich [Gu] and later by Takeuchi [T], of vector spaces with non-categorical symmetry. We remove the finiteness restriction from the objects, by using the terms of Hopf algebras and Lie coalgebras.

Introduction

To the author's best knowledge, Gurevich [Gu] was the first to propose a non-categorical approach to braided objects. A complete framework, which we will follow below, for the approach was given by Takeuchi [T]. Throughout we work over a fixed field k. We refer as a *τ-space* to a vector space V given a linear isomorphism $\tau : V \otimes V \xrightarrow{\simeq} V \otimes V$ satisfying the Yang-Baxter equation. The τ is said to be *symmetric* provided $\tau \circ \tau = \mathrm{id}$. Braided structures on a τ-space can be defined non-categorically, just as is done in a braided category, by using τ as a braiding. Braided objects thus defined are called with τ prefixed, so as *τ-bialgebra*, *τ-Hopf algebra*, etc. If τ is symmetric, a τ-Lie (co)algebra as well as τ-(co)commutativity is defined in the obvious way. As main examples of those τ-objects we have in mind objects in such a k-abelian braided (or symmetric) tensor category that is given a faithful exact, tensor-product preserving k-linear functor to the category of vector spaces; a typical example of such a symmetric category is that of super-vector spaces. But, compared with the categorical framework, the non-categorical one is more general, and

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even more convenient especially when we apply, as we will do, operations such as completion and dualization; it is also more flexible as was realized by Andruskiewitsch and Schneider [AS] in their study of Nichols algebras. Generalizing the notion of τ-objects, we will define *complete (topological) τ-objects*, in Section 3.

Suppose that the characteristic $\operatorname{ch} k = 0$, and the τ of any τ-space is symmetric, unless otherwise stated. Recently, Kharchenko [K] generalized Kostant's Theorem [S, Thm. 3.10; Ko, Prop. 3.2] to τ-objects, proving that $L \mapsto U(L)$ (the universal envelope) gives a category equivalence

$$(\tau\text{-Lie Algebras}) \approx \begin{pmatrix}\text{Irreducible } \tau\text{-Cocommutative} \\ \text{Hopf Algebras}\end{pmatrix},$$

see Theorem 1.1. Our objective is to generalize two classical category equivalences,

$$(\text{Formal Groups}) \approx \begin{pmatrix}\text{Finite-Dimensional} \\ \text{Lie Algebras}\end{pmatrix},$$
$$\begin{pmatrix}\text{Unipotent Algebraic} \\ \text{Affine Groups}\end{pmatrix} \approx \begin{pmatrix}\text{Finite-Dimensional Nilpotent} \\ \text{Lie Algebras}\end{pmatrix},$$

which are both dual to Kostant's Theorem, and given by the Lie algebra functor; see [Se, Part I, Chap. V, Sect. 6, Thm. 3], [DG, IV, Sect. 2, 4.5] and [H, XVI, Thm. 4.2] (The referee kindly informed that Cartier [C] was probably the first to prove the results). We will remove the finiteness restriction from the objects, by using the terms of Hopf algebras and Lie coalgebras as dual objects of groups and Lie algebras, respectively. As a result, finite-dimensional (nilpotent) Lie algebras, formal groups and unipotent algebraic affine groups will be replaced by (locally nilpotent) τ-Lie coalgebras (Definition 2.1), complete τ-commutative Hopf algebras (Definition 3.2) and irreducible τ-commutative Hopf algebras, respectively. Our main results are thus two category equivalences,

$$\begin{pmatrix}\text{Complete } \tau\text{-Commutative} \\ \text{Hopf Algebras}\end{pmatrix} \approx (\tau\text{-Lie Coalgebras}),$$
$$\begin{pmatrix}\text{Irreducible } \tau\text{-Commutative} \\ \text{Hopf Algebras}\end{pmatrix} \approx \begin{pmatrix}\text{Locally Nilpotent} \\ \tau\text{-Lie Coalgebras}\end{pmatrix},$$

see Theorems 4.4 and 6.7. The first equivalence restricts to an equivalence between the finite-type objects (see Corollary 4.5), which was formulated by Gurevich [Gu, Thm. 2] without detailed proof. The second was proved by Nichols [N, Thms. 12, 14] in the ordinary situation, on which paper we largely depend; see also [MO, Thm. 3.2]. Just as in the ordinary situation, a discrete (resp., complete) τ-commutative Hopf algebra H can represent a group-functor, $\mathrm{Sp}\, H$ (resp., Spf H), which may be called a *τ-affine group* (resp., *τ-formal group*). But, we will take very little this group-functorial view-point, since the group of 'rational points', including 'non-categorical' ones, is somewhat disordered. We remark also that in contrast with ordinary commutative Hopf algebras, a τ-commutative Hopf algebra is not necessarily a directed union of finitely generated τ-Hopf subalgebras.

Given a τ-space V, let $S(V)$ denote the τ-symmetric algebra, and let $\hat{S}(V)$ denote its completion; see Example 4.2. The proofs of the two main theorems are parallel, and divided into two steps. As the first step, given a complete (resp., irreducible) τ-commutative Hopf algebra H, we prove a canonical isomorphism $H \simeq \hat{S}(Q)$ (resp., $H \simeq S(Q)$) of τ-algebras, where Q denotes the (locally nilpotent) τ-Lie coalgebra corresponding to H; see Propositions 4.8 and 6.8. As the second step, given an arbitrary (resp., a locally nilpotent) τ-Lie coalgebra Q, we make $\hat{S}(Q)$ (resp., $S(Q)$) into a complete (resp., an irreducible) τ-commutative Hopf algebra, denoted by $\hat{H}(Q)$ (resp., $H(Q)$), by giving a coproduct according to the structure of Q; see Proposition 4.11 and Lemma 6.9. For this the Campbell-Hausdorff formal power series (see (7)) play an essential role. Notice that $\hat{S}(V)$ or $S(V)$ has the trivial coproduct for which each element in V is primitive. Therefore we can say that our theorems determine all possible deformations of the coproduct on $\hat{S}(V)$ or $S(V)$.

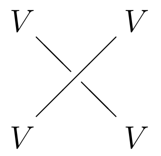
Among our proofs, the most difficult is probably to prove the isomorphism $H \simeq S(Q)$, where H is an irreducible τ-commutative Hopf algebra. This is derived from the following unipotent-group-like property of such H: if $K \subsetneqq H$ is a proper τ-Hopf subalgebra, H includes a τ-Hopf subalgebra $\tilde{K}$ properly including K, such that $K \hookrightarrow \tilde{K} \twoheadrightarrow \tilde{K}/K^+\tilde{K}$ is a τ-cocentral extension; see Proposition 6.1. The property is proved by using a cleftness result, Proposition 5.1, on τ-Hopf algebras in general, for which the τ may not be symmetric.

Let L be a τ-Lie algebra. Dualizing the construction of $\hat{H}(Q), H(Q)$ above, it is possible to deform the product on the τ-Hopf algebra $B(L)$, which is by definition the largest τ-cocommutative subcoalgebra in the τ-shuffle Hopf algebra $\mathrm{Sh}(L)$; see Proposition 7.1. It is easy to see that the resulting τ-Hopf algebra $B(L)$ is canonically isomorphic to $U(L)$. This gives an alternative proof of the crucial part of Kharchenko's Theorem; see Remark 7.2.

In a recent, interesting paper [AMS], Ardizzoni et al. generalize the notion of τ-Lie algebras as well as Kharchenko's Theorem, for those τ-objects in which the τ satisfies the equation $(\tau - 1)(\tau + q) = 0$ with q generic. We remark that in their generalized context, the first step of our proofs goes well whereas the second step, that is, the construction of $\hat{H}(Q), H(Q)$ fails at the moment; see Remark 7.6. This failure seems to arise by the same reason as the lack of non-trivial examples of the *braided Lie algebras* as defined in [AMS].

1 Basics on discrete τ-objects

We work over a fixed field k. The characteristic $\operatorname{ch} k$ may be arbitrary, though we will often suppose $\operatorname{ch} k = 0$. A *τ-space* is a vector space V given a linear isomorphism $\tau : V \otimes V \xrightarrow{\simeq} V \otimes V$, denoted by the crossing



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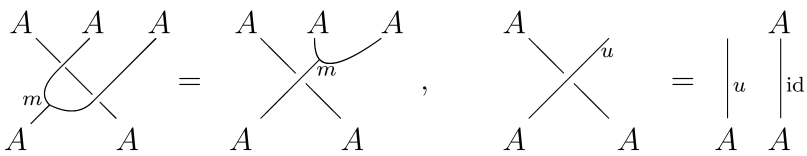
which satisfies the Yang-Baxter equation; we will use for simplicity this short term taken from [K], though this was called a *YB-space* by Takeuchi [T]. If a τ-space V is finite-dimensional, the dual vector space V^* is a τ-space with the dual τ^* of the original τ. A *τ-algebra* is a τ-space and at the same time algebra A whose structure maps $m : A \otimes A \to A$, $u : k \to A$ *pass through* A; this means that by diagrams,



and the mirror-image equations hold true. Notice that the crossing $A \otimes k \to k \otimes A$ is supposed to be trivial. A *τ-coalgebra* is defined dually; the coalgebra structure will be denoted by $\Delta : C \to C \otimes C$, $\varepsilon : C \to k$, as usual. A τ-bialgebra and other τ-objects as well, are defined by using τ as a braiding, just as is in a braided category. A *τ-Hopf algebra* is a τ-bialgebra which has an antipode $S : H \to H$, i.e., a convolution-inverse of the identity map. As important facts, S necessarily passes through H [T, Prop. 5.5], and it is a τ-anti-bialgebra map; see [Mj, Lemma 14.4], [T, (4.4)]. If a τ-bialgebra is irreducible as a coalgebra, it is necessarily a τ-Hopf algebra, by [S, Thm. 9.2.2]. A morphism between two τ-objects of the same kind is a τ-preserving linear map which commutes with the structures. For example a τ-algebra map is precisely a τ-preserving algebra map.

A subspace W of a τ-space V is said to be *categorical* [T, Sect. 6], if $\tau(V \otimes W) = W \otimes V$, $\tau(W \otimes V) = V \otimes W$. This is equivalent to saying that τ induces $V \otimes V/W \xrightarrow{\simeq} V/W \otimes V$, $V/W \otimes V \xrightarrow{\simeq} V \otimes V/W$. A quotient space of V of the form V/W, where W is categorical, is said to be *categorical*. In this case we say that $W \hookrightarrow V \twoheadrightarrow V/W$ is a *short exact sequence of τ-spaces*. Notice that W, V/W are τ-spaces by themselves.

Sub- and quotient objects of any τ-object are supposed to be categorical, and will be called with τ prefixed. For example a *τ-subspace* is a categorical subspace, and a τ-*ideal* of a τ-algebra is such an ideal that is a categorical subspace.

In this paper graded objects will play an important role. The gradation will be always by the non-negative integers $0, 1, 2, \ldots$. A *graded τ-space* is a graded vector space $V = \bigoplus_{n\geq 0} V(n)$ which is a τ-space such that $\tau(V(m) \otimes V(n)) = V(n) \otimes V(m)$ for all m, n. Structures in a graded τ-object are, of course, required to preserve the gradation.

In what follows, except in Section 3 and 5, we suppose that the τ of any τ-space is *symmetric* in the sense $\tau \circ \tau = \mathrm{id}$. Then a τ-(co)algebra is said to be τ-*(co)commutative* if $m = m \circ \tau$ ($\Delta = \tau \circ \Delta$). The tensor product of two copies of a τ-(co)algebra is naturally a τ-(co)algebra, and the structure maps of a τ-(co)commutative (co)algebra are τ-(co)algebra maps. Hence just as in the ordinary situation, a τ-(co)commutative Hopf algebra represents a group-functor on the category of τ-(co)commutative (co)algebras. It results that its antipode is involutory. But, we will not take this group-functorial view-point elsewhere.

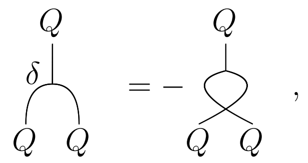
Given a τ-space V, the tensor algebra $T(V)$ forms a graded τ-Hopf algebra, in which every element in V is primitive, and $\tau : V^{\otimes m} \otimes V^{\otimes n} \to V^{\otimes n} \otimes V^{\otimes m}$ is constructed iteratedly from the τ of V. A τ-*Lie algebra* is a τ-space L given a bracket $[\ ,\ ] : L \otimes L \to L$ which passes through L, and satisfies the braided anti-symmetry and the braided Jacobi identity; see [K, (21)]. The last two conditions are represented by the inverted images of the diagrams (1), (2), below. In a τ-Hopf algebra H, the primitives form a τ-subspace, denoted by $P(H)$, which is in fact a τ-Lie algebra with respect to the braided commutator. The *universal envelope* $U(L)$ of a τ-Lie algebra L is the quotient τ-Hopf algebra of $T(L)$ divided by the ideal generated by $x \otimes y - \tau(x \otimes y) - [x, y]$, where $x, y \in L$; see [K, (22)]. This has the obvious universal mapping property. The τ-Hopf algebras $T(V)$, $U(L)$ are τ-cocommutative and irreducible. If L is abelian (i.e., $[\ ,\ ] = 0$), then $U(L)$ is a quotient graded τ-Hopf algebra of $T(L)$, and is denoted by $S(L)$.

Theorem 1.1 (Kharchenko [K, Thm. 6.1]) *Suppose that the characteristic* $\mathrm{ch}\, k = 0$*. Then,* $L \mapsto U(L)$ *gives an equivalence from the category of τ-Lie algebras to the category of irreducible τ-cocommutative Hopf algebras.*

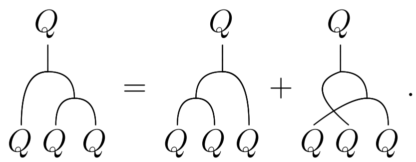
It is proved in [K] that $U \mapsto P(U)$ gives a quasi-inverse, and the canonical map $L \to U(L)$ gives an isomorphism $L \xrightarrow{\simeq} P(U(L))$ of τ-Lie algebras. It is also pointed out in [K, Sect. 7] that the PBW basis fails to exist, by giving the following example: if V is a τ-space given as τ the identity map id of $V \otimes V$, then $S(V) = T(V)$. We add: if $\tau = -\mathrm{id}$, then $S(V) = k \oplus V$, in which $xy = 0$ for $x, y \in V$. As is seen from these examples with V infinite-dimensional, a τ-commutative Hopf algebra is not necessarily a directed union of finitely generated τ-Hopf subalgebras. (Indeed, each non-zero element in V generates a τ-Hopf algebra, but it is not a categorical subspace of $S(V)$, and hence not a τ-Hopf subalgebra.)

2 τ-Lie coalgebras

As the dual notion of τ-Lie algebras, a *τ-Lie coalgebra* is defined to be a τ-space Q given such a linear map $\delta : Q \to Q \otimes Q$, called a *cobracket*, that passes through Q, and satisfies the axioms dual to the braided anti-symmetry and the braided Jacobi identity. By diagrams the axioms are represented by:



(1)



(2)

Here we have leveled the crossings since the τ is supposed to be symmetric.

Let $Q = (Q, \delta)$ be a τ-Lie coalgebra. If Q is finite-dimensional, the dual vector space Q^* is naturally a τ-Lie algebra. If $R \subset Q$ is a τ-Lie subcoalgebra, then Q/R is a quotient τ-Lie coalgebra of Q, so that we have a short exact sequence of τ-Lie coalgebras, $R \hookrightarrow Q \twoheadrightarrow Q/R$.

For $n \geq 0$, let $\delta^n : Q \to Q^{\otimes(n+1)}$ denote the n times iterated cobracket

$$\delta^n = (\mathrm{id}^{\otimes n} \otimes \delta) \circ \cdots \circ (\mathrm{id} \otimes \delta) \circ \delta,$$

and define

$$Q_n := \mathrm{Ker}(\delta^n : Q \to Q^{\otimes(n+1)}). \tag{3}$$

This is a τ-Lie subcoalgebra of Q. We have $0 = Q_0 \subset Q_1 \subset Q_2 \subset \cdots$. In Q, Q_n is the largest τ-Lie subcoalgebra including Q_{n-1} such that the Q-coaction $Q_n/Q_{n-1} \to Q \otimes Q_n/Q_{n-1}$ induced by δ is zero.

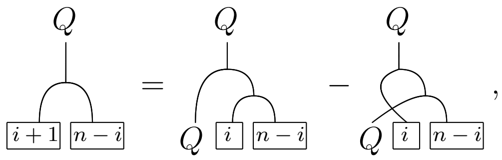
Definition 2.1 ([N, p. 75]) *Q is said to be* locally nilpotent*, if each element vanishes under δ^n for some n, or equivalently if $Q = \bigcup_{n>0} Q_n$.*

Lemma 2.2 *$\delta(Q_n) \subset \sum_{i=1}^{n} Q_i \otimes Q_{n-i}$ ($n = 1, 2, \dots$).*

PROOF. Let $Q_{-1} = 0$. We first prove

$$\delta(Q_n) \subset Q_i \otimes Q + Q \otimes Q_{n-1-i} \quad (0 \leq i \leq n) \tag{4}$$

by induction on i. When $i = 0$, the desired result $\delta(Q_n) \subset Q \otimes Q_{n-1}$ is easy to see. When $i > 0$, notice from (2) that



where $\boxed{j}$ indicates $\delta^{j-1} : Q \to Q^{\otimes j}$. We then see that the desired (4) in $i + 1$ follows from the result in i. By (4) and [S, Lemma 9.1.5], we see

$$\delta(Q) \subset \bigcap_{i=0}^{n} (Q_i \otimes Q + Q \otimes Q_{n-1-i}) = \sum_{i=1}^{n} Q_i \otimes Q_{n-i}.$$

$\square$

Corollary 2.3 *The positively graded τ-space*

$$\Gamma Q := \bigoplus_{n>0} Q_n / Q_{n+1}$$

naturally forms a graded τ-Lie coalgebra.

PROOF. This follows from Lemma 2.2, just as the corresponding result [S, p. 228] for coalgebras follows from [S, Cor. 9.1.7]. $\square$

Definition 2.4 *Let $\bar{Q}$ be a quotient τ-Lie coalgebra of Q. The quotient map $\pi : Q \to \bar{Q}$ is said to be τ-*cocentral *if $(id \otimes \pi) \circ \delta = 0$, or equivalently if $(\pi \otimes \mathrm{id}) \circ \delta = 0$.*

In this case the cobracket of $\bar{Q}$ is necessarily zero. The following is easy to prove.

Lemma 2.5 *If $\pi : Q \to \bar{Q}$ is τ-cocentral, then the kernel $R := \mathrm{Ker}\,\pi$ is a τ-Lie subcoalgebra of Q. If R is locally nilpotent, Q is, too.*

3 Complete τ-objects

By a *topological vector space* we mean a vector space V which is linearly topologized by such a basis of neighbourhood of 0, that consists of filtered subspaces $V = V_0 \supset V_1 \supset V_2 \supset \cdots$; such a basis $(V_n)_n$ will be called a *defining filtration*. The topological vector spaces in our consideration will be all *complete* in the sense that $V \simeq \varprojlim V/V_n$; such V will be called just a *complete vector space*. Given another complete vector space W with a defining filtration $(W_n)_n$, the *complete tensor product* is defined by $V\hat{\otimes}W = \varprojlim V/V_n \otimes W/W_n$. This is a complete vector space with a defining filtration $(V_n\hat{\otimes}W + V\hat{\otimes}W_n)_n$; we remark that $V_n\hat{\otimes}W = \varprojlim_{m\geq n} V_n/V_m \otimes W/W_m$. We regard k as discrete. Notice that $V\hat{\otimes}k = V$, since V is complete. The complete vector spaces thus form a tensor category.

A *complete τ-space* is a complete vector space V given a linear isomorphism $\tau : V\hat{\otimes}V \xrightarrow{\simeq} V\hat{\otimes}V$, such that (1) τ satisfies the Yang-Baxter equation, and (2) V has a defining filtration $(V_n)_n$ consisting of τ-subspaces in the sense $\tau(V_n\hat{\otimes}V) = V\hat{\otimes}V_n$, $\tau(V\hat{\otimes}V_n) = V_n\hat{\otimes}V$. Consequently, (3) each V/V_n is a discrete τ-space, (4) $V/V_{n+1} \to V/V_n$ is a quotient τ-space, and (5) the τ of V is the projective limit of the τ of V/V_n $(n \geq 0)$, and is hence a homeomorphism. We do not here assume that $\tau \circ \tau = \mathrm{id}$. If a complete vector space V with a defining filtration $(V_n)_n$ satisfies (3), (4), then the projective limit of the τ of V/V_n $(n \geq 0)$ makes V into a complete τ-space.

A complete τ-objects are defined just as in the discrete situation, but by replacing $\otimes$ with $\hat{\otimes}$. A *complete τ-(co)algebra* is thus a complete τ-space given such continuous structure maps that satisfy the (co)algebra axiom, and pass through itself.

Example 3.1 *The direct product $\prod_{n\geq 0} V(n)$ of τ-spaces $V(n)$ is a complete τ-space with a defining filtration $(\prod_{m\geq n} V(m))_n$. This is the completion of the graded τ-space $\bigoplus_{n\geq 0} V(n)$, and will be called a* complete graded τ-space. *Similarly, the completion $\prod_{n\geq 0} A(n)$ of a graded τ-algebra $\bigoplus_{n\geq 0} A(n)$ will be called a* complete graded τ-algebra.

A complete τ-bialgebra is defined as follows in a more restricted sense than might be guessed.

Definition 3.2 *A* complete τ-bialgebra *is a complete τ-algebra and τ-coalgebra H, such that (1) each power I^n $(n > 0)$ of the augmentation ideal $I := H^+$ is a τ-subspace of H, (2) $H \supset I \supset I^2 \supset \cdots$ gives a defining filtration, and (3) the coalgebra structures $\Delta : H \to H\hat{\otimes}H$, $\varepsilon : H \to k$ are algebra maps. The conditions (1), (2) means that $H = \varprojlim H/I^n$ as a complete τ-algebra.*

Let C (resp., A) be a complete coalgebra (resp., algebra); we may here forget τ, or may suppose that τ is trivial. Let

$$\mathrm{Hom_c}(C, A) \tag{5}$$

denote the vector space consisting of all continuous linear maps $C \to A$. The following is easy to prove.

Lemma 3.3 $\mathrm{Hom}_c(C, A)$ *is a complete algebra with respect to the convolution product,* $fg = m \circ (f \hat{\otimes} g) \circ \Delta$, *and the defining filtration* $(\mathrm{Hom}_c(C, A_n))_n$, *where* $(A_n)_n$ *is a defining filtration of* A*; the topology thus defined on* $\mathrm{Hom}_c(C, A)$ *is independent of choice of* $(A_n)_n$.

Proposition 3.4 *Every complete* τ*-bialgebra* H *has an antipode* S*, i.e., an inverse of the identity map* id *in* $\mathrm{Hom}_c(H, H)$*; it may be therefore called a complete* τ*-Hopf algebra. Necessarily,* S *passes through* H*, and is a* τ*-anti-bialgebra map.*

PROOF. Let $I := H^+$. Notice that

$$\mathrm{Hom}_c(H, H) = \varprojlim \mathrm{Hom}_c(H, H/I^n).$$

For the first assertion it is enough to prove that for each n, the natural image of id in $\mathrm{Hom}_c(H, H/I^n)$ is invertible. This is obvious when $n = 1$. The image in $\mathrm{Hom}_c(H, H/I^n)$ is then invertible, since it is so modulo the nilpotent ideal $\mathrm{Hom}_c(H, I/I^n)$. The second assertion follows in the same way as in the discrete situation; see [T, Prop. 5.5] and [Mj, Lemma 14.4]. □

Let H be a complete τ-commutative Hopf algebra, i.e., such a complete τ-Hopf algebra that is τ-commutative, assuming that $\tau \circ \tau = \mathrm{id}$. Such H represents a group-functor on the category of complete τ-commutative algebras. It results that the antipode of H is necessarily involutory.

4 Complete τ-commutative Hopf algebras

In this section we suppose that the τ of any τ-space is symmetric. We will soon add the assumption that $\mathrm{ch}\, k = 0$.

Let H be a complete τ-Hopf algebra, and write $I := H^+$. Define

$$Q = Q(H) := I/I^2.$$

This is a discrete sub-quotient τ-space of H.

Lemma 4.1 *The* τ*-preserving linear map* $(\mathrm{id} - \tau) \circ \Delta : H \to H \hat{\otimes} H$ *induces such a map* $\delta : Q \to Q \otimes Q$*, by which* Q *forms a* τ*-Lie coalgebra.*

PROOF. If $a \in I$, then $\Delta(a) - 1\hat{\otimes}a - a\hat{\otimes}1$ is in $I\hat{\otimes}I$. This implies that $D := (\mathrm{id} - \tau) \circ \Delta$ maps I into $I\hat{\otimes}I$, and I^2 into $I\hat{\otimes}I^2 + I^2\hat{\otimes}I$, whence such δ as above is induced. By using braid diagrams we see that (H, D) satisfies the τ-Lie coalgebra axioms (1), (2), with $\otimes$ replaced by $\hat{\otimes}$. This implies that (Q, δ) is a τ-Lie coalgebra. $\square$

We see that $H \mapsto Q(H)$ is functorial.

Example 4.2 *Let V be a τ-space. Recall from Section 1 that $S(V) = \bigoplus_{n\geq 0} S^n(V)$ is a (discrete) graded τ-commutative Hopf algebra in which $S^1(V) = V$ consists of primitives. The completion*

$$\hat{S}(V) := \prod_{n\geq 0} S^n(V)$$

of $S(V)$ is a complete τ-commutative Hopf algebra. We see that $Q(\hat{S}(V)) = V$ with zero cobracket.

Let H be a complete τ-Hopf algebra with $I := H^+$, and define

$$\mathrm{gr}\, H := \bigoplus_{n\geq 0} I^n/I^{n+1}.$$

This is a graded τ-algebra with $\mathrm{gr}^0 H = k$, $\mathrm{gr}^1 H = Q(H)$. The topology on $H\hat{\otimes}H$ can be defined by the filtration $(\sum_{i+j=n} I^i\hat{\otimes}I^j)_n$. The structure maps $\Delta : H \to H\hat{\otimes}H$, $\varepsilon : H \to k$, $S : H \to H$ then preserve the defining filtration consisting of τ-ideals, so that they induce graded τ-algebra or τ-anti-algebra maps, $\bar{\Delta} : \mathrm{gr}\, H \to \mathrm{gr}\, H \otimes \mathrm{gr}\, H$, $\bar{\varepsilon} : \mathrm{gr}\, H \to k$, $\bar{S} : \mathrm{gr}\, H \to \mathrm{gr}\, H$. We see easily the following.

Lemma 4.3 $\mathrm{gr}\, H$ *together with $\bar{\Delta}, \bar{\varepsilon}, \bar{S}$ forms a graded τ-Hopf algebra, in which* $\mathrm{gr}^1 H = Q(H)$ *consists of primitives, and generates the whole* $\mathrm{gr}\, H$.

In what follows in this section we suppose that $\mathrm{ch}\, k = 0$ and aim to prove the following.

Theorem 4.4 *$H \mapsto Q(H)$ gives an equivalence from the category of complete τ-commutative Hopf algebras to the category of τ-Lie coalgebras.*

A complete τ-commutative Hopf algebra H is said to be *of finite type*, if $Q(H)$ is finite-dimensional, or equivalently if $\mathrm{gr}\, H$ is finitely generated.

Corollary 4.5 *$H \mapsto Q(H)^*$ gives an equivalence from the category of complete τ-commutative Hopf algebras of finite type to the category of finite-dimensional τ-Lie algebras.*

This follows immediately from the theorem above. The corollary is essentially the same as Theorem 2 of Gurevich [Gu], in which detailed proofs are not given. Gurevich's construction of such an object, $\hat{H}(Q)$ as we will construct below, that corresponds to a τ-Lie algebra looks different from ours.

Let H be a complete τ-commutative Hopf algebra, and let $Q := Q(H)$. The τ-commutativity is equivalent to saying that for every n, l,

$$\begin{tikzcd} H/I^n \otimes H/I^l \arrow[rr, "\tau"] \arrow[dr, "m"'] & & H/I^l \otimes H/I^n \arrow[dl, "m"] \\ & H/I^{n+l} & \end{tikzcd}$$

commutes. Therefore, $\operatorname{gr} H$ is τ-commutative.

Proposition 4.6 *The inclusion $Q = \operatorname{gr}^1 H \hookrightarrow \operatorname{gr} H$ extends uniquely to an isomorphism $S(Q) \xrightarrow{\simeq} \operatorname{gr} H$ of graded τ-Hopf algebras.*

PROOF. By the obvious universal property of $S(Q)$, the inclusion extends uniquely to a graded τ-algebra map $S(Q) \to \operatorname{gr} H$, which is surjective since $\operatorname{gr} H$ is generated by Q. This is a coalgebra map since $Q \subset P(\operatorname{gr} H)$, and is injective since $P(S(Q)) = Q$ by Kharchenko's Theorem 1.1. $\square$

Let
$$\partial : H = k \oplus H^+ \to Q = Q(H)$$
denote the natural projection. This is a τ-preserving, linear ε-*derivation* in the sense that $\partial(ab) = \varepsilon(a)(\partial b) + (\partial a)\varepsilon(b)$; it is necessarily continuous since $\partial(I^2) = 0$. For the complete graded τ-algebra $\hat{S}(Q) = \prod_{n \geq 0} S^n(Q)$ (see Example 4.2),
$$\operatorname{Hom}_c(H, \hat{S}(Q)) = \prod_{n \geq 0} \operatorname{Hom}_c(H, S^n(Q))$$
is a complete graded algebra, in which we can exponentiate the ∂ of degree 1 to define
$$\theta = e^{\partial} : H \to \hat{S}(Q), \quad \theta(a) = \sum_{n=0}^{\infty} \frac{1}{n!} \partial^n(a).$$
This is a counit-preserving (and hence continuous) τ-algebra map since we see the following by modifying the proof of [N, Lemma 7].

Lemma 4.7 *Let J be a complete τ-Hopf algebra, and let A be a complete τ-commutative algebra. Suppose that a defining filtration $(A_n)_n$ of A can be chosen so as $A_1^n \subset A_n$ for all $n \gg 0$. If $\gamma : J \to A$ is a τ-preserving, linear ε-derivation such that $\gamma(J) \subset A_1$, it can be exponentiated to a continuous τ-algebra map $e^{\gamma} : J \to A$.*

Proposition 4.8 *$\theta : H \to \hat{S}(Q)$ is a counit-preserving isomorphism of complete τ-algebras.*

PROOF. As is easily seen, θ preserves the counit, and hence the defining filtration. The associated graded τ-algebra map $\operatorname{gr}\theta : \operatorname{gr} H \to S(Q)$ must be an isomorphism by Proposition 4.6, since $\operatorname{gr}\theta$ restricts to the identity map of Q. By induction on n, we see that θ induces isomorphisms $H/I^n \xrightarrow{\simeq} \hat{S}(Q)/(\hat{S}(Q)^+)^n$. As their projective limit, θ is an isomorphism. $\square$

Let X be a non-empty set. Let $T_X = \bigoplus_{n\geq 0} T_X^n$ denote the free algebra on X, or the tensor algebra of the vector space kX. This forms an ordinary graded Hopf algebra in which each element in X is primitive. Let $\hat{T}_X := \prod_{n\geq 0} T_X^n$ denote its completion. Define

$$L_X := P(T_X), \quad \hat{L}_X := P(\hat{T}_X),$$

the spaces of all primitives. It is known (see [Se, Part I, Thm. 7.1]) that L_X is the free Lie algebra on X, and is a positively graded Lie algebra, so as $L_X = \bigoplus_{n>0} L_X^n$. We have $\hat{L}_X = \prod_{n>0} L_X^n$, which may be called a *complete graded Lie algebra*; cf. Example 3.1. We will often use the abbreviated notation

$$[x_1, x_2, \dots, x_n] = [x_1, [x_2, \dots, [x_{n-1}, x_n] \cdots]] \tag{6}$$

for the iterated bracket.

The Campbell-Hausdorff formal power series [Se, Part I, Chap. IV, Sect. 8]

$$\Phi(x_1, x_2) = x_1 + x_2 + \frac{1}{2}[x_1, x_2] + \frac{1}{12}[x_1, [x_1, x_2]] + \frac{1}{12}[x_2, [x_2, x_1]] + \cdots, \tag{7}$$

where $x_1, x_2 \in X$, will play an important role. This is characterized as the element z in $\hat{L}_X$ such that $e^z = e^{x_1}e^{x_2}$ in $\hat{T}_X$. Therefore,

$$\Phi(x, 0) = \Phi(0, x) = x, \tag{8}$$

$$\Phi(\Phi(x_1, x_2), x_3) = \Phi(x_1, \Phi(x_2, x_3)). \tag{9}$$

These formal power series in (9) will be denoted by $\Phi(x_1, x_2, x_3)$. In general let $\Phi(x_1, x_2, \dots, x_n)$ denote $\Phi(x_1, \Phi(x_2, \dots, \Phi(x_{n-1}, x_n) \cdots))$.

Lemma 4.9 *Fix $n > 0$. The homogeneous component of $\Phi(x_1, x_2, \dots, x_n)$ in L_X^n equals*

$$\frac{1}{n}[x_1, x_2, \dots, x_n]$$

plus a linear combination of those $[x_{i_1}, x_{i_2}, \dots, x_{i_n}]$ in which $1 \leq i_s \leq n$ ($1 \leq s \leq n$), and $i_1, i_2, \dots, i_n$ are not distinct.

PROOF. Notice $\Phi(x_1, \ldots, x_n) = \log(e^{x_1} \cdots e^{x_n})$. The lemma follows from the expansion of this element; see [Se, p. 29]. $\square$

Let $Q = (Q, \delta)$ be a τ-Lie coalgebra. Fix an integer $N > 0$. Let $A = \prod_{n \geq 0} A(n)$ denote the complete graded algebra given by

$$A := S(Q)^{\hat{\otimes} N}, \quad A(n) := \bigoplus_{i_1 + \cdots + i_N = n} S^{i_1}(Q) \otimes \cdots \otimes S^{i_N}(Q).$$

On the complete graded vector space

$$\mathrm{Hom}(Q, A) = \prod_{n \geq 0} \mathrm{Hom}(Q, A(n)),$$

define a bilinear product $[\ ,\ ]$ by

$$[f, g] := m \circ (f \otimes g) \circ \delta,$$

where $f \in \mathrm{Hom}(Q, A(n))$, $g \in \mathrm{Hom}(Q, A(m))$. We see that $[f, g] \in \mathrm{Hom}(Q, A(n+m))$. For $1 \leq i \leq N$, let

$$d_i : Q \to A(1), \quad d_i(a) = 1 \otimes \cdots \otimes \overset{i}{\check{a}} \otimes \cdots \otimes 1$$

denote the linear map which embeds each element in Q into the i-th position. Set

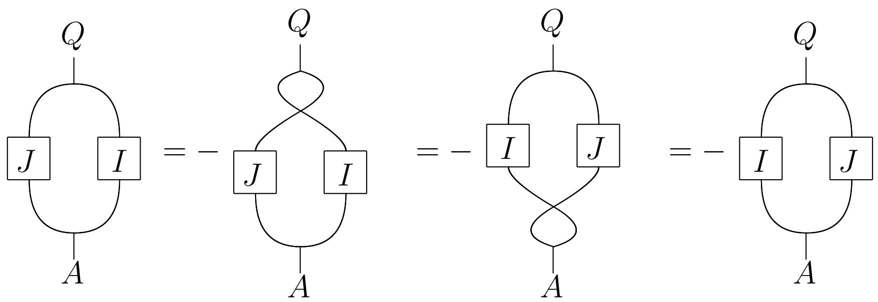
$$D = D_N := \{d_1, d_2, \ldots, d_N\}.$$

For each $n > 0$, let L_D^n denote the subspace of $\mathrm{Hom}(Q, A(n))$ spanned by all $[d_{i_1}, d_{i_2}, \ldots, d_{i_n}]$ $(1 \leq i_s \leq N)$ (see (6)), and define

$$\hat{L}_D := \prod_{n > 0} L_D^n.$$

Proposition 4.10 *$\hat{L}_D$ forms a complete graded Lie algebra with respect to* $[\ ,\ ]$.

PROOF. Take $I = [d_{i_1}, \ldots, d_{i_n}]$, $J = [d_{j_1}, \ldots, d_{j_m}]$, $K = [d_{k_1}, \ldots, d_{k_l}]$. As a crucial point, all d_i and hence I, J, K can pass through $Q, S(Q)^{\otimes r}, A(r)$ $(r \geq 0)$. By using (1), the anti-symmetry $[J, I] = -[I, J]$ is verified as follows.



Similarly, by using (2), the Jacobi identity $[I, [J, K]] = [[I, J], K] + [J, [I, K]]$ is verified. Suppose in particular, $I = x_{i_1}$. Then the last result proves $[J, K] \in \hat{L}_D$, by induction on the length m of J. Therefore, $\hat{L}_D$ is closed under $[\ ,\ ]$. $\square$

Let $x_1, x_2, \ldots, x_N$ be N letters, and set $X = X_N := \{x_1, x_2, \ldots, x_N\}$. By Proposition 4.10, $\phi(x_i) = d_i$ $(1 \le i \le N)$ defines a surjection

$$\phi = \phi_N : \hat{L}_X \to \hat{L}_D$$

of complete graded Lie algebras. Therefore, given an equation in $\hat{L}_X$, we can substitute d_i for x_i, to obtain an equation in $\hat{L}_D$.

When $N = 2$, let us use the Campbell-Hausdorff formal power series (7), to define

$$\Delta := \Phi(d_1, d_2) : Q \to S(Q)\hat{\otimes}S(Q). \tag{10}$$

This passes through $\hat{S}(Q)$, since each term in $\Phi(d_1, d_2)$ does. Consequently, Δ preserves the τ. In addition the image $\Delta(Q)$ is included in positive components. By the obvious universal property of $\hat{S}(Q)$, Δ uniquely extends to a continuous τ-algebra map

$$\Delta : \hat{S}(Q) \to \hat{S}(Q)\hat{\otimes}\hat{S}(Q).$$

Let $\varepsilon : \hat{S}(Q) \to k$ be the natural projection (or the original counit).

Proposition 4.11 $(\hat{S}(Q), \Delta, \varepsilon)$ *is a complete τ-commutative Hopf algebra.*

PROOF. It is now easy to see that Δ, ε pass through $\hat{S}(Q)$. It remains to see that $\hat{S}(Q)$ is a coalgebra. For the coassociativity of Δ, consider the following diagram of complete Lie algebras.

$$\begin{array}{ccc} \hat{L}_{X_2} & \xrightarrow{\phi_2} & \hat{L}_{D_2} \\ \downarrow & & \downarrow \\ \hat{L}_{X_3} & \xrightarrow{\phi_3} & \hat{L}_{D_3} \end{array}$$

The vertical arrow on the left-hand side is the continuous Lie algebra map determined by $x_1 \mapsto \Phi(x_1, x_2)$, $x_2 \mapsto x_3$. That of the right-hand side is induced by the continuous algebra map $\Delta\hat{\otimes}\mathrm{id} : S(Q)^{\hat{\otimes}2} \to S(Q)^{\hat{\otimes}3}$; it is a continuous Lie algebra map, too. By chasing the generators x_1, x_2 in $\hat{L}_{X_2}$, we see that the diagram commutes. By chasing $\Phi(x_1, x_2)$, we see that $(\Delta\hat{\otimes}\mathrm{id}) \circ \Delta = \Phi(\Phi(d_1, d_2), d_3)$. Similarly, but by replacing the vertical arrows, we see that $(\mathrm{id}\hat{\otimes}\Delta) \circ \Delta = \Phi(d_1, \Phi(d_2, d_3))$. The coassociativity now follows by (9). By using commutative diagrams containing ϕ_1, ϕ_2, the counit property follows from (8). $\square$

Given a τ-Lie coalgebra Q, let $\hat{H}(Q)$ denote the complete τ-commutative Hopf algebra $(\hat{S}(Q), \Delta, \varepsilon)$ defined as above. We see that $Q \mapsto \hat{H}(Q)$ is functorial; it will be seen below to be a quasi-inverse of the τ-Lie coalgebra functor.

Proof of Theorem 4.4. We see that the τ-Lie coalgebra associated to $\hat{H}(Q)$ equals Q, since for $a \in Q$,

$$\Delta(a) = a \otimes 1 + 1 \otimes a + \frac{1}{2}\delta(a) + \text{terms of degree} > 2,$$

which is projected to $\frac{1}{2}\delta(a)$ in $Q \otimes Q$.

Let H be a complete τ-commutative Hopf algebra, and set $Q := Q(H)$. To conclude the proof, we wish to prove that the natural isomorphism $\theta = e^{\partial} : H \xrightarrow{\simeq} \hat{H}(Q)$ given by Proposition 4.8 preserves the coproduct. Consider the following diagram consisting of complete vector spaces and continuous linear maps.

$$\begin{tikzcd} \hat{L}_{X_2} \arrow[r, hook] \arrow[d, "\phi_2"'] & \hat{T}_{X_2} \arrow[d] \\ \mathrm{Hom}(Q, S(Q)^{\hat{\otimes}2}) \arrow[r] & \mathrm{Hom}_{\mathrm{c}}(H, S(Q)^{\hat{\otimes}2}) \end{tikzcd} \tag{11}$$

Here the horizontal arrow on the bottom is induced by $\partial : H \to Q$, and the vertical arrow on the right-hand side is the graded algebra map defined by $x_i \mapsto d_i \circ \partial$ $(i = 1, 2)$. The four arrows all preserve the bracket, where we define $[f, g] := m \circ (f \hat{\otimes} g) \circ (\mathrm{id} - \tau) \circ \Delta$ on $\mathrm{Hom}_{\mathrm{c}}(H, S(Q)^{\hat{\otimes}2})$. In particular, the one on the bottom preserves it, since ∂ is a continuous τ-Lie coalgebra map, or more precisely since it makes

$$\begin{tikzcd} H \arrow[r, "(\mathrm{id}-\tau)\circ\Delta"] \arrow[d, "\partial"'] & H \hat{\otimes} H \arrow[d, "\partial \hat{\otimes} \partial"] \\ Q \arrow[r, "\delta"'] & Q \otimes Q \end{tikzcd} \tag{12}$$

commute; see Lemma 4.1. By chasing the generator x_1, x_2 in $\hat{L}_{X_2}$, we see that the diagram (11) commutes. Since $\exp \Phi(x_1, x_2) = e^{x_1} e^{x_2}$ in $\hat{T}_{X_2}$, it follows that $\exp(\Delta \circ \partial) = \exp(d_1 \circ \partial) \exp(d_2 \circ \partial)$ in $\mathrm{Hom}_{\mathrm{c}}(H, S(Q)^{\hat{\otimes}2})$, whence

$$\Delta \circ \theta = \Delta \circ e^{\partial} = (d_1 \circ e^{\partial})(d_2 \circ e^{\partial}) = (\theta \hat{\otimes} \theta) \circ \Delta,$$

as desired. $\square$

Let Q be a τ-Lie coalgebra, and let $\partial : \hat{H}(Q) \to Q$ denote the projection. Then,

$$e^{\partial} = \mathrm{id}, \tag{13}$$

since one sees that e^∂ is identical on Q. The pair $(\hat{H}(Q), \partial)$ has the following couniversal property.

Proposition 4.12 *If C is a complete τ-coalgebra, and if $\gamma : C \to Q$ is a continuous τ-Lie coalgebra map (see (12)), then the exponential $\varphi = e^\gamma : C \to \hat{H}(Q)$ is a unique continuous τ-coalgebra map satisfying $\gamma = \partial \circ \varphi$.*

PROOF. The uniqueness follows by exponentiating $\gamma = \partial \circ \varphi$; see (13). Obviously, φ is continuous and preserves τ. In the same way just as in the last proof, φ is proved to be a coalgebra map. $\square$

Let L be a τ-Lie algebra. Let $(U_n)_n$ denote the coradical filtration of the universal envelope $U(L)$. Explicitly, $U_n = (k + L)^n$ $(n \geq 0)$, which is a τ-subcoalgebra; see [K, Thm. 3.5]. Suppose that L is finite-dimensional. Then the dual vector space $U(L)^*$ of $U(L)$ forms a complete τ-commutative Hopf algebra, since $U(L)^*/I^{n+1} \simeq U_n^*$, where $I := (U(L)^*)^+$. Notice that L^* is a τ-Lie coalgebra.

Proposition 4.13 $U(L)^* \simeq \hat{H}(L^*)$ *as a complete τ-Hopf algebras.*

PROOF. This follows by Theorem 4.4, since we see $Q(U(L)^*) = L^*$. $\square$

It follows that in Corollary 4.5, $L \mapsto U(L)^*$ is a quasi-inverse of $H \mapsto Q(H)^*$.

5 A cleftness result on τ-Hopf algebras

Let us go back to the discrete situation. In this section we prove such a result as in the title above. It might be of independent interest, and is hence proved in a more generalized situation than will be needed for later use. The characteristic $\mathrm{ch}\, k$ may be arbitrary. We do not assume that the τ of τ-spaces is symmetric.

Let H be a τ-Hopf algebra. The coproduct will be denoted by $\Delta(a) = \sum a_1 \otimes a_2$. Let $K \subset H$ be a τ-Hopf subalgebra which includes the coradical of H. Let $J := H/K^+H$. This is a quotient τ-right H-module coalgebra of H, which is pointed irreducible as a coalgebra. Notice that H is a left K-module and right J-comodule, and the two structures commute with each other. As in the ordinary situation, we have an isomorphism $H \otimes H \xrightarrow{\simeq} H \otimes H$ given by $a \otimes b \mapsto \sum ab_1 \otimes b_2$, which induces

$$H \otimes_K H \xrightarrow{\simeq} H \otimes J. \tag{14}$$

The wedge products $F_nH := \wedge^{n+1}K$ $(n \geq 0)$ in H form an ascending chain $K = F_0H \subset F_1H \subset \cdots$ of τ-left K-module subcoalgebras, such that $H = \bigcup_{n\geq 0} F_nH$. The associated graded coalgebra is denoted by

$$\Gamma_K H := \bigoplus_{n\geq 0} F_nH/F_{n-1}H \quad (F_{-1}H = 0).$$

This is in fact a graded τ-Hopf algebra; see Montgomery [Mo, Lemma 5.2.8]. Let $\pi : \Gamma_K H \to \Gamma_K H(0) = K$ denote the projection. Let $\mathcal{J} := \Gamma_K H/K^+\Gamma_K H$; this is a graded Yetter-Drinfeld τ-bialgebra over K (in the sense that it satisfies the condition obtained by appropriately modifying (b) in [R, Thm. 1.1]). Just as in the ordinary situation, we can construct the bosonization $K \ltimes \mathcal{J}$, so that $a \mapsto \sum \pi(a_1) \otimes \bar{a}_2$ gives an isomorphism

$$\tilde{\pi} : \Gamma_K H \xrightarrow{\simeq} K \ltimes \mathcal{J} \tag{15}$$

of graded τ-Hopf algebras; see Radford [R, Theorem 3], and Andruskiewitsch and Schneider [AS, Sect. 1]. In fact, $x \otimes a \mapsto \sum xS(\pi(a_1))a_2$, $K \otimes \Gamma_K H \to \Gamma_K H$ induces an inverse of $\tilde{\pi}$. Let $J_n := F_nH/K^+F_nH$, a quotient τ-coalgebra of F_nH.

Proposition 5.1 *Let the notation be as above.*

(1) The canonical maps $J_n \to J$ $(n \geq 0)$ are injective, so that we have $k = J_0 \subset J_1 \subset J_2 \subset \cdots$. This coincides with the coradical filtration in J, and the associated graded τ-coalgebra $\Gamma J := \bigoplus_{n\geq 0} J_n/J_{n-1}$ coincides with $\mathcal{J}$.

(2) There is a unit and counit-preserving, left K-linear and right J-colinear isomorphism

$$H \xrightarrow{\simeq} K \otimes J$$

which restricts to such isomorphisms $F_nH \xrightarrow{\simeq} K \otimes J_n$ $(n \geq 0)$ that induce $\tilde{\pi}$.

(3) H is free as a left K-module, and cofree as a right J-comodule.

PROOF. Part 3 follows immediately from Part 2.

Let us write a diagram of left K-modules,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_{n-1}H & \longrightarrow & F_nH & \longrightarrow & \Gamma_K H(n) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \tilde{\gamma}_{n-1}} & & \big\downarrow{\scriptstyle \tilde{\gamma}_n} & & {\scriptstyle \simeq}\big\downarrow{\scriptstyle \tilde{\pi}(n)} & & \\
0 & \longrightarrow & K \otimes J_{n-1} & \longrightarrow & K \otimes J_n & \longrightarrow & K \otimes \mathcal{J}(n) & \longrightarrow & 0.
\end{array}$$

The first row is exact, and splits since $\Gamma_K H(n)$ is K-free. Hence the second is exact, too. By using the first row, we can construct inductively those K-linear retractions $\gamma_n : F_nH \to K$ which commute with the inclusions $F_{n-1}H \hookrightarrow F_nH$.

We may suppose that γ_n preserves counit since $(F_nH)^+ \to \Gamma_K H(n)$ $(n > 0)$ are surjective. Define $\tilde{\gamma}_n(a) = \sum \gamma_n(a_1) \otimes \bar{a}_2$, where $a \in F_nH$. Then the diagram above commutes, and each $\tilde{\gamma}_n$ is proved by induction to be an isomorphism. Now, γ_n $(n \geq 0)$ amount to a counit-preserving, K-linear retraction $\gamma : H \to K$. Define $\tilde{\gamma} : H \to K \otimes J$ by $\tilde{\gamma}(a) = \sum \gamma(a_1) \otimes \bar{a}_2$. Once $\tilde{\gamma}$ is proved to be an isomorphism, we will see that $J_n \subset J$, and $\tilde{\gamma}$ is such as desired in Part 2. We wish to prove that $\tilde{\gamma}$ is an isomorphism. Notice from the isomorphism (14) that H is injective as a right J-comodule and the socle $\mathrm{Soc}\, H$ equals K, since the opposite-side argument of the above shows that H is right K-free. Since $\tilde{\gamma}$ is such a J-comodule map between injectives that is identical on the socles, it must be an isomorphism.

To conclude the proof we will see that if $n > 1$, $\mathcal{J}(n)$ does not contain any non-zero primitive in $\mathcal{J}$. Suppose that $n > 1$, and a is a primitive contained in $\mathcal{J}(n)$. Notice that each $\mathcal{J}(m)$ $(m \geq 0)$ is costable under the coaction by K. It follows that the coproduct $\Delta(a)$ in $\Gamma_K H$ is contained in $K \otimes \mathcal{J}(n) + \mathcal{J}(n) \otimes K$, and hence vanishes through the projection $\mathrm{pr}_{1,n-1} : \Gamma_K H \otimes \Gamma_K H \to \Gamma_K H(1) \otimes \Gamma_K H(n-1)$. Since by construction, the composite $\mathrm{pr}_{1,n-1} \circ \Delta : \Gamma_K H(n) \to \Gamma_K H(1) \otimes \Gamma_K H(n-1)$ is injective, we must have $a = 0$. This concludes the proof of Part 1, and of the proposition. $\square$

6 Irreducible τ-commutative Hopf algebras

In this section we suppose that the characteristic $\mathrm{ch}\, k = 0$, and the τ of any τ-space is symmetric.

Let H be an irreducible τ-commutative Hopf algebra, and let $K \subset H$ be a τ-Hopf subalgebra. Recall from Proposition 5.1 the result and the notation. The τ-commutativity implies $K^+H = HK^+$, so that $J = H/K^+H$ is now a quotient τ-Hopf algebra of H. We say that $K \hookrightarrow H \twoheadrightarrow J$ is a *short exact sequence*. Notice that $\mathcal{J} = \Gamma J$ is now a quotient graded τ-Hopf algebra of $\Gamma_K H$. In the bosonization in (15), the associated K-action $\mathcal{J} \otimes K \to \mathcal{J}$ is trivial, and the K-coaction $\mathcal{J} \to \mathcal{J} \otimes K$ is a τ-right K-comodule bialgebra structure which preserves the gradation on $\mathcal{J}$. Thus, $K \rhd\!\!< \mathcal{J}$ is replaced by the smash coproduct $K \blacktriangleright\!\!< \mathcal{J}$.

Fix such an isomorphism

$$H \xrightarrow{\simeq} K \otimes J \tag{16}$$

as in Proposition 5.1 (2), through which we will identify $H = K \otimes J$. Define

$$V := \mathcal{J}(1) \ (= P(J)).$$

Let $\rho : H \to H \otimes J$ denote the composite of the coproduct Δ with $\mathrm{id} \otimes \mathrm{proj}$:

$H \otimes H \to H \otimes J$, and define

$$K * V := \{a \in H \mid \rho(a) = a \otimes 1 + 1 \otimes v \ \text{ for some } \ v \in V\}. \tag{17}$$

We see that this is a τ-subspace of H, and is identified so as

$$K * V = K \otimes k + k \otimes V.$$

Let $K\langle V\rangle$ denote the subalgebra in H generated by $K * V$.

Proposition 6.1 *Let the notation be as above.*

(1) $K\langle V\rangle$ is a τ-Hopf subalgebra of H, which gives rise to a short exact sequence $K \hookrightarrow K\langle V\rangle \twoheadrightarrow S(V)$.

(2) The same statement holds true with V replaced by

$$V_0 := \mathcal{J}(1)^{\mathrm{co}\,K},$$

the coinvariants in $\mathcal{J}(1)$ under the K-coaction on $\mathcal{J}$. In this case the quotient $\pi : K\langle V_0\rangle \twoheadrightarrow S(V_0)$ is τ-cocentral in the sense that $(\mathrm{id} \otimes \pi) \circ (\mathrm{id} - \tau) \circ \Delta = 0$.

PROOF. (1) From the coproduct of $\Gamma_K H$, we see that the image $\Delta(V)$ of $V = k \otimes V$ by the coproduct of H is included in $K \otimes V + V \otimes K + K \otimes K$, whence $K * V$ is a τ-subcoalgebra. By Kharchenko's Theorem 1.1, the τ-Hopf subalgebra in J generated by V is $S(V)$. These imply Part 1.

(2) The τ-cocentrality follows, since for $v \in V_0$, $\Delta(v) \in 1 \otimes v + v \otimes 1 + K^+ \otimes K^+$. $\square$

Remark 6.2 *If $K \subsetneqq H$, V and V_0 are both non-zero.*

Given an irreducible τ-commutative Hopf algebra H, define

$$Q(H) := H^+/(H^+)^2,$$

just as in Section 4. This is a τ-Lie coalgebra, and the natural projection $\partial : H = k \oplus H^+ \to Q(H)$ is a τ-Lie coalgebra map; see (12). For example, $Q(S(V)) = V$ for a τ-space V.

Lemma 6.3 *Suppose $K\langle V\rangle = H$ in the situation of Proposition 2.1, so that we have a short exact sequence $K \hookrightarrow H \twoheadrightarrow S(V)$. Then this induces a short exact sequence $Q(K) \hookrightarrow Q(H) \twoheadrightarrow V$ of τ-Lie coalgebras.*

PROOF. Write $J := S(V)$. Define $A := H/(K^+)^2 H$; this is a quotient of H as a τ-right J-comodule algebra with augmentation. The isomorphism (16) induces

$$A \xrightarrow{\simeq} K/(K^+)^2 \otimes J.$$

In particular, A is right J-injective. We see $Q(H) = A^+/(A^+)^2$, where $A^+ = \operatorname{Ker}\varepsilon$ in A. Let $\rho_A : A \to A \otimes J$ denote the right J-comodule structure, and define

$$Q := \{a \in A^+ \mid \rho_A(a) = a \otimes 1 + 1 \otimes v \ \text{ for some } \ v \in V\}.$$

This is a τ-subspace of A, and generates A; see the next isomorphism. Let $R := Q(K)$; this is a τ-subspace of Q. The last isomorphism restricts to

$$Q \xrightarrow{\simeq} R \otimes k + k \otimes V. \tag{18}$$

Let $I = (R^2)$ be the graded τ-ideal in $S(Q)$ generated by the τ-space R^2 in degree 2. The canonical map $S(Q) \to A$ induces a counit-preserving surjection $\xi : S(Q)/I \to A$, say, of τ-right J-comodule algebras. Since we have a short exact sequence $S(R) \hookrightarrow S(Q) \twoheadrightarrow S(V)$, it follows as before that $S(Q)/I$ is right J-injective. Since ξ restricts to the identity map $S(R)/(R^2) = K/(K^+)^2 \to K/(K^+)^2$ on the J-socles, it is an isomorphism. It follows now easily that $Q \simeq A^+/(A^+)^2 = Q(H)$, and $Q(K) \hookrightarrow Q(H) \twoheadrightarrow V$ is short exact. $\square$

Proposition 6.4 *Every short exact sequence $K \hookrightarrow H \twoheadrightarrow J$ of irreducible τ-commutative Hopf algebras induces a short exact sequence $Q(K) \hookrightarrow Q(H) \twoheadrightarrow Q(J)$ of τ-Lie coalgebras.*

PROOF. Given a short exact sequence as above, suppose that $Q(K) \to Q(H)$ is injective. Then we see easily that $Q(K) \subset Q(H)$ is categorical, and

$$Q(J) = H^+/((H^+)^2 + K^+H) = \operatorname{Coker}(Q(K) \to Q(H)).$$

Therefore it remains to prove the injectivity supposed above.

Given an algebra A with augmentation $\varepsilon : A \to k$, the space $Q(A) = A^+/(A^+)^2$, where $A^+ = \operatorname{Ker}\varepsilon$, represents the functor which associates to each vector space M, the space $\operatorname{Der}_\varepsilon(A, M)$ of k-linear ε-derivations $A \to M$. It follows that $\varinjlim Q(A_\lambda) = Q(\varinjlim A_\lambda)$ for an inductive system $\{A_\lambda\}$ of augmented algebras. Therefore, given $K \subset H$ as above, Zorn's lemma gives a maximal τ-Hopf subalgebra $\tilde{K}$, say, of H including K, such that $Q(K) \to Q(\tilde{K})$ is injective. If $\tilde{K} \subsetneqq H$, we have such a τ-Hopf subalgebra $\tilde{K}\langle V\rangle$ as in Proposition 6.1 (1), which includes $\tilde{K}$ properly; see Remark 6.2. But, $Q(\tilde{K}) \to Q(\tilde{K}\langle V\rangle)$, and hence $Q(K) \to Q(\tilde{K}\langle V\rangle)$ are injective, by Lemma 6.3. Hence we must have $\tilde{K} = H$, which completes the proof. $\square$

Corollary 6.5 *For every irreducible τ-commutative Hopf algebra H, the primitives $P(H)$ in H form a τ-Lie subcoalgebra of $Q(H)$, which has a zero cobracket.*

PROOF. Apply Proposition 6.4 to $K = S(P(H))$, the τ-Hopf subalgebra generated by $P(H)$. $\square$

Proposition 6.6 *For every irreducible τ-commutative Hopf algebra H, the τ-Lie coalgebra $Q(H)$ is locally nilpotent; see Definition 2.1.*

PROOF. Since a directed union of locally nilpotent τ-Lie coalgebra is locally nilpotent, we see from the last proof that there is a maximal τ-Hopf subalgebra $K \subset H$ such that $Q(K)$ is locally nilpotent. To prove $K = H$, we suppose on the contrary that $K \subsetneqq H$. We then have such a τ-Hopf subalgebra $K\langle V_0\rangle$ ($\supsetneqq K$) as described in Proposition 6.1 (2). Since a τ-cocentral τ-Hopf algebra quotient induces such a τ-Lie coalgebra quotient, it follows from Lemmas 2.5 and 6.3 that $Q(K\langle V_0\rangle)$ is locally nilpotent; this contradicts the maximality of K. $\square$

We aim to prove the following.

Theorem 6.7 *$H \mapsto Q(H)$ gives an equivalence from the category of irreducible τ-commutative Hopf algebras to the category of locally nilpotent τ-Lie coalgebras.*

This was proved by Nichols [N, Thms. 12,14] in the ordinary situation.

Let H be an irreducible τ-commutative Hopf algebra with $Q := Q(H)$. Let $\partial : H = k \oplus H^+ \to Q$ denote the projection. If $(H_n)_n$ denotes the coradical filtration of H, then $\mathrm{Hom}(H, S(Q)) = \varprojlim \mathrm{Hom}(H_n, S(Q))$, which is a complete algebra. Therein the τ-preserving linear ε-derivation $\partial : H \to Q$ is exponentiated to

$$\theta = e^{\partial} : H \to S(Q), \quad \theta(a) = \sum_{n=0}^{\infty} \frac{1}{n!}\partial^n(a).$$

This is a counit-preserving τ-algebra map; see [N, Lemma 7], and also Lemma 4.7.

Proposition 6.8 *θ is an isomorphism.*

This was proved by Nichols [N, Thm. 8] in the ordinary situation. Applying our structural results above, we will give below an alternative, hopefully simpler and more conceptual proof in our generalized situation.

PROOF. Let us write θ_H for θ. Since θ_H is natural in H, Zorn's lemma gives a maximal τ-Hopf algebra K, say, of H for which θ_K is an isomorphism.

To prove $K = H$, we suppose $K \subsetneqq H$, and take such $K\langle V_0\rangle$ just as in the last proof. We wish to prove that the θ for $K\langle V_0\rangle$ is an isomorphism. We may suppose $K\langle V_0\rangle = H$. Recall that $V_0 = k \otimes V_0$ is embedded in H^+, and maps isomorphically into Q; see (18). Let $R := Q(K)$. If $v \in V_0$, we see from the expression of $\Delta(v)$ given in the proof of Proposition 6.1 (2) that $\theta(v) \in v + S(R)$, and so $v \in \operatorname{Im}\theta_H$. This implies $Q \subset \operatorname{Im}\theta_H$, whence θ_H is surjective. For the injectivity let $I := K^+$, and identify so as $K = S(R)$, $I = (R)$ via θ_K. The K-linear map θ_H onto the free left module is an isomorphism modulo I, and so modulo I^n for all $n > 0$. For the θ_H modulo I is precisely $\theta_{S(V_0)}$, which is the identity map of $S(V_0)$. It follows that all I^nH $(n > 0)$ include $\operatorname{Ker}\theta_H$. This must be zero, whence θ_H is injective, since $\bigcap_{n\geq 0} I^n = 0$ and so $\bigcap_{n\geq 0} I^nH = 0$. $\square$

Let Q be a locally nilpotent τ-Lie coalgebra. Then the image of the map $\Delta : Q \to S(Q)\hat{\otimes}S(Q)$ defined by (10) is included in $S(Q)\otimes S(Q)$. In fact, if $a \in Q_n$ (see (3)), then it vanishes under all homogeneous terms in $\Phi(d_1, d_2)$ of degree $> n$, so that $\Delta(a) \in \sum_{i+j\leq n} S^i(Q) \otimes S^j(Q)$. Therefore, Δ extends uniquely to a τ-algebra map $\Delta : S(Q) \to S(Q) \otimes S(Q)$. It follows by Proposition 4.11 that $(S(Q), \Delta)$ together with the projection $\varepsilon : S(Q) \to k$ forms a τ-bialgebra, since the inclusion $S(Q) \subset \hat{S}(Q)$ is compatible with Δ, ε. Let us denote $(S(Q), \Delta, \varepsilon)$ by $H(Q)$.

Lemma 6.9 *$H(Q)$ is irreducible as a coalgebra, and is hence a τ-Hopf algebra.*

PROOF. The iterated coproduct $\Delta^{n-1}|_Q$, restricted onto Q, equals $\Phi(d_1, d_2, \ldots, d_n)$; see the proof of Proposition 4.11. This implies that $Q_n \subset \wedge^{n+1}k$. To be more precise for later use, it follows by Lemma 4.9 that if $a \in Q_n$, then $\Delta^{n-1}(a)$ is mapped to

$$\frac{1}{n}\delta^{n-1}(a) + \text{higher terms}$$

under the projection $H(Q)^{\otimes n} \to (H(Q)/k)^{\otimes n}$; $\frac{1}{n}\delta^{n-1}(a) \in S^1(Q)^{\otimes n}$, and hence the higher terms are those in some $S^{i_1}(Q) \otimes \cdots \otimes S^{i_n}(Q)$ with $i_s > 0$ $(1 \leq s \leq n)$, $i_1 + \cdots + i_n > n$. This implies that

$$Q_n = Q \cap \wedge^{n+1}k. \tag{19}$$

Therefore, Q is included in the irreducible component $H^1 := \bigcup_{n\geq 0} \wedge^{n+1}k$ of $H(Q)$. One sees that the τ-coalgebra $H^1 \otimes H^1$ is irreducible, since the natural filtration on it that arises from $(\wedge^{n+1}k)_n$ is a coalgebra filtration. It follows that $H^1 \subset H(Q)$ is a subalgebra including Q, whence $H^1 = H(Q)$. $\square$

Proof of Theorem 6.7. Notice that the completions of those $H(Q), \theta$ in this section are precisely those $\hat{H}(Q), \theta$ in Section 4. Then Theorem 6.7 follows easily from the proof of Theorem 4.4. $\square$

When we say that C is a pointed irreducible τ-coalgebra it will be always assumed that the unique grouplike, say 1_C, spans a τ-subspace. The assumption is automatically satisfied if C is an irreducible τ-Hopf algebra.

Proposition 6.10 *Let Q be a locally nilpotent τ-Lie coalgebra. If C is a pointed irreducible τ-coalgebra, and if $\gamma : C \to Q$ is a τ-Lie coalgebra map with $\gamma(1_C) = 0$, then the exponential $\varphi = e^{\gamma} : C \to H(Q)$ is a unique τ-coalgebra map such that $\gamma = \partial \circ \varphi$.*

PROOF. Modify the proof of Proposition 4.12. $\square$

If $\mathcal{Q} = \bigoplus_{n>0} \mathcal{Q}(n)$ is a positively graded τ-Lie coalgebra, which is necessarily locally nilpotent, then $H(\mathcal{Q})$ is graded so that $\mathcal{Q}(n) \subset H(\mathcal{Q})(n)$, and is in fact a graded τ-Hopf algebra, as is easily seen. Let Q be a locally nilpotent τ-Lie coalgebra, and let ΓQ denote the associated, positively graded τ-Lie coalgebra as defined in Corollary 2.3. On the other hand, let $\Gamma H(Q)$ denote the graded τ-Hopf algebra arising from the coradical filtration, say $(H_n)_n$, of $H(Q)$; thus, $H_n = \wedge^{n+1} k$.

Proposition 6.11 $\Gamma H(Q) \simeq H(\Gamma Q)$ *as graded τ-Hopf algebras.*

PROOF. The τ-Lie coalgebra map $\partial : H(Q) \to Q$ induces a graded τ-Lie coalgebra map. $\Gamma\partial : \Gamma H(Q) \to \Gamma Q$, which is surjective since we see

$$\partial(H_n) = Q_n \quad (n \geq 0),$$

as follows. The one inclusion $\subset$ is easy to see. The other follows by (19). We see that $\Gamma\partial$ is exponentiated to a map $\Gamma H(Q) \to H(\Gamma Q)$ of graded τ-Hopf algebras, which is surjective since $\Gamma\partial$ is. It is injective, since by Corollary 6.5, it is injective, restricted on the primitives. $\square$

7 Dual argument for the universal envelope

We still suppose that $\operatorname{ch} k = 0$, and the τ of any τ-space is symmetric.

The construction of $\hat{H}(Q)$ or $H(Q)$ is dualized as follows; see [N, Remark, p. 71]. Let V be a τ-space. Let $B(V) = \bigoplus_{n\geq 0} B^n(V)$ denote the largest τ-cocommutative subcoalgebra, necessarily graded τ-Hopf subalgebra, in the

τ-shuffle Hopf algebra $\mathrm{Sh}(V)$; this last is by definition the tensor coalgebra of the vector space V, endowed with the shuffle product with respect to τ (see [S, Chap. XII]). We have $B^0(V) = k$, $B^1(V) = P(B(V)) = V$; if $n > 1$, $B^n(V)$ consists of those elements in $V^{\otimes n}$ which are fixed by all $\mathrm{id}^{\otimes(i-1)} \otimes \tau \otimes \mathrm{id}^{\otimes(n-i)}$ ($1 \le i \le n$). Let $\pi : B(V) \to B^1(V) = V$ denote the projection. Then, $(B(V), \pi)$ is couniversal among the pairs (C, ϖ) in which C is a pointed irreducible τ-cocommutative coalgebra, and $\varpi : C \to V$ is a τ-preserving linear map such that $\varpi(1_C) = 0$. Since $\mathrm{ch}\, k = 0$ as we supposed, we have $S(V) \simeq B(V)$ canonically; see [K, Sect. 7], for example. Forget once the product on $B(V)$.

Let L be a τ-Lie algebra. Define $b_i : B(L) \otimes B(L) \to L$ ($i = 1, 2$) by $b_1 = \pi \otimes \varepsilon$, $b_2 = \varepsilon \otimes \pi$. Dualizing Proposition 4.10, we see that in the complete graded algebra

$$\mathrm{Hom}(B(L)^{\otimes 2}, L) = \prod_{n \ge 0} \mathrm{Hom}(\bigoplus_{i+j=n} B^i(L) \otimes B^j(L), L),$$

b_1, b_2 generate a complete graded Lie algebra with respect to the bracket $[f, g] = [\ \ ,\ \ ] \circ (f \otimes g) \circ \Delta$, where the vacant $[\ \ ,\ \ ]$ denotes the bracket on L. Using the Campbell-Hausdorff formal power series (7), define

$$\bar{m} := \Phi(b_1, b_2) : B(L)^{\otimes 2} \to L.$$

By the couniversal property of $B(L)$, this gives rise to a τ-coalgebra map $m : B(L)^{\otimes 2} \to B(L)$ such that $\bar{m} = \pi \circ m$. Let $u : k = B^0(L) \hookrightarrow B(L)$ denote the inclusion (or the original unit).

Proposition 7.1 *$(B(L), m, u)$ forms an irreducible τ-cocommutative Hopf algebra including L, in which $P(B(L)) = L$, and the braided commutator $m \circ (\mathrm{id} - \tau)(x \otimes y)$ equals the bracket $[x, y]$ in L, where $x, y \in L$.*

PROOF. If $x, y \in L$, the homogeneous terms in $\Phi(b_1, b_2)$ of degree other than 2 annihilate $x \otimes y$, and so $\bar{m}(x \otimes y) = \frac{1}{2}[x, y]$. This implies the last equality claimed above, since we know that $m \circ (\mathrm{id} - \tau)(x \otimes y) \in L$. By definition, $P(B(L)) = L$. For the remaining claim, dualize the proof of Proposition 4.10. $\square$

Remark 7.2 *By Proposition 7.1, the universal property of $U(L)$ gives a unique τ-Hopf algebra map $\sigma : U(L) \to B(L)$ that preserves the canonical map from L. It follows that the canonical $L \to U(L)$ is injective. This gives an alternative proof of the crucial part of Kharchenko's Theorem 1.1; cf. the proof of [K, Thm. 5.2]. It also follows from the general result [K, Thm. 3.5] that σ is an isomorphism.*

Let L be a τ-Lie algebra. Let $I := U(L)^+$, and define

$$\operatorname{gr} U(L) := \bigoplus_{n \geq 0} I^n/I^{n+1}.$$

This naturally forms a graded τ-Hopf algebra, which is irreducible and τ-cocommutative. On the other hand the descending central series

$$L^1 = L, L^2 = [L, L], \ldots, L^n = [L^{n-1}, L], \ldots$$

form τ-Lie ideals of L, and we have a graded τ-Lie algebra

$$\operatorname{gr} L := \bigoplus_{n>0} L^n/L^{n+1}.$$

The universal envelope $U(\operatorname{gr} L)$ is naturally graded, so that the embedding $\operatorname{gr} L \to U(\operatorname{gr} L)$ preserves the degree.

Proposition 7.3 $U(\operatorname{gr} L) \simeq \operatorname{gr} U(L)$ *as graded τ-Hopf algebras.*

PROOF. We have a natural map $\operatorname{gr} L \to \operatorname{gr} U(L)$ of graded τ-Lie algebras. Since this is surjective in degree 1, the induced $U(\operatorname{gr} L) \to \operatorname{gr} U(L)$ is a surjection of graded τ-Hopf algebras. To prove that this is injective, we may identify $U(L) = B(L)$, $I = B(L)^+$; see Remark 7.2. It suffices to prove $\pi(I^n) = L^n$ ($n > 0$), where $\pi : B(L) \to L$ is the projection, as before. For this implies that π induces $I^n/I^{n+1} \to L^n/L^{n+1}$; this is necessarily a retraction of the natural map $L^n/L^{n+1} \to I^n/I^{n+1}$, which then must be injective, proving the desired injectivity. Let $\mu : I^{\otimes n} \to I \to L$ denote the composite of the product $I^{\otimes n} \to I$ with π; this is given by $\Phi(b_1, b_2, \ldots, b_n)$, where $b_i := \varepsilon^{\otimes(i-1)} \otimes \pi \otimes \varepsilon^{\otimes(n-i)}$. Suppose $x_1 \in B^{i_1}(L), \ldots, x_n \in B^{i_n}(L)$, where $i_s > 0$ ($1 \leq s \leq n$). Then by Lemma 4.9,

$$\mu(x_1 \otimes \cdots \otimes x_n) = \begin{cases} \dfrac{1}{n}[x_1, \ldots, x_n] & \text{if } i_1 = \cdots = i_n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $\pi(I^n) = L^n$, as desired. □

Given an algebra A with augmentation $\varepsilon : A \to k$, let A' denote the largest irreducible subcoalgebra containing ε in the dual coalgebra A°. Let $I := \operatorname{Ker} \varepsilon$. If A is finitely generated, each A/I^n ($n > 0$) is finite-dimensional, and

$$A' = \bigcup_{n>0} (A/I^n)^*,$$

the directed union of the dual coalgebras $(A/I^n)^*$; see [Mo, Sect. 9.2].

Let H be a finitely generated τ-Hopf algebra with $I := H^+$. Then H' is naturally an irreducible τ-Hopf algebra, and the τ-Lie algebra $P(H')$ of primitives in H' is dual to $Q(H) = I/I^2$, that is, $P(H') = Q(H)^*$. Moreover, the graded τ-Hopf algebra $\Gamma(H')$ arising from the coradical filtration $k = (H/I)^* \subset (H/I^2)^* \subset \cdots$ in H' coincides with the graded dual $(\mathrm{gr}\, H)^{\mathrm{g}} = \bigoplus_{n\geq 0}(I^n/I^{n+1})^*$ of $\mathrm{gr}\, H = \bigoplus_{n\geq 0} I^n/I^{n+1}$.

Let L be a finite-dimensional τ-Lie algebra which is *nilpotent* in the sense that $L^n = 0$ for $n \gg 0$. Let $Q := L^*$; this is naturally a τ-Lie coalgebra which is (locally) nilpotent.

Proposition 7.4 *$U(L) \simeq H(Q)'$, $H(Q) \simeq U(L)'$ as τ-Hopf algebras.*

PROOF. Since $P(H(Q)') = L$, as was seen above, the first isomorphism follows from Theorem 1.1. The embedding $L \hookrightarrow U(L)$ induces $U(L)' \to L^* = Q$, whence we have a natural map $Q(U(L)') \to Q$ of τ-Lie coalgebras. By Theorem 6.7, it is enough to prove that the last map, say α, is an isomorphism. The isomorphism $\mathrm{gr}\, L \simeq P(\mathrm{gr}\, U(L))$ obtained by Proposition 7.3 is dualized to an isomorphism $Q(\Gamma(U(L)')) = Q((\mathrm{gr}\, U(L))^{\mathrm{g}}) \simeq \Gamma Q$ of graded τ-Lie coalgebras. Since by Proposition 6.11, this is identified with $\Gamma\alpha : \Gamma Q(U(L)') \to \Gamma Q$, it follows that $\Gamma\alpha$ and hence α are isomorphisms. $\square$

Corollary 7.5 *$H \mapsto P(H')$ and $L \mapsto U(L)'$ give a category equivalence, as mutual quasi-inverses, between the category of finitely generated, irreducible τ-commutative Hopf algebras H and the category of finite-dimensional nilpotent τ-Lie algebras L.*

PROOF. This follows by Theorem 6.7 and Proposition 7.4. $\square$

In the ordinary situation the corollary, translated into the language of affine groups, is precisely the result [DG, IV, Sect. 2, 4.5] probably due to Cartier; see also Hochschild [H, XVI, Thm. 4.2]. The same result was proved by [MO, Thm. 3.2] in the symmetric category of super-vector spaces. For related results on (co)commutative Hopf algebras in that category, refer to [M], and especially [Ko, Thm. 3.3].

Remark 7.6 *In a recent, interesting paper [AMS] by Ardizzoni, Menini and Stefan, τ-spaces with Hecke-type τ are studied; τ is thus supposed to satisfy $(\tau - 1)(\tau + q) = 0$ (with q generic), and is symmetric if $q = -1$, in particular. In this generalized context they especially define the notion of braided Lie algebras, and generalize Kharchenko's Theorem 1.1 for those objects. In the dual situation of ours, the author can prove some of our results, including Propositions 4.8 and 6.8, in their generalized context. But, our construction of $\hat{H}(Q)$, $H(Q)$*

cannot be directly generalized. The author also feels it difficult to construct the free braided Lie algebra (with τ of Hecke-type), and even to find non-trivial examples of those Lie algebras.

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