Optimal Stopping Problems with Reservation

1999

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Submitted in Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy
in Management Science and Engineering

by

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January 1999

Abstract

Almost all decision making processes which we encounter in the real world have deadlines up to when a decision must be made. Typical of these is the so-called *optimal stopping problem*, for example:

Suppose a buyer must purchase an asset, say a house, a plot of land, or the like, within a given number of days and he can find a seller every day by paying a cost. When a seller appears, the buyer decides on whether or not to purchase the asset from him after considering whether the offered price is acceptable or not.

To sum up, the optimal stopping problem is reduced to a problem of when and how to make a decision to maximize the expected reward in the situation where an opportunity has to be taken among ones appearing subsequently up to the deadline. Many different models of the problem have been investigated so far. However, almost all of these assume that the future availability of an opportunity appearing in the past is determined independently of the will of the decision maker, in other words, little attention has been given to the question of what happens if the decision maker himself can control the availability, or reserve an opportunity, by paying a cost.

The purpose of this thesis is to propose three models of a discrete-time optimal stopping problem in which such control of availability for any opportunity, or reservation of an opportunity, is taken into account and to examine the properties of the optimal decision rule for each of them. A major finding in these studies is that no reserved opportunity should be accepted while it remains available at the next point in time, in other words, the time when accepting a reserved opportunity can become an optimal decision is restricted only to the maturity of its reservation.

Acknowledgments

The author would like to express his great appreciation to Professor Seizo Ikuta of the University of Tsukuba for his thoughtful and valuable advice. It was he who first introduced the author to the field of stochastic decision processes. The author wishes to express his gratitude to Professor Hideaki Takagi, Professor Mamoru Kaneko, Professor Masato Koda, Professor Takeshi Koshizuka, Professor Yoshitsugu Yamamoto, Professor Akiko Yoshise, Professor Kazuhisa Takemura, Professor Ryo Sato, Professor Hitoshi Takehara, and Professor Maiko Shigeno of the University of Tsukuba for their comments. The author would also like to thank his colleagues Dr. Byung-Kook Kang, Dr. Henry Osadoro Aigbedo, Dr. Peng-Sheng You, Mr. Yu-Hung Peng, Mrs. Feng-Bo Shi, Mrs. Jing-Xiao Lei of the University of Tsukuba, and Dr. Masahiro Sato of the Ground Self Defense Force for their kind and friendly support during his stay at the University of Tsukuba. Mr. Michael Cullen has checked the English in detail in this thesis. Furthermore, thanks are due to Mr. Sotaro Kamide of the Tokyu Land Corporation for providing the author with sample data on land trading. Finally, the author would like to express his sincere gratitude to his family for their affection and understanding.

Contents

Chapter 1

Introduction

1.1 Conventional Optimal Stopping Problems

To begin with, let us consider the following two decision making problems:

- **House Purchasing Problem**: Suppose that you would like to purchase a house at as low a price as possible before you have to move and have just found a house for sale. The price of the house seems pretty good but you reject it on account that you feel you could find a better deal within the periods remaining. If you continually repeated such rejections and reached the deadline, you would face a dire situation where the price of the house just found is too high while you may have no choice but to purchase it because the houses you already rejected have been sold. However, if a too high purchasing price is accepted early in the time frame because of the fear of the dire situation, there is every chance that better houses which may have been found later would be missed. To avoid these extremes, when and how do you select the house to be purchased?

- **Secretary Problem**: Suppose that you are to employ the best person as a secretary among a certain number of applicants with the provision that the result of acceptance or rejection has to be told to each applicant just after his or her interview. The question that bothers you at every interview is whether the best applicant is a person already rejected, the person in front of you, or a person not interviewed yet.

The core of the above problems is given in the following three points: The first is that you are searching for offers (say, houses for sale, applicants for the secretary, and so on) with the intention to accept one of them up to the deadline (say, the date when you move house, the final interview you can conduct, and so on). The second is that offers appear one by one and the value of each is not known or cannot be estimated before the appearance. The third is that a decision whether or not to stop the search must be made every time when you inspect an offer.

The decision making problem with these three properties is referred to as the *optimal stopping problem*, and many models have been presented and examined ([11] [24] [9] [57] [18] [4]).

In general, models of optimal stopping problems can be categorized into two groups in terms of the objective of the search: The first group aims to maximize the expected reward from the search process where the value of an offer is considered like in the house purchasing problem ([20] [21] [28] [27] [14]). The second group aims to maximize the probability of accepting the best offer where only the relative rank of an offer is noted like in the secretary problem ([54] [43] [33] [2] [10]). In addition, there are compound models of the two groups ([31] [26] [53] [8] [3]). For example, in Lorenzen [26] it is assumed that each interview needs an interview cost, and the objective is to minimize the sum of the expectation of the loss sustained depending on the absolute rank of the accepted offer and that of the total interview costs.

In any model belonging to these groups, the question whether or not a rejected offer will become available again in the future dominates the structure of the optimal decision rule. The following three models, which are classified according to this viewpoint, roughly cover almost all the models proposed so far:

Model with No Recall ([56] [29] [23] [28] [14]): Every offer once rejected is supposed to disappear instantly and become unavailable forever, thus at any point in time, the latest offer found is only the offer available then. Lippman et al. [23] treats a model where the offer value distribution changes according to the progress of time. In Ikuta [14] there are several areas in which you search for offers and the traveling cost is considered.

Model with Recall ([29] [28] [59] [41] [58]): Every rejected offer is supposed to stay there forever, thus at any point in time, all the offers having appeared up to then remain available. Kang [17] [16] assumes that the value of a rejected offer deteriorates as time goes by. In Morgan et al. [29] two or more offers can be found at a point in time and the decision maker is allowed each time to decide how many offers he draws at the next time.

The most simple models with no recall and with recall are described in [22] [15] [46] [7] [12].

Model with Uncertain Recall ([21] [25] [34] [32] [52]): The future availability of a rejected offer is supposed to be determined by a certain probability, thus at any point in time, it is uncertain whether an offer available then remains available at the next point in time. In Karni et al. [19] the probability of a successful recall is assumed to decrease with the time elapsed since the offer appearance. In Ikuta [13] the uncertainty is defined by the probability of a presently available offer becoming unavailable at the next point in time.

Furthermore, there have been other different models proposed which relate to those cited above. For instance, model with uncertain deadline ([55] [37] [5] [48] [35]) where the decision maker does not know in advance when the deadline will come, model with unknown offer value distribution ([19] [42] [40] [30] [49]) where the offer value distribution is unknown and is updated by learning as the search process proceeds, model with multiple accepted offers ([36] [47] [6]

[50] [1]) where more than one offer must be accepted, and model with a free search order ([56] [41]) where a certain number of offers with unknown value are given in advance and the decision maker inspects these one by one while considering the inspection order in addition to deciding when to stop further inspection.

1.2 Our Models

The point to be emphasized here is that, in all the above conventional models, the recallability of each offer is determined independently of the will of the decision maker. However, what if the decision maker gets the right to keep any offer available, or to reserve any offer, for certain periods in exchange for a certain compensation? Now, let us consider the following two problems:

- **House Purchasing Problem**: Suppose that you would like to purchase the best possible house within this month and that today an estate agent has introduced you to a house with potential. In order to contract the purchase of the house, or to obtain a right to purchase the house in the future, you have to make a deposit to the owner and pay a brokerage fee to the estate agent. Of course, if you do not make the contract, it is likely that the house will be sold to another person and a better house will not appear in the search process ahead. However, having made the contract, as long as you are prepared to give up the fees and cancel the contract, if necessary, you keep the option to continue the search for other houses with the chance to purchase the house of your dreams. Now, do you make the contract today? When and how do you stop your search?

- **University Entrance Problem**: Suppose that you, a high school student, have taken a certain number of university entrance examinations and just got the news of the success in University X. If X is the one you hope to enter, your ordeal will be happily over. Although you can take an examination to another university by using X as insurance, the admission to X requires a certain amount of entrance fee, which would be wasted money if you succeeded in getting to a more favorable university. By not paying the fee, you will save money but lose the chance of admission. You face a dire situation where you could end up with no university to enter. Now, how do you decide whether or not to pay the fee to University X? When and how do you decide what university to enter?

Like examples as the two above, we often encounter the problems that can be explained as optimal stopping problems in which ideas of reservation are introduced. However, only few attempts have so far been made to solve such problems. In Rose [39] each offer is allowed to be reserved for k periods in return for a cost bk with a given $b > 0$ where only one offer can be reserved at any point in time and it is prohibited to renew the reservation of an offer at the time of its maturity. A similar model is treated in Rose [38] where $k = 1$ and renewals are permitted.

However, his models belong to secretary problems and the optimal decision rules presented are characterized only by some numerical calculations.

In this thesis, in order to systematize optimal stopping problems from the viewpoint of recallability, we shall establish a mathematical method to treat optimal stopping problems where the reservation of an offer is permitted by paying the reserving cost, which depends on the value of an offer to be reserved. We shall also summarize the basic properties of the optimal decision rule for these problems.

This thesis consists of the following three basic models: The first is a modified model of the one treated in Saito [45], in which any offer once reserved is kept available at any point of time in the future. The second is a modified model of the one treated in Saito [44], in which any reservation holds effective for only finite k periods independently of the value of an offer or the reserving cost. The last can be said to be the same as the first model except for the point that the remaining time value is considered where the remaining time value is the expected reward that can be gained at the time of stopping the search. The reason why such an idea is adopted in the model is that as in the proverb "time is money," if the search for offers stops early in the process, we can ordinarily use the remaining periods for other economic activities and gain some reward.

A major result from these three models is that you must not recall and accept any reserved offer if it remains available at the next point in time, in other words, the time when recalling and accepting a reserved offer can become an optimal decision is restricted only to the maturity of its reservation.

This thesis only deals with basic properties of reservation in the three models above, in other words, the results obtained will be useful only under some restricted situations. So that, in order to fit the results to our activities in the real world and to check the sensitivity of the results with respect to the parameters of the models, we have to add more realistic assumptions as stated in Chapter 7 to our models. However, such generalized models will be treated by applying the methodology utilized in this thesis.

In Chapter 2 definitions of the models in the thesis are clearly presented. Chapter 3 is devoted to preliminaries for mathematical analysis over the thesis. Chapters 4 to 6 treat the first, second, and third models, respectively. In Chapter 7, the final chapter, we will summarize the conclusions obtained throughout this thesis and state some research subjects that have not been dealt with and are thought to be worthwhile for future studies.

Chapter 2

Definitions of the Models

In this chapter we first present two standard models of optimal stopping problems and then define the three models dealt with in the thesis.

2.1 Two Standard Models of Optimal Stopping Problems

2.1.1 Optimal Stopping Problem with No Recall

Consider a person who periodically searches for offers with the intention to accept one of them up to the deadline. For convenience, let points in time t be equally spaced and numbered backward from the deadline $t = 0$, thus t also represents the number of periods remaining.

Now, suppose that the person, the searcher, is at the point in time t, simply called *time t* from now on. Then he can find an offer with value w, simply referred to as *offer w* later on, if the *search cost* $s > 0$ was paid at the previous time, or time $t+1$ (Figure 2.1.1). He does not know in advance, however, what offer will come up. The only information available for him is that values of subsequent offers w, w', w'', $\cdots$ are independent and identically distributed random variables following a known *offer value distribution function* F, satisfying

$$\begin{cases} 0 = F(w), & w < a, \\ 0 < F(w) < 1, & a \le w < b, \\ F(w) = 1, & b \le w, \end{cases} \tag{2.1.1}$$

where a and b are real numbers such that $0 \le a < b < \infty$. Obviously, there exists the expectation of the offer value μ with $a < \mu < b$.

After inspecting the offer found at that time, the so-called *current offer*, the searcher has to decide either to stop the search by accepting it or to continue the search by passing it up. If he decides to accept the current offer w, the search stops with getting the value w. If he decides to pass up the current offer w, it disappears instantly and become unrecallable forever. Here, it is assumed that the present value of q monetary units obtained at the next time, or time $t-1$, is given by βq monetary units where β is the *discount factor* such that $0 < \beta \le 1$.

The objective of the search is to maximize the total expected discounted present net profit which will be gained in the future search process, that is, the expectation of the present discounted value of an accepted offer minus that of the amount of search costs paid over the periods from the present point in time to the termination of the search by accepting an offer.

Since no offers passed up in the past are supposed to be unrecallable, only the current offer is an available offer at any time. So, $\breve{u}_t(w)$, the maximum total expected present discounted net profit attainable by starting the search for offers from time t with the current offer w, can be expressed as

$$\breve{u}_0(w) = w, \tag{2.1.2}$$

$$\breve{u}_t(w) = \max\{w,\ -s + \beta\breve{v}_{t-1}\}, \quad t \geq 1, \tag{2.1.3}$$

where

$$\breve{v}_t = \int_a^b \breve{u}_t(w)dF(w), \quad t \geq 0. \tag{2.1.4}$$

Now, let θ be the solution of

$$\beta\int_a^b \max\{w,x\}dF(w) - x - s = 0$$

where its existence and properties will be presented in Chapter 3.

Lemma 2.1.1 $\breve{u}_t(w) \to \max\{w,\theta\}$ *as* $t \to \infty$.

PROOF. See Ikuta [13]. ∎

2.1.2 Optimal Stopping Problem with Recall

The model is exactly the same as the model with no recall except for the point that any offer once passed up can be recalled at any time in the future, thus available offers at any time are the

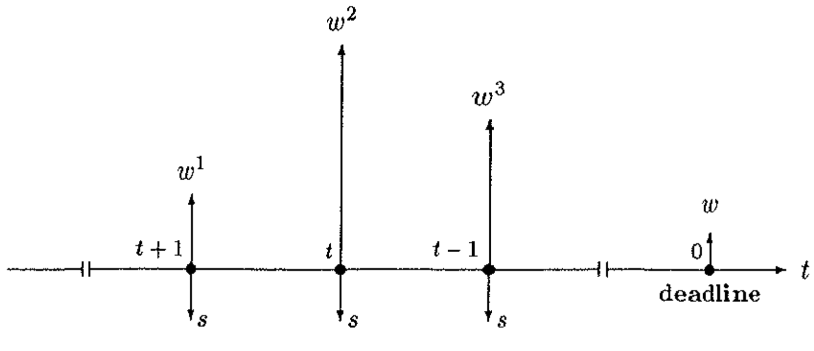


Figure 2.1.1: Offers and costs on the time frame: Model with no recall

current offer and all the offers drawn in the past. Here note that, among all the offers found so far, only the best of them is needed to be considered. So, $\tilde{u}_t(w, x)$, the maximum total expected present discounted net profit attainable by starting the search for offers from time t with the current offer w and the best offer in the past x, can be expressed as

$$\tilde{u}_0(w,x) = \max\{w,x\}, \tag{2.1.5}$$

$$\tilde{u}_t(w,x) = \max\{w,\ x,\ -s+\beta\tilde{v}_{t-1}(\max\{w,x\})\}, \quad t \geq 1, \tag{2.1.6}$$

where

$$\tilde{v}_t(x) = \int_a^b \tilde{u}_t(w,x)dF(w), \quad t \geq 0. \tag{2.1.7}$$

Lemma 2.1.2 $\tilde{u}_t(w,x) \to \max\{w,x,\theta\}$ *as* $t \to \infty$.

Proof. See Ikuta [13]. ∎

2.2 The Models in the Thesis

2.2.1 Common Assumptions

The general structure of the models dealt with in the thesis is the same as that of the standard optimal stopping problem with no recall except for the condition that the searcher is allowed each time not only to accept or pass up the current offer w but also to *reserve* it by paying the *reserving cost* $r(w)$. The structure of the reserving cost $r(w)$ will be explained later. If he reserves an offer, he gets the right to recall and accept it until its maturity (Figure 2.2.1). Of course, if he does not pay the reserving cost for an offer, it is considered to be passed up, that is, it disappears instantly and become unavailable forever.

Without loss of generality, the searcher is allowed to have a certain number of reserved offers at the beginning of the search as initial offers given before entering the search.

Here, let the *leading offer* mean the most lucrative of all available reserved offers. Then, the choices which can be taken at each time except for the deadline are the following four:

1. `AS`: Accepting the current offer and stopping the search,
2. `RC`: Reserving the current offer and continuing the search,
3. `PS`: Passing up the current offer and stopping the search by accepting the leading offer,
4. `PC`: Passing up the current offer and continuing the search,

where `AS`, `RC`, `PS`, and `PC` represent the above four choices, respectively. Of course, at the deadline, only decisions `AS` and `PS` are permitted.

The objective here is to find a rule to guide us to which action should be taken for each offer appearing so as to maximize the total expected discounted present net profit obtainable in the process ahead, that is, the expectation of the present discounted value of an accepted offer minus that of the amount of search costs and reserving costs paid over the periods from the present point in time to the termination of the search by accepting an offer.

2.2.2 The Reserving Cost

The reserving cost $r(w)$ is assumed to be continuous and nondecreasing in the offer value w with

$$0 < r(w) < \infty. \tag{2.2.1}$$

The first assumption, continuity, is made only to avoid unnecessary complication.

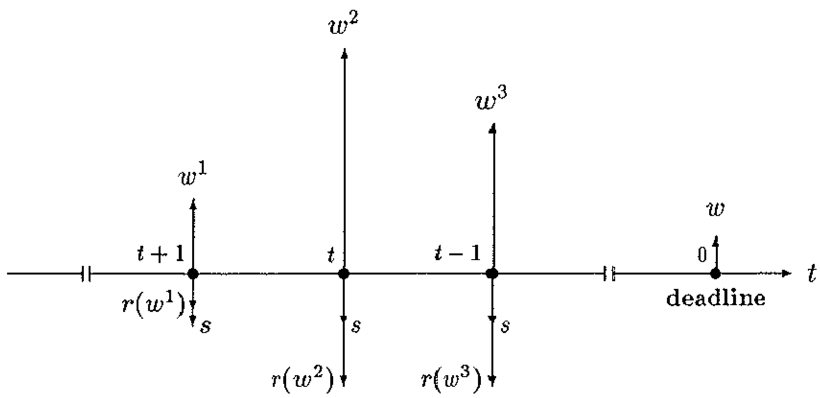
The second assumption, nondecreasing property, indicates a natural situation such that the better the offer is, the higher the reserving cost is incurred.

The final assumption, Eq. (2.2.1), allows the case $w < r(w)$ for a certain w. Although it seems strange, there does exist a case where you would have to require the searcher to pay the reserving cost even if it is higher than the offer value in order to follow the designate procedure for the reservation.

2.2.3 The Models in the Thesis

The thesis deals with the following three models:

1. Model with infinite-period reservation – The basic model – (Chapter 4).
   In the model, any offer once reserved can be recalled and accepted at any time in the future. Hence, the leading offer changes at only the time when a new offer with higher



($r(w^i)$ will be paid only for reserving the offer w^i)

Figure 2.2.1: Offers and costs on the time frame: Model with reservation

value is reserved. Thus, no reserved offers except for the leading offer are needed to be remembered at any time (the author [45]).

2. **Model with finite-period reservation.** (Chapter 5).

   In the model, any reserved offer can be recalled and accepted at any time within only k periods from its reservation to the maturity where k is finite and is fixed independently of the offer value. In the model, it is possible that the leading offer changes to a reserved offer with lower value because of the expiration of the leading offer. Hence, the latest k reserved offers must be remembered at each time (the author [44]).

3. **Model with the remaining time value.** (Chapter 6).

   The model is exactly the same as the model with infinite-period reservation in Chapter 4 except for the point that the *remaining time value* d_t is considered where it is postulated that $d_t > 0$ and d_t is strictly increasing in t. Note that the length of the remaining periods is expressed by t if we are at time t. In the model, two types of the remaining time value are treated. The one is *convex* type[1], which indicates the case where the remaining time value rises steeply as the remaining periods become longer. The other is a special case of concave type, expressed by $(\beta + \beta^2 + \cdots + \beta^t)\sigma$. We call it the *$\beta$-additive* type, implying the case where a certain reward σ will be gained per period after the termination of the search.

[1] In this thesis, "d_t is convex (concave) in t" means that $d_t - d_{t-1}$ is nondecreasing (nonincreasing) in t.

Chapter 3

Preliminaries

In this chapter we present some properties of two functions S and K, which will be often used for analyses in the subsequent chapters.

3.1 Definitions

Let β, s, a, and b denote real numbers such that $0 < \beta \leq 1$, $0 < s < \infty$, and $0 \leq a < b < \infty$. Further, let $F(w)$ be a distribution function, satisfying Eq. (2.1.1) on p.5.

Now, for any real number x, let us define two functions $S(x)$ and $K(x)$ as follows:

$$S(x) = \int_a^b \max\{w, x\} dF(w), \tag{3.1.1}$$

$$K(x) = \beta \int_a^b \max\{w, x\} dF(w) - x - s \tag{3.1.2}$$

$$= \beta S(x) - x - s. \tag{3.1.3}$$

3.2 Analysis

Lemma 3.2.1

(a) $S(x) = \mu$ *for* $x \leq a$, $x < S(x)$ *for* $x < b$, *and* $S(x) = x$ *for* $b \leq x$.

(b) $S(x) - x$ *is nonincreasing in* x, *and strictly decreasing in* $x \leq b$.

(c) $S(x)$ *is nondecreasing in* x, *and strictly increasing in* $x \geq a$.

(d) $S(x)$ *is continuous in* x.

(e) $S(x)$ *is convex in* x.

PROOF.

(a) First, let $x \leq a$. Then, for any $w \in [a, b]$ we have $x \leq w$, thus

$$S(x) = \int_a^b \max\{w, x\} dF(w) = \int_a^b w \, dF(w) = \mu.$$

Next, let $x < b$. Choose an x^1 so that $x < x^1 < b$ and let $x^2 = \max\{a, x^1\}$. Then $a \le x^2 < b$, and thus $F(b) - F(x^2) = 1 - F(x^2) > 0$ due to Eq. (2.1.1). In addition, for any $w \ge x^2$ ($\ge x^1$) we obtain $w - x \ge x^1 - x > 0$. Hence,

$$\begin{aligned} S(x) - x &= \int_a^{x^2} \max\{w - x, 0\} dF(w) + \int_{x^2}^b \max\{w - x, 0\} dF(w) \\ &\ge \int_a^{x^2} 0 \, dF(w) + \int_{x^2}^b (w - x) dF(w) \\ &\ge 0 + \int_{x^2}^b (x^1 - x) dF(w) \\ &= (x^1 - x)\big(1 - F(x^2)\big) > 0, \end{aligned}$$

from which we get $x < S(x)$ for $x < b$.

Finally, let $x \ge b$. Then, for any $w \in [a, b]$ we have $w \le x$, hence

$$S(x) = \int_a^b \max\{w, x\} dF(w) = \int_a^b x \, dF(w) = x.$$

(b) It follows from (a) that if $x \le a$, then $S(x) - x = \mu - x$, and if $b \le x$, then $S(x) - x = 0$. Thus, the cases $x \le a$ and $b \le x$ have been verified.

In order to complete the proof, it suffices to show that $S(x) - x$ is strictly decreasing in x on the interval (a, b). Choose x^1 and x^2 so that $a < x^1 < x^2 < b$. Then, $x^2 - x^1 > 0$ and $1 - F(x^2) > 0$. Further, for any $w \in [x^1, x^2]$ we have $w - x^1 \ge 0$. Thereby

$$\begin{aligned} S(x^1) - x^1 - S(x^2) + x^2 &= \int_a^b \max\{w - x^1, 0\} dF(w) - \int_a^b \max\{w - x^2, 0\} dF(w) \\ &= \int_{x^1}^{x^2} (w - x^1) dF(w) + \int_{x^2}^b (w - x^1) dF(w) - \int_{x^2}^b (w - x^2) dF(w) \\ &\ge 0 + \int_{x^2}^b (x^2 - x^1) dF(w) \\ &= (x^2 - x^1)\big(1 - F(x^2)\big) > 0, \end{aligned} \tag{3.2.1}$$

yielding $S(x^1) - x^1 > S(x^2) - x^2$. We have thus confirmed the assertion.

(c) The case $x \le a$ or $b \le x$ can be easily proven by using (a).

For the case $a < x < b$, let x^1 and x^2 be any numbers such that $a < x^1 < x^2 < b$. Then, in almost the same way as in Eq. (3.2.1), we get $S(x^1) - S(x^2) \le (x^1 - x^2)F(x^1) < 0$, which shows that the assertion holds true.

(d) For any x^1 and x^2 with $x^1 \le x^2$, it follows from (c) and (b) that $0 \le S(x^2) - S(x^1) \le x^2 - x^1$. Hence, given any $\epsilon > 0$, by letting $\delta \le \epsilon$ we deduce that if $|x^1 - x^2| < \delta$, then $|S(x^1) - S(x^2)| \le |x^1 - x^2| < \delta \le \epsilon$. Thereby, the assertion proves to be true.

(e) For any x^1, x^2, and $\rho \in (0, 1)$, we have

$$S(\rho x^1 + (1 - \rho)x^2) = \int_a^b \max\{\rho w + (1 - \rho)w, \rho x^1 + (1 - \rho)x^2\} dF(w)$$

$$\le \rho \int_a^b \max\{w, x^1\} dF(w) + (1-\rho) \int_a^b \max\{w, x^2\} dF(w)$$
$$= \rho S(x^1) + (1-\rho) S(x^2),$$

thus the assertion holds true. ∎

Here, let us define the following number α for analyses in subsequent discussions:

$$\alpha = \beta\mu - s \quad (< \mu < b). \tag{3.2.2}$$

Lemma 3.2.2

(a) $K(x) = \alpha - x$ *for* $x \le a$ *and* $K(x) = (\beta - 1)x - s$ *for* $b \le x$.

(b) $K(x)$ *is nonincreasing in* x *and strictly decreasing in* $x \le b$.

(c) $K(x)$ *is continuous in* x.

(d) $K(x)$ *is convex in* x.

PROOF.

(a-c) Immediate from Eq. (3.1.3) and Lemma 3.2.1(a,b,d), respectively.

(d) In almost the same fashion as in the proof of Lemma 3.2.1(e), we get the assertion. ∎

Now, by θ let us denote the root of $K(x) = 0$, if it exists. Then,

$$K(\theta) = \beta S(\theta) - \theta - s = 0. \tag{3.2.3}$$

Lemma 3.2.3

(a) θ *exists uniquely in* $[\alpha, b)$.

(b) $\theta = \alpha$ *if and only if* $\alpha \le a$.

PROOF.

(a) Since $S(x) \ge \mu$ for any x due to Lemma 3.2.1(a,c), it follows from Eq. (3.1.3) that $K(\alpha) = \beta S(\alpha) - (\beta\mu - s) - s = \beta(S(\alpha) - \mu) \ge 0$. Due to Lemma 3.2.2(a) we have $K(b) = (\beta - 1)b - s \le -s < 0$. From these two relations and Lemma 3.2.2(c,b) we deduce that $K(x) = 0$ has a unique root $\theta \in [\alpha, b)$.

(b) First, if $\alpha \le a$, then $K(\alpha) = \beta\mu - \alpha - s = 0$ from Lemma 3.2.2(a) and Eq. (3.2.2), thus $\theta = \alpha$ due to the uniqueness of θ. Conversely, if $a < \alpha$, then $\mu = S(a) < S(\alpha)$ due to Lemma 3.2.1(a,c), thus $K(\alpha) = \beta(S(\alpha) - \mu) > 0$ by Eq. (3.1.3). Hence, we get $\theta \in (\alpha, b)$ in almost the same manner as in the proof of (a). ∎

3.3 Technical Terms and Symbols

Finally, we shall confirm the main technical terms and symbols used in the thesis. First, let us check the meanings of the terms, current offer and leading offer:

- current offer : the offer found at the present point in time.
- leading offer : the best of all the offers that are reserved prior to the present point in time and recallable at the present point in time.

Furthermore, we list the symbols used in the paper:

t	:	point in time (length of the remaining planning horizon), $t \geq 0$.
k	:	the number of the periods for which a reservation is available (Chapter 5).
w	:	value of current offer.
x	:	value of leading offer, $x = 0$ if there are no recallable offers (Chapters 4 and 6).
$\hat{x}$	:	value of leading offer, $\hat{x} = 0$ if there are no recallable offers (Chapter 5).
F	:	distribution function of w.
a	:	minimum value of offers, $0 \leq a$.
b	:	maximum value of offers, $a < b < \infty$.
μ	:	expectation of w, thus $a < \mu < b$.
s	:	search cost, $s > 0$.
$r(w)$	:	reserving cost for the current offer with value w, continuous and nondecreasing in w with $0 < r(w) < \infty$.
β	:	discount factor, $0 < \beta \leq 1$.
α	:	$\alpha = \beta\mu - s$, defined on p.12.
θ	:	unique root of equation $K(x) = 0$ where $K(x)$ is defined on p.10.

Chapter 4

Model 1: Infinite-Period Reservation

— The Basic Model —

This chapter is devoted to the discrete-time optimal stopping problem where any of offers appearing subsequently can be reserved by paying the reserving cost and any reserved offer is allowed to be recalled and accepted at any time in the future. A major finding is that no reserved offer should be recalled and accepted prior to the deadline of the search process.

4.1 Model

Suppose that a person periodically searches for offers with the intention to accept one of them over the t periods from time t to the deadline $t = 0$. If he pays the search cost $s > 0$, an offer with value w appears where the value w is a random variable following a known offer distribution function $F(w)$, which produces $w \in [a, b]$ and has the mean μ. For an offer, he can not only accept or pass up but also reserve it if the reserving cost $r(w) > 0$ is paid. The reservation of an offer is effective forever independently of the reserving cost spent for it. His objective is to maximize the total expected present discounted net profit.

4.2 Analysis

Without loss of generality, we can consider the value of an offer to be in $(-\infty, b]$. So, for expressional simplicity, let "for any w" and "for any x" mean "for any w with $w \leq b$" and "for any x with $x \leq b$," respectively.

4.2.1 Optimal Equation

In this model, each offer once reserved is assumed to be available at any time after its reservation. Hence, the leading offer of each time is the highest of all offers reserved up to that time, and the other reserved offers can be forgotten.

By $u_t(w,x)$ let us denote the maximum total expected present discounted net profit attainable by starting the search for offers from time t with the current offer w and the leading offer x.

In addition, by $v_t(x)$ let us denote the expectation of $u_t(w,x)$ with respect to w, that is,

$$v_t(x) = \int_a^b u_t(w,x)dF(w), \quad t \geq 0. \tag{4.2.1}$$

So as to formulate $u_t(w,x)$, we shall calculate the profit attainable by taking four decisions, AS, RC, PS, and PC, respectively.

Suppose that we are at time t with the leading offer x and have just drawn an offer w.

AS: If we accept the current offer w, we receive the value w and the search is stopped.

RC: If we reserve the current offer w, we must pay the reserving cost $r(w)$. Furthermore, since we are to continue the search, the search cost s must be spent to find the next offer. Of course, the leading offer of the next time becomes the more lucrative between the current offer w and the present leading offer x, so let it be denoted by $\max\{w,x\}$. Hence, $\beta v_{t-1}(\max\{w,x\})$ is the maximum total expected present discounted net profit attainable from time $t-1$. Therefore, the maximum total expected present discounted net profit by reserving the offer w at time t is given by $-r(w) - s + \beta v_{t-1}(\max\{w,x\})$.

PS: If we pass up the current offer w and accept the leading offer x, we will stop the search with getting the value x.

PC: If we pass up the current offer w and continue the search, the search cost s is incurred. Since no offer is reserved, the leading offer of time $t-1$ remains to be x. Hence, $-s + \beta v_{t-1}(x)$ represents the maximum total expected present discounted net profit by taking the decision PC.

Of course, at the deadline $t = 0$, the search must be stopped, thus decisions RC and PC are prohibited.

On these grounds, $u_t(w,x)$ can be expressed as follows:

$$u_0(w,x) = \max\left\{\begin{array}{lll} \text{AS} & : & w, \\ \text{PS} & : & x \end{array}\right\}, \tag{4.2.2}$$

$$u_t(w,x) = \max\left\{\begin{array}{lll} \text{AS} & : & w, \\ \text{RC} & : & -r(w) - s + \beta v_{t-1}(\max\{w,x\}), \\ \text{PS} & : & x, \\ \text{PC} & : & -s + \beta v_{t-1}(x) \end{array}\right\}, \quad t \geq 1. \tag{4.2.3}$$

Note that, due to Eqs. (4.2.2), (4.2.1), and (3.1.1), we have

$$v_0(x) = \int_a^b u_0(w,x)dF(w) = \int_a^b \max\{w,x\}dF(w) = S(x). \tag{4.2.4}$$

Lemma 4.2.1 *Let c and d be real numbers with $c \leq a < b \leq d$, hence $[a,b] \subseteq [c,d]$.*

(a) *Suppose that* $u_t(w,x)$ *is continuous in* w *on* $[a,b]$ *and* x *on* $[c,d]$, *and nondecreasing and convex in* x *on* $[c,d]$ *for any* $w \in [a,b]$. *Then,* $v_t(x)$ *is continuous, nondecreasing, and convex in* x *on* $[c,d]$.

(b) *Suppose that* $v_{t-1}(x)$ *is continuous, nondecreasing, and convex in* x *on* $[c,d]$. *Then,* $u_t(w,x)$ *is continuous in* w *on* $[a,b]$ *and* x *on* $[c,d]$, *and nondecreasing and convex in* x *on* $[c,d]$ *for any* $w \in [a,b]$.

PROOF.

(a) Let x^1 and x^2 be real numbers such that $c \leq x^1 < x^2 \leq d$.

First, we shall show the continuity of $v_t(x)$ in x. Since $u_t(w,x)$ is assumed to be continuous on the compact set $[a,b] \times [c,d] \subseteq R^2$, we find that $u_t(w,x)$ is uniformly continuous there. Hence, given any $\epsilon > 0$, there exists a $\delta > 0$ such that, for any points (w^1,x^1) and (w^2,x^2) in $[a,b] \times [c,d] \subseteq R^2$, it follows that

$$\Big\|(w^1,x^1)-(w^2,x^2)\Big\| < \delta \implies \Big|u_t(w^1,x^1)-u_t(w^2,x^2)\Big| < \epsilon. \tag{4.2.5}$$

From Eq. (4.2.5) we deduce that, for any $w \in [a,b]$, if $|x^1 - x^2| < \delta$, then $\|(w,x^1)-(w,x^2)\| < \delta$, thus $|u_t(w,x^1) - u_t(w,x^2)| < \epsilon$. Hence, it follows that if $|x^1 - x^2| < \delta$, then

$$\begin{aligned}
\Big|v_t(x^1) - v_t(x^2)\Big| &= \left|\int_a^b u_t(w,x^1)dF(w) - \int_a^b u_t(w,x^2)dF(w)\right| \\
&\leq \int_a^b \Big|u_t(w,x^1) - u_t(w,x^2)\Big|\, dF(w) \\
&< \epsilon \int_a^b dF(w) \;=\; \epsilon\Big(F(b)-F(a)\Big) \;=\; \epsilon.
\end{aligned}$$

As a result, we conclude that $v_t(x)$ is continuous in x on $[c,d]$.

Next, for any $w \in [a,b]$, since $u_t(w,x)$ is assumed to be nondecreasing in x on $[c,d]$, we have $u_t(w,x^1) \leq u_t(w,x^2)$, thus

$$v_t(x^1) = \int_a^b u_t(w,x^1)dF(w) \leq \int_a^b u_t(w,x^2)dF(w) = v_t(x^2),$$

which shows that $v_t(x)$ is nondecreasing in x on $[c,d]$.

Finally, for any $w \in [a,b]$, since $u_t(w,x)$ is assumed to be convex in x on $[c,d]$, it follows for any $\rho \in (0,1)$ that $u_t(w,\rho x^1 + (1-\rho)x^2) \leq \rho u_t(w,x^1) + (1-\rho)u_t(w,x^2)$, thus

$$\begin{aligned}
v_t(\rho x^1 + (1-\rho)x^2) &= \int_a^b u_t(w,\rho x^1 + (1-\rho)x^2)dF(w) \\
&\leq \int_a^b \Big(\rho u_t(w,x^1) + (1-\rho)u_t(w,x^2)\Big)dF(w) \\
&= \rho \int_a^b u_t(w,x^1)dF(w) + (1-\rho)\int_a^b u_t(w,x^2)dF(w) \\
&= \rho v_t(x^1) + (1-\rho)v_t(x^2),
\end{aligned}$$

which indicates the convexity of $v_t(x)$.

(b) Here, we shall show the continuity of $u_t(w,x)$ in w. For any $x \in [c,d]$ we know that $\max\{w,x\}$ is continuous in w on $[a,b]$ and satisfies $c \le a \le \max\{w,x\} \le d$ for any $w \in [a,b]$, hence $v_{t-1}(\max\{w,x\})$ is continuous in w on $[a,b]$ due to the premise of the assertion. From this and the continuity of $r(w)$, we find that $-r(w) - s + \beta v_{t-1}(\max\{w,x\})$, the second expression in the braces of Eq. (4.2.3), is continuous in w on $[a,b]$ for any $x \in [c,d]$. In addition, the other three expressions in the braces of Eq. (4.2.3) are also continuous in w on $[a,b]$ for any $x \in [c,d]$ due to the premise, thus so also is $u_t(w,x)$.

The other assertions, that is, continuity, nondecreasing property, and convexity of $u_t(w,x)$ with respect to x, can be proven in a like manner. ∎

Corollary 4.2.1

(a) $u_t(w,x)$ *is :*

1. *continuous in w and x,*
2. *nondecreasing in x,*
3. *convex in x,*
4. *nondecreasing in t.*

(b) $v_t(x)$ *is :*

1. *continuous in x,*
2. *nondecreasing in x,*
3. *convex in x,*
4. *nondecreasing in t.*

PROOF. First, let us verify the properties 1 to 3 of assertions (a) and (b). Clearly, $u_0(w,x)$ has the three properties of (a) due to Eq. (4.2.2). Thereby, applying Lemma 4.2.1(a,b), and noting that c and d are arbitrary numbers with $[a,b] \subseteq [c,d]$, we conclude that the properties hold true for every t.

Next, we turn to the property 4 of assertions (a) and (b). From Eqs. (4.2.2) and (4.2.3) we have $u_0(w,x) = \max\{w,x\} \le u_1(w,x)$ for any w and x, thus for any x we get

$$v_0(x) = \int_a^b u_0(w,x)dF(w) \le \int_a^b u_1(w,x) = v_1(x). \tag{4.2.6}$$

Therefore, due to Eqs. (4.2.3) and (4.2.6) we deduce $u_1(w,x) \le u_2(w,x)$ for any w and x. Repeating this argument completes the proof. ∎

Lemma 4.2.2 *For $t \ge 0$:*

(a) $x < v_t(x)$ *for* $x < b$.

(b) $v_t(b) = b$.

(c) $\mu \le v_t(x) \le b$ *for any* x.

(d) $\beta v_t(x) - x$ *is strictly decreasing in* x.

PROOF.

(a) From Lemma 3.2.1(a), Eq. (4.2.4), and Corollary 4.2.1(b4), we obtain $x < S(x) = v_0(x) \le v_1(x) \le \cdots$ for any $x < b$.

(b) From Eq. (4.2.4) and Lemma 3.2.1(a) we get $v_0(b) = S(b) = b$. Now, suppose $v_{t-1}(b) = b$. Then, for any w we deduce $v_{t-1}(\max\{w, b\}) = v_{t-1}(b) = b$. From this and Eq. (4.2.3) we have, for any w,

$$u_t(w, b) = \max\left\{\begin{array}{l} w, \\ -r(w) - s + \beta v_{t-1}(\max\{w, b\}), \\ b, \\ -s + \beta v_{t-1}(b) \end{array}\right\} = b,$$

implying

$$v_t(b) = \int_a^b b\, dF(w) = b.$$

(c) Since $w \le u_t(w, x)$ for any w and x, we get

$$\mu = \int_a^b w\, dF(w) \le \int_a^b u_t(w, x) dF(w) = v_t(x).$$

From assertion (b) and Corollary 4.2.1(b2) we claim $v_t(x) \le b$ for any x.

(d) Choose x^1 and x^2 so that $x^1 < x^2 \le b$. Then, it follows from Corollary 4.2.1(b3) that

$$\beta \frac{v_t(x^2) - v_t(x^1)}{x^2 - x^1} \le \beta \frac{v_t(b) - v_t(x^1)}{b - x^1}. \tag{4.2.7}$$

Since $v_t(b) = b$ and $v_t(x^1) > x^1$ due to assertions (b) and (a), respectively, we get

$$\frac{v_t(b) - v_t(x^1)}{b - x^1} = \frac{b - v_t(x^1)}{b - x^1} < \frac{b - x^1}{b - x^1} = 1. \tag{4.2.8}$$

From Eqs. (4.2.7) and (4.2.8) we get $\beta v_t(x^2) - \beta v_t(x^1) < x^2 - x^1$, which immediately implies that the assertion holds true. ∎

Lemma 4.2.3 *For any* w, x, *and* $t \ge 1$,

$$\max\left\{\begin{array}{l} w, \\ -r(w) - s + \beta v_{t-1}(\max\{w, x\}), \\ x, \\ -s + \beta v_{t-1}(x) \end{array}\right\} = \max\left\{\begin{array}{l} w, \\ -r(w) - s + \beta v_{t-1}(w), \\ x, \\ -s + \beta v_{t-1}(x) \end{array}\right\}. \tag{4.2.9}$$

PROOF. If $x < w$, then $\max\{w, x\} = w$, thus Eq. (4.2.9) evidently holds true.

Let $w \le x$. Then, due to Corollary 4.2.1(b2) and $r(w) > 0$ we get

$$\begin{aligned} -r(w) - s + \beta v_{t-1}(w) &\le -r(w) - s + \beta v_{t-1}(\max\{w, x\}) \\ &= -r(w) - s + \beta v_{t-1}(x) \\ &< -s + \beta v_{t-1}(x), \end{aligned}$$

which implies that either side of Eq. (4.2.9) becomes $\max\{w, x, -s+\beta v_{t-1}(x)\}$. Consequently, Eq. (4.2.9) holds for any w and x. ∎

From Lemma 4.2.3 we conclude that $u_t(w,x)$, defined by Eq. (4.2.3), can be rewritten as

$$u_t(w,x) = \max\left\{\begin{array}{lcl} \texttt{AS} & : & w, \\ \texttt{RC} & : & -r(w) - s + \beta v_{t-1}(w), \\ \texttt{PS} & : & x, \\ \texttt{PC} & : & -s + \beta v_{t-1}(x) \end{array}\right\}, \quad t \geq 1. \tag{4.2.10}$$

4.2.2 Optimal Decision Rule

Let us define the two functions $z_t^o(x)$ and $z_t^r(w)$ as follows:

$$z_t^o(x) = \max\{x,\ -s + \beta v_{t-1}(x)\}, \qquad t \geq 1, \tag{4.2.11}$$

$$z_t^r(w) = \max\{w,\ -r(w) - s + \beta v_{t-1}(w)\}, \qquad t \geq 1, \tag{4.2.12}$$

$z_0^o(x) = x$, and $z_0^r(w) = w$. Notice that $z_0^o(x)$ is the latter term in the braces of Eq. (4.2.2), and $z_t^o(x)$ for $t \geq 1$ consists of the third and fourth terms in the braces of Eq. (4.2.10). Hence, $z_t^o(x)$ represents the maximum total expected present discounted net profit from the search by passing up the current offer w at time t with the leading offer is x. Similarly, since $z_0^r(w)$ is the former term in the braces of Eq. (4.2.2), and $z_t^r(w)$ for $t \geq 1$ consists of the first and second terms in the braces of Eq. (4.2.10), we know that $z_t^r(w)$ means the maximum total expected present discounted net profit from the search by either accepting or reserving the current offer w at time t.

As a result, if we are at time t with the leading offer x and have just drawn an offer w such that $z_t^o(x) \leq z_t^r(w)$, a higher profit can be expected by either accepting or reserving the offer w than by passing it up.

Now, by $W_t(x)$ let us denote the set of current offers that should not be passed up at time t with the leading offer x, that is,

$$W_t(x) = \{w \mid z_t^o(x) \leq z_t^r(w)\}, \quad t \geq 0. \tag{4.2.13}$$

By use of these notations, $u_t(w,x)$, defined by Eqs. (4.2.2) and (4.2.10), can be rewritten as

$$u_t(w,x) = \max\{z_t^r(w),\ z_t^o(x)\} \tag{4.2.14}$$

$$= \begin{cases} z_t^r(w) & \text{if} \quad w \in W_t(x), \\ z_t^o(x) & \text{if} \quad w \notin W_t(x), \end{cases} \qquad t \geq 0, \tag{4.2.15}$$

and thus $v_t(x)$, defined by Eq. (4.2.1), can be rewritten as

$$v_t(x) = \int_a^b \max\{z_t^r(w),\ z_t^o(x)\} dF(w) \tag{4.2.16}$$

$$= \int_{W_t(x)} z_t^r(w)dF(w) + \int_{W_t(x)^c} z_t^o(x)dF(w), \quad t \geq 0. \tag{4.2.17}$$

Theorem 4.2.1 *If $\alpha \leq a$, where α is defined on* p.12, *then $u_t(w,x) = \max\{w,x\}$ for any $w \geq a$, $x \geq 0$, and $t \geq 0$.*

PROOF. Suppose $\alpha \leq a$, or $\alpha - a \leq 0$.

The assertion is evident for $t = 0$ due to Eq.(4.2.2). Suppose $u_{t-1}(w,x) = \max\{w,x\}$ for any $w \geq a$ and $x \geq 0$. Then, $v_{t-1}(x) = S(x)$ from Eq.(4.2.1), thus $z_t^o(x) = \max\{x, -s + \beta S(x)\}$ and $z_t^r(w) = \max\{w, -r(w) - s + \beta S(w)\}$ from Eqs.(4.2.11) and (4.2.12), respectively.

For $x \leq a$, Lemma 3.2.1(a) and Eq.(3.2.2) yield $-s + \beta S(x) = -s + \beta\mu = \alpha \leq a$, hence

$$z_t^o(x) = \max\{x, -s + \beta S(x)\} \leq a. \tag{4.2.18}$$

For $x \geq a$, Eq.(3.1.3) and Lemma 3.2.2(b,a) imply $-s + \beta S(x) - x = K(x) \leq K(a) = \alpha - a \leq 0$, thus

$$z_t^o(x) = \max\{x, -s + \beta S(x)\} = x. \tag{4.2.19}$$

For $w \geq a$, from $r(w) > 0$ and Eq.(4.2.19) we get $-r(w) - s + \beta S(w) < -s + \beta S(w) \leq w$, thus

$$z_t^r(w) = \max\{w, -r(w) - s + \beta S(w)\} = w \geq a. \tag{4.2.20}$$

Consequently, from Eqs.(4.2.18) to (4.2.20) we obtain

$$\begin{cases} z_t^o(x) \leq a \leq w = z_t^r(w) & \text{if} \quad x < a \ (\leq w), \\ z_t^o(x) = x \leq w = z_t^r(w) & \text{if} \quad a \leq x \leq w, \\ z_t^r(w) = w < x = z_t^o(x) & \text{if} \ (a \leq)\ w < x. \end{cases} \tag{4.2.21}$$

Due to Eqs.(4.2.14) and (4.2.21) we arrive at

$$u_t(w,x) = \max\{z_t^r(w), z_t^o(x)\} = \begin{cases} z_t^r(w) = w & \text{if} \ x \leq w, \\ z_t^o(x) = x & \text{if} \ w < x, \end{cases}$$

from which we find that the assertion holds true. ∎

From Theorem 4.2.1, the optimal decision rule for the case $\alpha \leq a$ can be prescribed as follows:

◇ **Optimal Decision Rule:** In the case where $\alpha \leq a$, if $w \geq x$, accept the current offer w and stop the search, or else accept the leading offer x and stop the search.

From now on, let us assume $a < \alpha$.

Lemma 4.2.4

(a) *$z_t^o(x)$ is continuous, nondecreasing, and convex in x, and nondecreasing in t.*

(b) $z^r_t(w)$ *is continuous in* w, *and nondecreasing in* t.

(c) $z^r_{t+1}(w) - z^r_t(w) \le z^o_{t+1}(w) - z^o_t(w)$ *for any* w *and* $t \ge 0$.

(d) $z^o_t(x) = z^o_t(a)$ *for any* $x \le a$ *and* $t \ge 1$.

(e) $W_t(x)$ *is a closed set such that* $a \notin W_t(x)$ *and* $b \in W_t(x)$ *for any* x *and* $t \ge 1$.

PROOF.

(a,b) Both assertions are evident from Eqs. (4.2.11), (4.2.12), and Corollary 4.2.1(b).

(c) By definitions of $z^o_t(x)$ and $z^r_t(w)$ we have, for any w,

$$\begin{aligned} z^r_1(w) - z^r_0(w) &= \max\{0, -r(w) - s + \beta v_0(w) - w\} \\ &\le \max\{0, -s + \beta v_0(w) - w\} = z^o_1(w) - z^o_0(w), \end{aligned}$$

thus the assertion proves to be true for $t = 0$. For $t \ge 1$, note that the inequality of the assertion is equivalent to $z^o_t(w) - z^r_t(w) \le z^o_{t+1}(w) - z^r_{t+1}(w)$, that is,

$$\max\{w, A\} - \max\{w, A - r\} \le \max\{w, B\} - \max\{w, B - r\} \tag{4.2.22}$$

where $A = -s + \beta v_{t-1}(w)$, $B = -s + \beta v_t(w)$, and $r = r(w)$. From Corollary 4.2.1(b4) we get

$$A = -s + \beta v_{t-1}(w) \le -s + \beta v_t(w) = B. \tag{4.2.23}$$

Let L and R denote the left and right sides of Eq. (4.2.22), respectively. Then, clearly $0 \le L$ and $0 \le R$. If $w < A$, then $w < B$ by Eq. (4.2.23), thus

$$\begin{aligned} L = A - \max\{w, A - r\} &= \min\{A - w, r\} \\ &\le \min\{B - w, r\} = B - \max\{w, B - r\} = R. \end{aligned}$$

If $A \le w$, then $A - r \le w$, thus $L = 0 \le R$. Since Eq. (4.2.22) is verified, the assertion holds.

(d) If $x \le a$, then $v_0(x) = S(x) = S(a) = v_0(a)$ $(= \mu)$ from Eq. (4.2.1) and Lemma 3.2.1(a). Suppose $v_{t-1}(x) = v_{t-1}(a)$ for $x \le a$. Then, due to $a < \alpha$ and Lemma 4.2.2(c) we get

$$x \le a < \alpha = -s + \beta\mu \le -s + \beta v_{t-1}(a) = -s + \beta v_{t-1}(x). \tag{4.2.24}$$

Hence, it follows from Eqs. (4.2.11) and (4.2.24) that

$$z^o_t(x) = -s + \beta v_{t-1}(x) = -s + \beta v_{t-1}(a) = z^o_t(a). \tag{4.2.25}$$

Since $z^r_t(w)$ is independent of x, from Eqs. (4.2.16) and (4.2.25) we get $v_t(x) = v_t(a)$ for $x \le a$. Hence, $v_t(a) \le v_t(x)$ holds for any $x \le a$ and $t \ge 0$, thus the assertion proves to be true.

(e) Given any x, let $\{w^1, w^2, \cdots, w^n, \cdots\}$ and w^* be a sequence and a number, respectively, such that $w^n \in W_t(x)$ for all n and $w^n \to w^*$ as $n \to \infty$.

Suppose $w^* \notin W_t(x)$, that is, $z^r_t(w^*) < z^o_t(x)$. Then, due to $z^o_t(x) - z^r_t(w^*) > 0$ and (b), we can pick an $\epsilon > 0$ such that if $|w - w^*| < \epsilon$, then $|z^r_t(w) - z^r_t(w^*)| < z^o_t(x) - z^r_t(w^*)$. For such an $\epsilon > 0$ we have

$$|w - w^*| < \epsilon \implies z_t^r(w) < z_t^o(x) \iff w \notin W_t(x). \tag{4.2.26}$$

Since $w^n \in W_t(x)$ is assumed for every n, it follows from the contraposition of Eq. (4.2.26) that $|w^n - w^*| \geq \epsilon > 0$ for all n, which contradicts the premise that $w^n \to w^*$ as $n \to \infty$. Thereby, we conclude that w^* must be an element of $W_t(x)$, which implies that $W_t(x)$ is a closed set.

Next, it follows from $a < \alpha$ and Lemma 4.2.2(c) that

$$a < \alpha = -s + \beta\mu \leq -s + \beta v_{t-1}(x). \tag{4.2.27}$$

Since (d) immediately implies $v_t(a) \leq v_t(x)$ for any x, it follows from $r(a) > 0$ that

$$-r(a) - s + \beta v_{t-1}(a) < -s + \beta v_{t-1}(x). \tag{4.2.28}$$

Hence, due to Eqs. (4.2.27) and (4.2.28) we arrive at

$$z_t^r(a) = \max\{a, -r(a) - s + \beta v_{t-1}(a)\} < -s + \beta v_{t-1}(x) \leq z_t^o(x),$$

which suggests $a \notin W_t(x)$ from Eq. (4.2.13).

Finally, due to Lemma 4.2.2(b,c) we get $v_{t-1}(x) \leq v_{t-1}(b) = b$ for any x, thus

$$z_t^o(x) \leq \max\{b, -s + \beta b\} = b = \max\{b, -r(b) - s + \beta b\} = z_t^r(b).$$

Therefore, $b \in W_t(x)$. ∎

Here, we define the two functions $g_t(x)$ and $f_t(w)$ with $t \geq 1$ as follows:

$$g_t(x) = -s + \beta v_{t-1}(x) - x, \qquad t \geq 1, \tag{4.2.29}$$

$$f_t(w) = -r(w) - s + \beta v_{t-1}(w) - w, \qquad t \geq 1. \tag{4.2.30}$$

Corollary 4.2.2 *For $t \geq 1$:*

(a) *$g_t(x)$ is continuous and strictly decreasing in x.*

(b) *$f_t(w)$ is continuous and strictly decreasing in w.*

PROOF. Since $r(w)$ is assumed to be continuous and nondecreasing in w, the assertions hold true from Eqs. (4.2.29), (4.2.30), Corollary 4.2.1(b1), and Lemma 4.2.2(d). ∎

Let us introduce θ_t and λ_t with $t \geq 1$ as the respective roots of $g_t(x) = 0$ and $f_t(w) = 0$, if they exist, that is,

$$g_t(\theta_t) = -s + \beta v_{t-1}(\theta_t) - \theta_t = 0, \qquad t \geq 1, \tag{4.2.31}$$

$$f_t(\lambda_t) = -r(\lambda_t) - s + \beta v_{t-1}(\lambda_t) - \lambda_t = 0, \qquad t \geq 1. \tag{4.2.32}$$

From Eq. (4.2.11) we find that θ_t is a point of indifference between accepting the leading offer x

and continuing the search. From Eq. (4.2.12) we know that λ_t is a point of indifference between accepting the current offer w and reserving it.

Lemma 4.2.5 *For $t \geq 1$:*

(a) θ_t *exists uniquely with* $\alpha \leq \theta_t < b$.

(b) λ_t *exists uniquely with* $\alpha - r(b) \leq \lambda_t < \theta_t$.

PROOF.

(a) By using Lemma 4.2.2(c) and Eq. (3.2.2), we have

$$g_t(\alpha) = -s + \beta v_{t-1}(\alpha) - \alpha = \beta\Big(v_{t-1}(\alpha) - \mu\Big) \geq 0. \tag{4.2.33}$$

Due to Lemma 4.2.2(b) we get

$$g_t(b) = -s + \beta v_{t-1}(b) - b = -s + \beta b - b = (\beta - 1)b - s \leq -s < 0. \tag{4.2.34}$$

From Eqs. (4.2.33), (4.2.34), and Corollary 4.2.2(a), we conclude that equation $g_t(x) = 0$ has a unique root $\theta_t \in [\alpha, b)$.

(b) Since $\alpha - r(b) < b$, we get $r(\alpha - r(b)) \leq r(b)$. From this and Lemma 4.2.2(c) we have

$$\begin{aligned} f_t(\alpha - r(b)) &= -r(\alpha - r(b)) - s + \beta v_{t-1}(\alpha - r(b)) - (\alpha - r(b)) \\ &= r(b) - r(\alpha - r(b)) + \beta\Big(v_{t-1}(\alpha - r(b)) - \mu\Big) \geq 0. \end{aligned} \tag{4.2.35}$$

Since $g_t(\theta_t) = 0$ by Eq. (4.2.31), it follows from Eqs. (4.2.29) and (4.2.30) that

$$f_t(\theta_t) = -r(\theta_t) + g_t(\theta_t) = -r(\theta_t) < 0. \tag{4.2.36}$$

Due to Eqs. (4.2.35), (4.2.36), and Corollary 4.2.2(b), we claim that equation $f_t(w) = 0$ has a unique root $\lambda_t \in [\alpha - r(b), \theta_t)$. ∎

Corollary 4.2.3

(a) *For $t \geq 1$:*

1. *If $x < \theta_t$, then $x < -s + \beta v_{t-1}(x)$.*
2. *If $x = \theta_t$, then $x = -s + \beta v_{t-1}(x)$.*
3. *If $x > \theta_t$, then $x > -s + \beta v_{t-1}(x)$.*

(b) *For $t \geq 1$:*

1. *If $w < \lambda_t$, then $w < -r(w) - s + \beta v_{t-1}(w)$.*
2. *If $w = \lambda_t$, then $w = -r(w) - s + \beta v_{t-1}(w)$.*
3. *If $w > \lambda_t$, then $w > -r(w) - s + \beta v_{t-1}(w)$.*

PROOF. Both assertions are evident from Corollary 4.2.2 and the uniqueness of θ_t and λ_t. ∎

According to Eq. (4.2.15) and Corollary 4.2.3 we obtain, for $t \geq 1$,

$$u_t(w,x) = \begin{cases} z_t^r(w) = \begin{cases} w & \text{if } w \in W_t(x) \text{ and } \lambda_t < w, \\ -r(w) - s + \beta v_{t-1}(w) & \text{if } w \in W_t(x) \text{ and } w \leq \lambda_t, \end{cases} \\ z_t^o(x) = \begin{cases} x & \text{if } w \notin W_t(x) \text{ and } \theta_t < x, \\ -s + \beta v_{t-1}(x) & \text{if } w \notin W_t(x) \text{ and } x \leq \theta_t. \end{cases} \end{cases} \tag{4.2.37}$$

Now, let $\theta_0 = \lambda_0 = -\infty$ for convenience. Then, in general, by Eq. (4.2.37) we can prescribe the optimal decision rule as follows:

◇ **Optimal Decision Rule:** Suppose that you are at time t with the leading offer x and have just drawn an offer w. Then, the choices are:

(a) If $w \in W_t(x)$, then:

1. If $\lambda_t < w$, then `AS` (accept the current offer w and stop the search).
2. If $w \leq \lambda_t$, then `RC` (reserve the current offer w and continue the search).

(b) If $w \notin W_t(x)$, then:

1. If $\theta_t < x$, then `PS` (pass up the current offer w and stop the search by accepting the leading offer x).
2. If $x \leq \theta_t$, then `PC` (pass up the current offer w and continue the search).

Lemma 4.2.6 *For $t \geq 1$ we have*

$$z_t^r(w) \begin{cases} < \theta_t & \textit{if } w < \theta_t, \\ = w & \textit{if } \theta_t \leq w. \end{cases} \tag{4.2.38}$$

Proof. If $w < \theta_t$, it follows from Corollaries 4.2.1(b2) and 4.2.3(a2) that

$$-r(w) - s + \beta v_{t-1}(w) \leq -r(w) - s + \beta v_{t-1}(\theta_t) < -s + \beta v_{t-1}(\theta_t) = \theta_t,$$

from which

$$z_t^r(w) = \max\{w, -r(w) - s + \beta v_{t-1}(w)\} < \max\{\theta_t, \theta_t\} = \theta_t. \tag{4.2.39}$$

If $\theta_t \leq w$, then $\lambda_t < w$ since $\lambda_t < \theta_t$ due to Lemma 4.2.5(b). Hence, it follows from Corollary 4.2.3(b3) that

$$z_t^r(w) = \max\{w, -r(w) - s + \beta v_{t-1}(w)\} = w. \tag{4.2.40}$$

By using Eqs. (4.2.39) and (4.2.40), we immediately confirm Eq. (4.2.38). ∎

Lemma 4.2.7 *For $t \geq 1$ we have $\theta_t = \theta_{t+1}$ if and only if $v_{t-1}(\theta_t) = v_t(\theta_t)$.*

Proof. If $\theta_t = \theta_{t+1}$, due to Eq. (4.2.31) we obtain $g_t(\theta_t) = 0$ and $g_{t+1}(\theta_t) = g_{t+1}(\theta_{t+1}) = 0$, so $g_t(\theta_t) = g_{t+1}(\theta_t)$, or $-s + \beta v_{t-1}(\theta_t) - \theta_t = -s + \beta v_t(\theta_t) - \theta_t$, from which $v_{t-1}(\theta_t) = v_t(\theta_t)$.

Conversely, if $v_{t-1}(\theta_t) = v_t(\theta_t)$, it follows from Eq. (4.2.31) that

$$g_{t+1}(\theta_t) = -s + \beta v_t(\theta_t) - \theta_t = -s + \beta v_{t-1}(\theta_t) - \theta_t = g_t(\theta_t) = 0. \tag{4.2.41}$$

Since θ_{t+1} is the unique root of $g_{t+1}(x) = 0$, we deduce $\theta_t = \theta_{t+1}$ from Eq. (4.2.41). ∎

Theorem 4.2.2

(a) $\theta_t = \theta$ *for each* $t \geq 1$ *where* θ *is a unique root of* $K(x) = 0$, *defined on* p.12.

(b) λ_t *is nondecreasing in* t.

PROOF.

(a) It follows from Eq. (4.2.4) that

$$v_0(\theta_1) = \int_a^b \max\{w, \theta_1\} dF(w) = \int_a^{\theta_1} \theta_1 \ dF(w) + \int_{\theta_1}^b w \ dF(w). \tag{4.2.42}$$

From Corollary 4.2.3(a2) we have $z_1^o(\theta_1) = \theta_1$. Hence, due to Eq. (4.2.16) and Lemma 4.2.6 we get

$$\begin{aligned} v_1(\theta_1) &= \int_a^b \max\{z_1^r(w), \theta_1\} dF(w) \\ &= \int_a^{\theta_1} \max\{z_1^r(w), \theta_1\} dF(w) + \int_{\theta_1}^b \max\{z_1^r(w), \theta_1\} dF(w) \\ &= \int_a^{\theta_1} \theta_1 \ dF(w) + \int_{\theta_1}^b w \ dF(w). \end{aligned} \tag{4.2.43}$$

Since $v_0(\theta_1) = v_1(\theta_1)$ from Eqs. (4.2.42) and (4.2.43), we find $\theta_1 = \theta_2$ from Lemma 4.2.7.

Next, assume $\theta_t = \theta_{t+1}$, or equivalently, $v_{t-1}(\theta_t) = v_t(\theta_t)$. Then, in exactly the same way as in Eq. (4.2.43) we obtain

$$\begin{aligned} v_t(\theta_{t+1}) = v_t(\theta_t) &= \int_a^b \max\{z_t^r(w), \theta_t\} dF(w) \\ &= \int_a^{\theta_t} \theta_t \ dF(w) + \int_{\theta_t}^b w \ dF(w) \\ &= \int_a^{\theta_{t+1}} \theta_{t+1} \ dF(w) + \int_{\theta_{t+1}}^b w \ dF(w) \\ &= \int_a^b \max\{z_{t+1}^r(w), \theta_{t+1}\} dF(w) = v_{t+1}(\theta_{t+1}), \end{aligned}$$

from which $\theta_{t+1} = \theta_{t+2}$. Therefore, by induction we claim $\theta_t = \theta_{t+1}$ for all $t \geq 1$.

Now, from Eqs. (4.2.4) and (3.2.3) we have

$$g_1(\theta) = -s + \beta v_0(\theta) - \theta = -s + \beta S(\theta) - \theta = 0. \tag{4.2.44}$$

Since θ_1 exists uniquely, we get $\theta_1 = \theta$ due to Eq. (4.2.44). Hence, we conclude $\theta_t = \theta$ for each $t \geq 1$.

(b) Clearly, $f_t(w)$ is nondecreasing in t due to Eq. (4.2.30) and Corollary 4.2.1(b4). Hence, from Eq. (4.2.32) we get $0 = f_{t+1}(\lambda_{t+1}) = f_t(\lambda_t) \leq f_{t+1}(\lambda_t)$, thus $f_{t+1}(\lambda_t) \geq f_{t+1}(\lambda_{t+1})$. From this and Corollary 4.2.2(b) we get $\lambda_t \leq \lambda_{t+1}$, thus the assertion holds true. ∎

Theorem 4.2.3 *For $t \geq 0$:*

(a) *For any x, if $w \in W_t(x)$, then $x \leq w$.*

(b) *If $x^1 < x^2$, then $W_t(x^1) \supseteq W_t(x^2)$.*

PROOF.

(a) It is clear for $t = 0$ by definitions of $W_0(x)$, $z_0^o(x)$ and $z_0^r(w)$. The case for $t \geq 1$ is proven by contraposition. If $w < x$, then $v_{t-1}(w) \leq v_{t-1}(x)$ by Corollary 4.2.1(b2), thus $-r(w) - s + \beta v_{t-1}(w) \leq -r(w) - s + \beta v_{t-1}(x) < -s + \beta v_{t-1}(x)$, hence

$$z_t^r(w) = \max\{w, -r(w) - s + \beta v_{t-1}(w)\} < \max\{x, -s + \beta v_{t-1}(x)\} = z_t^o(x),$$

which indicates $w \notin W_t(x)$. Thereby, the assertion proves to be true.

(b) Let $x^1 < x^2$. We shall show that any $w \in W_t(x^2)$ is also an element of $W_t(x^1)$. Choose any $w \in W_t(x^2)$. Then, from Eq. (4.2.13) we get

$$z_t^o(x^2) \leq z_t^r(w). \tag{4.2.45}$$

Due to Lemma 4.2.4(a) we obtain

$$z_t^o(x^1) \leq z_t^o(x^2). \tag{4.2.46}$$

From Eqs. (4.2.45) and (4.2.46) we arrive at $z_t^o(x^1) \leq z_t^r(w)$, which means $w \in W_t(x^1)$. ∎

Theorem 4.2.4 *Let $\theta \leq x$. Then :*

(a) $W_t(x) = \{w \mid x \leq w\}$.

(b) $u_t(w, x) = \max\{w, x\}$

(c) $v_t(x) = S(x)$.

PROOF. Note that $\theta_t = \theta$ from Theorem 4.2.2(a). Then, since $\theta \leq x$ is assumed, it follows from Corollary 4.2.3(a2,a3) that

$$z_t^o(x) = \max\{x, -s + \beta v_{t-1}(x)\} = x \geq \theta. \tag{4.2.47}$$

(a) From Theorem 4.2.3(a) we have

$$W_t(x) \subseteq \{w \mid x \leq w\}. \tag{4.2.48}$$

Suppose $\theta \leq x \leq w$. Then, $\lambda_t < w$ since $\lambda_t < \theta$ by Lemma 4.2.5(b). Hence, due to Eq. (4.2.47) and Corollary 4.2.3(b2,b3) we have $z_t^o(x) = x \leq w = z_t^r(w)$, thus $w \in W_t(x)$. Consequently, if $\theta \leq x$, then

$$\{w \mid x \leq w\} \subseteq W_t(x). \tag{4.2.49}$$

Eqs. (4.2.48) and (4.2.49) prove the assertion to be true.

(b) For $t = 0$, the assertion is clear from Eq. (4.2.2). Using Lemma 4.2.6, we get

$$z_t^r(w) \begin{cases} < \theta \leq x & \text{if } -\infty < w < \theta, \\ = w < x & \text{if } \theta \leq w < x, \\ = w & \text{if } x \leq w \leq b, \end{cases}$$

from which

$$z_t^r(w) \begin{cases} < x & \text{if } w < x, \\ = w & \text{if } x \leq w. \end{cases} \tag{4.2.50}$$

Hence, if $w < x$, then $w \notin W_t(x)$ due to (a), thus $u_t(w,x) = z_t^o(x) = x$ from Eqs. (4.2.15) and (4.2.47). Contrarily, if $x \leq w$, then $w \in W_t(x)$ due to (a), hence $u_t(w,x) = z_t^r(w) = w$ from Eqs. (4.2.15) and (4.2.50). We have thus confirmed the assertion.

(c) It follows from Eq. (4.2.1) and (b) that, for any x with $\theta \leq x$,

$$v_t(x) = \int_a^b \max\{w, x\} dF(w) = S(x),$$

thus the assertion proves to be true. ∎

From Theorem 4.2.4(b), the optimal decision rule for the case $\theta \leq x$ can be prescribed as follows:

◇ **Optimal Decision Rule:** In the case where $\theta \leq x$, if $w \geq x$, accept the current offer w and stop the search, or else accept the leading offer x and stop the search.

Lemma 4.2.8

(a) $v_t(x) - v_{t-1}(x)$ *is nonincreasing in* $x \leq \theta$ *for any* $t \geq 1$.

(b) $z_{t+1}^o(x) - z_t^o(x)$ *is nonincreasing in* x *for any* $t \geq 0$.

PROOF.

(a) Choose x^1 and x^2 so that $x^1 < x^2 \leq \theta$, and let $W_1 = W_t(x^1)$ and $W_2 = W_t(x^2)$. Then, from Eq. (4.2.17) we get

$$v_t(x^1) = \int_{W_1} z_t^r(w) dF(w) + \int_{W_1^c} z_t^o(x^1) dF(w), \tag{4.2.51}$$

$$v_t(x^2) = \int_{W_2} z_t^r(w) dF(w) + \int_{W_2^c} z_t^o(x^2) dF(w). \tag{4.2.52}$$

Since $W_1 \supseteq W_2$ due to Theorem 4.2.3(b), we have $W_1 = W_2 \cup (W_1 \cap W_2^c)$ and $W_2^c = W_1^c \cup (W_1 \cap W_2^c)$. Hence, due to Eqs. (4.2.51) and (4.2.52) we get

$$v_t(x^2) - v_t(x^1) = \int_{W_1^c} \Big(z_t^o(x^2) - z_t^o(x^1)\Big) dF(w) + \int_{W_1 \cap W_2^c} \Big(z_t^o(x^2) - z_t^r(w)\Big) dF(w). \tag{4.2.53}$$

Since $z_t^o(x^1) \le z_t^r(w)$ for any $w \in W_1$ due to Eq. (4.2.13), we claim

$$z_t^o(x^2) - z_t^r(w) \le z_t^o(x^2) - z_t^o(x^1) \quad \text{if} \quad w \in W_1 \cap W_2^c \ (\subseteq W_1). \tag{4.2.54}$$

It follows from Corollary 4.2.3(a1,a2) that if $x^1 < x^2 \le \theta \ (= \theta_t)$, then

$$z_t^o(x^2) - z_t^o(x^1) = -s + \beta v_{t-1}(x^2) + s - \beta v_{t-1}(x^1) = \beta\Big(v_{t-1}(x^2) - v_{t-1}(x^1)\Big). \tag{4.2.55}$$

Hence, by using Eqs. (4.2.53) to (4.2.55), we arrive at

$$\begin{aligned}
v_t(x^2) - v_t(x^1) &\le \int_{W_1^c}\Big(z_t^o(x^2) - z_t^o(x^1)\Big)dF(w) + \int_{W_1\cap W_2^c}\Big(z_t^o(x^2) - z_t^o(x^1)\Big)dF(w)\\
&= \Big(z_t^o(x^2) - z_t^o(x^1)\Big)\int_{W_2^c} dF(w)\\
&= \beta\Big(v_{t-1}(x^2) - v_{t-1}(x^1)\Big)\int_{W_2^c} dF(w)\\
&\le v_{t-1}(x^2) - v_{t-1}(x^1),
\end{aligned}$$

from which $v_t(x^1) - v_{t-1}(x^1) \ge v_t(x^2) - v_{t-1}(x^2)$. We have thus confirmed the assertion.

(b) We have

$$z_1^o(x) - z_0^o(x) = \max\{0, -s + \beta v_0(x) - x\} = \max\{0, g_1(x)\}. \tag{4.2.56}$$

From Corollary 4.2.2(a) and Eq. (4.2.56), the assertion proves to be true for $t = 0$. For $t \ge 1$, since $\theta_t = \theta_{t+1} = \theta$ due to Theorem 4.2.2(a), it follows from Corollary 4.2.3(a) that

$$z_{t+1}^o(x) - z_t^o(x) = \begin{cases} \beta\Big(v_t(x) - v_{t-1}(x)\Big) & \text{if} \ \ x \le \theta,\\ 0 & \text{if} \ \ \theta \le x. \end{cases} \tag{4.2.57}$$

Due to (a) and Eq. (4.2.57) we have completed the proof of (b). ∎

Theorem 4.2.5 *$W_t(x) \supseteq W_{t+1}(x)$ for any x and $t \ge 0$.*

Proof. Suppose $x \le w$. Then, due to Lemmas 4.2.4(c) and 4.2.8(b) we have

$$z_{t+1}^r(w) - z_t^r(w) \le z_{t+1}^o(w) - z_t^o(w) \le z_{t+1}^o(x) - z_t^o(x),$$

yielding

$$z_{t+1}^r(w) - z_{t+1}^o(x) \le z_t^r(w) - z_t^o(x),$$

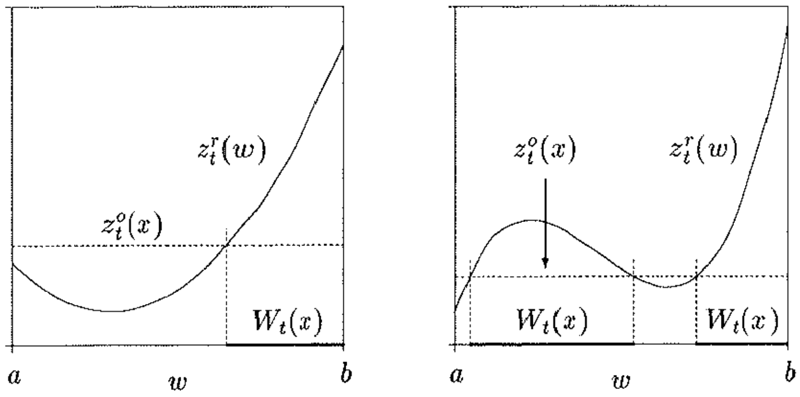
from which

$$0 \le z_{t+1}^r(w) - z_{t+1}^o(x) \implies 0 \le z_t^r(w) - z_t^o(x),$$

or equivalently,

$$z_{t+1}^o(x) \le z_{t+1}^r(w) \implies z_t^o(x) \le z_t^r(w). \tag{4.2.58}$$

By noting $x \le w$ for any $w \in W_{t+1}(x)$ from Theorem 4.2.3(a), we conclude that any $w \in W_{t+1}(x)$ also belongs to $W_t(x)$ due to Eq. (4.2.58). The proof is completed. ∎



(a) $r(w)$ is concave (b) $r(w)$ is not concave

Figure 4.2.1: Relations among $z_t^o(x)$, $z_t^r(w)$, and $W_t(x)$

Theorem 4.2.6 *If $r(w)$ is concave, $W_t(x)$ is a connected set for any x and $t \geq 0$.*

PROOF. We immediately get $W_0(x) = \{w \mid x \leq w\}$ for any $r(w)$ by Eq. (4.2.13), thus we have confirmed the assertion for $t = 0$.

Given any $t \geq 1$ and x, suppose that $W_t(x)$ is a union of two disjoint sets. Then, since $W_t(x)$ is a closed set and $a \notin W_t(x)$ from Lemma 4.2.4(e), we can rewrite $W_t(x)$ as

$$W_t(x) = [w^0, w^1] \cup [w^2, b] \tag{4.2.59}$$

where w^0, w^1 and w^2 are real numbers such that

$$a < w^0 \leq w^1 < w^2 \leq b. \tag{4.2.60}$$

Due to Eqs. (4.2.13) and (4.2.59) we get

$$\begin{cases} z_t^o(x) \leq z_t^r(w) & \text{if } \; w \in [w^0, w^1] \cup [w^2, b] \; (= W_t(x)), \\ z_t^r(w) < z_t^o(x) & \text{otherwise.} \end{cases} \tag{4.2.61}$$

It follows from Eq. (4.2.61) that if $w^1 < w < w^2$, then

$$z_t^r(w) < z_t^o(x) \leq z_t^r(w^2). \tag{4.2.62}$$

Since $z_t^r(w)$ is continuous in w by Lemma 4.2.4(b), we have

$$\lim_{w \to w^2} z_t^r(w) = z_t^r(w^2). \tag{4.2.63}$$

From Eqs. (4.2.62) and (4.2.63) we deduce $z_t^o(x) = z_t^r(w^2)$. Similarly, we get $z_t^o(x) = z_t^r(w^1)$. Due to Lemma 4.2.4(e) we obtain $a \notin W_t(x)$, or $z_t^r(a) < z_t^o(x)$. Accordingly,

$$z_t^r(a) < z_t^o(x) = z_t^r(w^1) = z_t^r(w^2),$$

from which

$$\frac{z_t^r(w^1) - z_t^r(a)}{w^1 - a} > 0 = \frac{z_t^r(w^2) - z_t^r(w^1)}{w^2 - w^1}. \tag{4.2.64}$$

Since $r(w)$ is assumed to be concave in w, we know that $-r(w) - s + \beta v_{t-1}(w)$ is convex in w by Corollary 4.2.1(b3), thus so also is $z_t^r(w)$. Thereby, it follows from Eq. (4.2.60) that

$$\frac{z_t^r(w^1) - z_t^r(a)}{w^1 - a} \leq \frac{z_t^r(w^2) - z_t^r(w^1)}{w^2 - w^1}, \tag{4.2.65}$$

contradicting Eq. (4.2.64). Hence, it is impossible that $W_t(x)$ is a union of two disjoint sets.

Even though $W_t(x)$ is assumed to be a union of more than three disjoint sets, we also have contradiction in the same way as above. Therefore, the assertion proves to be true. ∎

4.2.3 Relationships between the Model and Standard Two Models

Although $r(w) > 0$ has been assumed in Eq. (2.2.1), we allow $r(w) = 0$ only in the following theorem so as to check the relationships between this model and standard models with no recall and with recall, presented on p.6.

Theorem 4.2.7

(a) *If $r(w) \geq w$ for any w and there exist no initial offers, this model results in the conventional model with no recall.*

(b) *If $r(w) = 0$ for any w, this model results in the conventional model with recall.*

PROOF.

(a) Let $r(w) \geq w$ for any w. Then, it follows from Lemma 4.2.2(d) and Corollary 4.2.1(b2) that, for any $w > 0$ and $t \geq 1$,

$$-r(w) - s + \beta v_{t-1}(w) \leq -w - s + \beta v_{t-1}(w) < -0 - s + \beta v_{t-1}(0) \leq -s + \beta v_{t-1}(x), \tag{4.2.66}$$

which means that decision PC is better than RC for any w and x. In other words, no offers should be reserved throughout the search process. Hence, if initial offers do not exist, we shall have no reserved offers throughout the search process, thus we cannot recall any offer at any time. Note that the leading offer x of each time can be regarded as $x = 0$ in this case.

In order to complete the proof, it suffices to show $u_t(w, 0) = \check{u}_t(w)$ for any w. From Eqs. (4.2.2) and (2.1.2) we get $u_0(w, 0) = \max\{w, 0\} = w = \check{u}_0(w)$. If $u_{t-1}(w, 0) = \check{u}_{t-1}(w)$ for any w, then $v_{t-1}(0) = \check{v}_{t-1}$, thus it follows from Eqs. (4.2.10), (2.1.2), and (4.2.66) that

$$\begin{aligned} u_t(w, 0) &= \max \left\{ \begin{array}{l} w, \\ -r(w) - s + \beta v_{t-1}(w), \\ 0, \\ -s + \beta v_{t-1}(0) \end{array} \right\} \\ &= \max\{w, -s + \beta v_{t-1}(0)\} \\ &= \max\{w, -s + \beta \check{v}_{t-1}\} \\ &= \check{u}_t(w). \end{aligned}$$

(b) It suffices to show that all offers should be reserved and that $u_t(w, x) = \tilde{u}_t(w, x)$ for any w and x. From Eqs. (4.2.2) and (2.1.5) we get $u_0(w, x) = \tilde{u}_0(w, x)$.

Assume $u_{t-1}(w, x) = \tilde{u}_{t-1}(w, x)$ for any w and x. Then, $v_{t-1}(x) = \tilde{v}_{t-1}(x)$. Thus, it follows from Corollary 4.2.1(b2) and the premise $r(w) = 0$ that, for any w and x,

$$-r(w) - s + \beta v_{t-1}(\max\{w, x\}) = -s + \beta v_{t-1}(\max\{w, x\}) \geq -s + \beta v_{t-1}(x). \tag{4.2.67}$$

Hence, due to Eqs. (4.2.3), (4.2.67), and (2.1.6), we lead to

$$\begin{aligned} u_t(w, x) &= \max \left\{ \begin{array}{l} w, \\ -s + \beta v_{t-1}(\max\{w, x\}), \\ x, \\ -s + \beta v_{t-1}(x) \end{array} \right\} \\ &= \max\{w, x, -s + \beta v_{t-1}(\max\{w, x\})\} \qquad (4.2.68) \\ &= \max\{w, x, -s + \beta \tilde{v}_{t-1}(\max\{w, x\})\} \\ &= \tilde{u}_t(w, x). \qquad (4.2.69) \end{aligned}$$

Eq. (4.2.68) shows that any offer should be reserved when we continue the search. From this and Eq. (4.2.69) we have confirmed the assertion. ∎

Lemma 4.2.9 *$\breve{u}_t(w) \leq u_t(w, x) \leq \tilde{u}_t(w, x)$ for any w, x, and $t \geq 0$.*

PROOF. From Eqs. (4.2.2), (2.1.5), and (2.1.2), we have $\breve{u}_0(w) = w \leq \max\{w, x\} = u_0(w, x) = \tilde{u}_0(w, x)$. Suppose $\breve{u}_{t-1}(w) \leq u_{t-1}(w, x) \leq \tilde{u}_{t-1}(w, x)$ for any w and x. Then, $\breve{v}_{t-1} \leq v_{t-1}(x) \leq \tilde{v}_{t-1}(x)$ for any x. Thus, it follows from Eqs. (4.2.30) and (2.1.3) that

$$\breve{u}_t(w) = \max\{w, -s + \beta \breve{v}_{t-1}\} \leq \max\{w, -s + \beta v_{t-1}(x)\} \leq u_t(w, x). \tag{4.2.70}$$

Furthermore, due to Corollary 4.2.1(b2) we obtain

$$-r(w) - s + \beta v_{t-1}(\max\{w, x\}) < -s + \beta \tilde{v}_{t-1}(\max\{w, x\}) \tag{4.2.71}$$

and

$$-s + \beta v_{t-1}(x) \leq -s + \beta v_{t-1}(\max\{w, x\}) \leq -s + \beta \tilde{v}_{t-1}(\max\{w, x\}). \tag{4.2.72}$$

From Eqs. (4.2.3), (4.2.71), (4.2.72), and (2.1.6), we get

$$\begin{aligned} u_t(w, x) &\leq \max\{w, -s + \beta \tilde{v}_{t-1}(\max\{w, x\}), x, -s + \beta \tilde{v}_{t-1}(\max\{w, x\})\} \\ &= \max\{w, x, -s + \beta \tilde{v}_{t-1}(\max\{w, x\})\} \;=\; \tilde{u}_t(w, x). \qquad (4.2.73) \end{aligned}$$

Due to Eqs. (4.2.70) and (4.2.73) we obtain $\breve{u}_t(w) \leq u_t(w, x) \leq \tilde{u}_t(w, x)$. ∎

4.2.4 Infinite Planning Horizon

Theorem 4.2.8 *$u_t(w,x)$ converges to $u(w,x) = \max\{w,x,\theta\}$ as $t \to \infty$.*

PROOF. It follows from Lemmas 2.1.1, 2.1.2, and 4.2.9 that $u_t(w,x)$ converges to a function $u(w,x)$ such that

$$\max\{w,\theta\} \leq u(w,x) \leq \max\{w,x,\theta\}. \tag{4.2.74}$$

From Eq. (4.2.74) we get

$$u(w,x) = \max\{w,\theta\} \quad \text{if} \quad x < \theta. \tag{4.2.75}$$

Due to Theorem 4.2.4(b) we immediately obtain

$$u(w,x) = \max\{w,x\} \quad \text{if} \quad \theta \leq x. \tag{4.2.76}$$

From Eqs. (4.2.75) and (4.2.76) we have confirmed the theorem. ∎

This theorem yields the optimal decision rule for an infinite planning horizon:

◇ **Optimal Decision Rule:** In the case of an infinite planning horizon, if $\theta \leq \max\{w,x\}$, accept the more lucrative between the current offer w and the leading offer x, or else continue the search.

4.3 Numerical Example

We here depict an example of the optimal decision rule for $t = 1$ where $F(w)$ is the uniform distribution on $[0,1]$ (, or $a = 0$ and $b = 1$), $\beta = 0.97$, $s = 0.005$, and $r(w) = 0.01$ for $w < 0.4$, $0.9w - 0.35$ for $0.4 \leq w < 0.6$, and 0.19 for $0.6 \leq w$ (Figure 4.3.1). In this case it follows that $\theta = 0.760$ and $\lambda_1 = 0.501$. Figure 4.3.2 illustrates $z_1^r(w)$ and $z_1^o(0.300) = 0.523$ where the thick line on the horizontal axis indicates

$$W_1(0.300) = \{w \mid z_1^o(0.300) \leq z_1^r(w)\} = \{w \mid 0.333 \leq w \leq 0.449,\ 0.524 \leq w\}.$$

The line corresponds to the vertical thick line in Figure 4.3.3. Then, we find that the thick curved line in Figure 4.3.3 is the locus of points (x,w) satisfying $z_1^o(x) = z_1^r(w)$ and that the areas on the left and right sides of the thick curved line represent the sets of $\{(x,w) \mid w \in W_1(x)\}$ and $\{(x,w) \mid w \notin W_1(x)\}$, respectively. Accordingly, it follows that the entire region $[0,1] \times [0,1]$ can be divided into two regions corresponding to "AS and RC" and "PS and PC." Furthermore, either region is divided into two regions corresponding to PS and PC by the horizontal line λ_t and to PS and PC by the vertical line θ, respectively. Accordingly, when we have the leading offer $x = 0.300$ at $t = 1$, Figure 4.3.3 tells us that the best choice for a current offer w is:

If $0.524 \leq w \leq 1.000$, AS (accept the current offer w and stop the search).

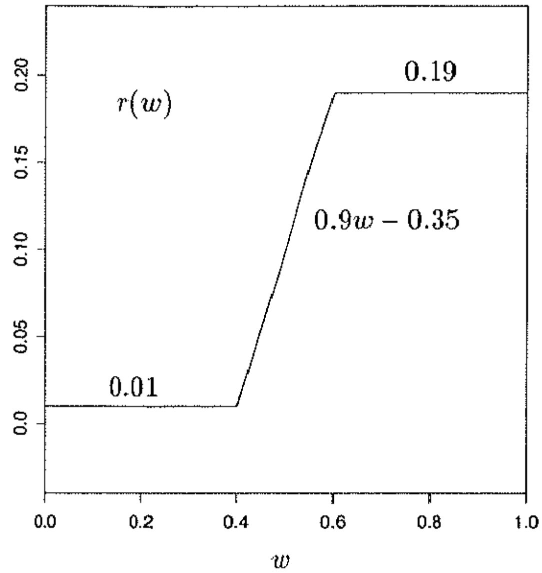


Figure 4.3.1: Reserving cost $r(w)$

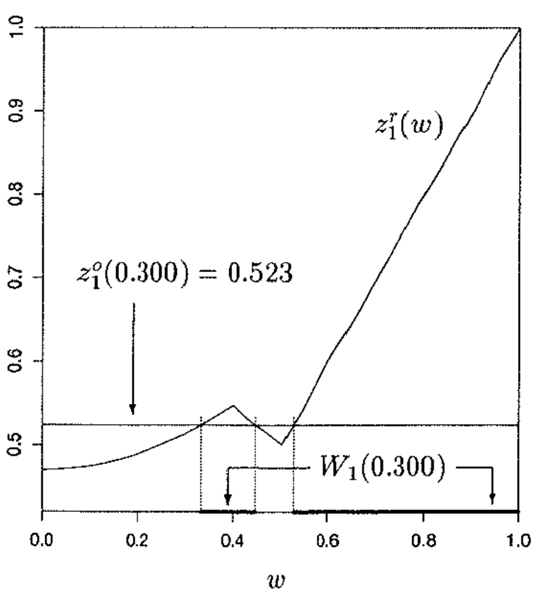


Figure 4.3.2: $z_1^o(0.300)$ and $z_1^r(w)$

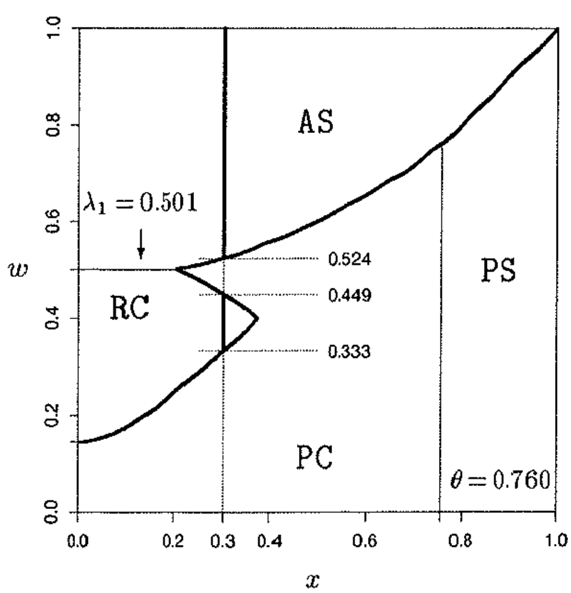


Figure 4.3.3: The optimal decision rule

If $0.449 < w < 0.524$, PC (pass up the current offer w and continue the search).

If $0.333 \le w \le 0.449$, RC (reserve the current offer w and continue the search).

If $0.000 \le w < 0.333$, PC (pass up the current offer w and continue the search).

We further observe the optimal decision rule by other examples. In Figures 4.3.4 to 4.3.6 the parameters except for $r(w)$ are the same as those used in Figure 4.2.1. In Figure 4.3.7 it is also

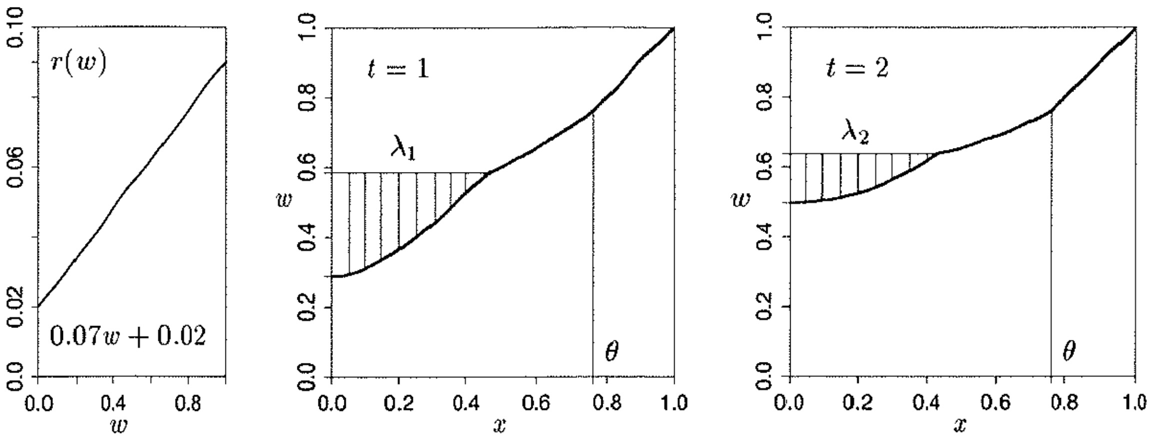


Figure 4.3.4: Example A.

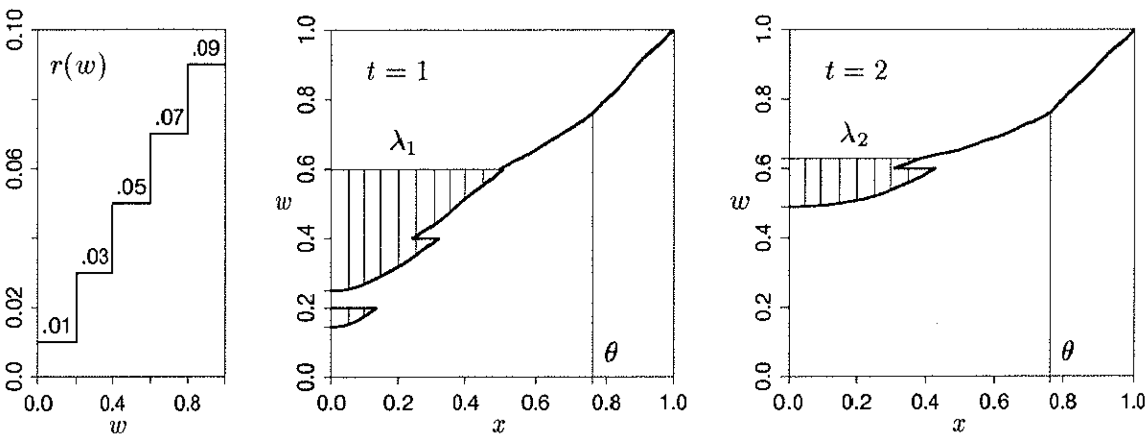


Figure 4.3.5: Example B.

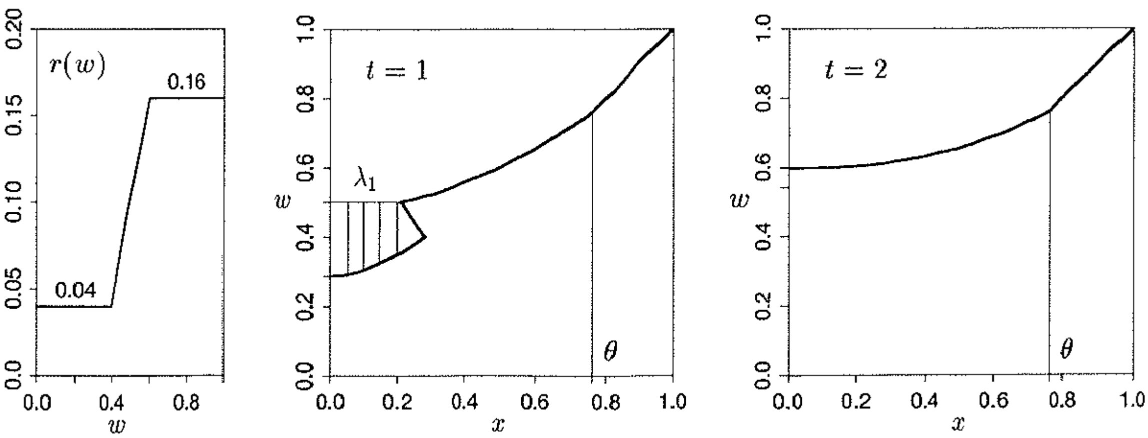


Figure 4.3.6: Example C.

used $\beta = 0.97$ and $s = 0.005$ but $F(w)$ is such that $F(w) = 9w$ for $0 \leq w \leq 0.1$ and $(w + 8)/9$ for $0.1 < w < 1$ (, or $a = 0$ and $b = 1$).

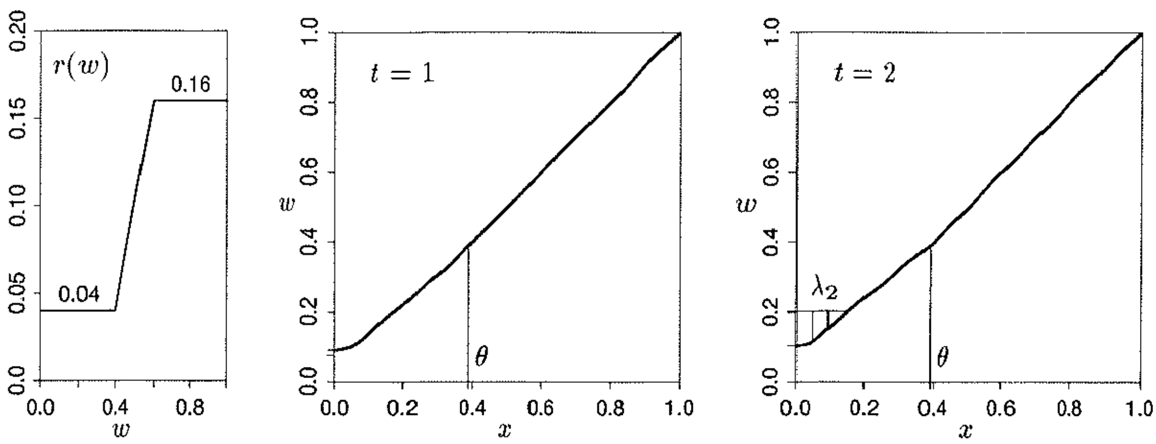


Figure 4.3.7: Example D.

4.4 Properties of the Optimal Decision Rule

A. *If $\alpha \leq a$, then the continuation of the search is not optimal at all.*

This is a restatement of the optimal decision rule on p.20. In the model, α can be interpreted as the expected discounted present net profit attainable from one more search. Hence, if it is inferior to the lowest value of offers, or if $\alpha \leq a$, no one is willing to engage in the search process.

The following properties are obtained for the case of $a < \alpha$.

B. *If the leading offer x is such that $\theta \leq x$, then accept the more lucrative between the leading offer x and the current offer w.*

This is already stated in the optimal decision rule on p.27. This implies that if an offer x with $\theta \leq x$ is given as an initial offer before entering the search, we have to stop the search by accepting it or the offer w found from the first search.

C. *No offer reserved during the search process should be recalled and accepted except at the deadline.*

If the value of the initial leading offer, which is the best of offers given before entering the search, is larger than or equal to θ, we will stop the search by accepting it due to Property B.

Below, suppose that the values of the initial leading offer is less than θ. According to the optimal decision rule (a2) on p.24 and Theorem 4.2.2(a), an offer w which should be reserved must satisfy $w \leq \lambda_t$. Further, at any $t \geq 1$, since $\lambda_t < \theta_t = \theta$ holds due to Lemma 4.2.5(b) and Theorem 4.2.2(a), the values of reserved offers are all less than θ. However, the optimal decision rule (b1) indicates that when we can stop the search by accepting the leading offer x is restricted only to the time satisfying $\theta \leq x$. Consequently, we have not to recall and accept the leading offer prior to the deadline $t = 0$.

In the model it is allowed to recall and accept the leading offer at any time over the whole planning horizon, and a search cost $s > 0$ must be spent in order to proceed the search. So, it is natural to think that recalling and accepting the leading offer in the middle of the search can become an optimal decision. The above result, however, tells that such an action is not optimal at all. Therefore, we conclude that the aim to reserve an offer is only to avoid any dire situation which may be awaiting at the deadline.

D. *At each time, any offer inferior to the leading offer should be passed up, and the range of offers to be passed up should be spread as the leading offer becomes better.*

This result is derived from the optimal decision rule (b) on p.24 and Theorem 4.2.3. Since each reserved offer is assumed to be available at any time in the model, there is no reason for reserving an offer inferior to the leading offer. So, this is a very convincing result.

E. *The range of offers to be passed up should be wider as the remaining periods become larger.*

This property is derived from Theorem 4.2.5. If there is a lot of periods to go, we can afford to wait for an excellent offer to be found.

F. *If $r(w)$ is concave, the indifferent point between reserving and passing up an offer is determined at one critical point.*

If $r(w)$ is concave in w, it follows from Theorem 4.2.6 that $W_t(x)$ can be depicted as in Figure 4.3.4 on p.34, which shows us an image of the property.

If $r(w)$ is not concave, $W_t(x)$ can be a union of some disjoint sets as in Figure 4.3.3. So, there may exist more than two indifferent points between the two actions. According to the numerical calculations the author made, this phenomenon tends to occur when $r(w)$ is steady or increases slightly up to a certain w and then rises steeply.

G. *If the planning horizon is infinite, we should continue the search with reserving no offers until an offer exceeding θ is found.*

This is a restatement of the optimal decision rule on p.32. From Lemmas 2.1.2 (p.7) and 2.1.1 (p.6) we find that this property is almost the same as the optimal decision rule for an infinite horizon of models with no recall and with recall.

From all the stated above, the optimal decision rule can be summarized as follows:

$\diamondsuit$ **Optimal Decision Rule:** Suppose that you are at time t with the leading offer x and have just drawn an offer w. Let x^0 be the initial leading offer, thus $x = x^0$ if time t is the start point of the search process. Then, the choices are:

(a) If $\alpha \le a$ or $\theta \le x^0$, then:

1. AS if the offer w found at the start is such that $x^0 \le w$ (accept it and stop the search).
2. PS otherwise (accept the initial offer x^0 and stop the search).

(b) If $a < \alpha$ and $x^0 < \theta$, then:

1. If $t = 0$ (deadline), then:
   - i AS if $x \le w$ (accept the current offer w and stop the search).
   - ii PS otherwise (accept the leading offer x and stop the search).
2. If $t \ge 1$, then:
   - i AS if $w \in W_t(x)$ and $\lambda_t < w$ (accept the current offer w and stop the search).

ii `RC` if $w \in W_t(x)$ and $w \leq \lambda_t$ (reserve the current offer w and continue the search).

iii `PC` if $w \notin W_t(x)$ (pass up the current offer w and continue the search).

3. If $t = \infty$ (infinite planning horizon), then:

i `AS` if $\theta \leq w$ (accept the current offer w and stop the search).

ii `PC` otherwise (pass up the current offer w and continue the search).

Chapter 5

Model 2: Finite-Period Reservation

This chapter is devoted to the discrete-time optimal stopping problem where any of offers appearing subsequently can be not only accepted or passed up but also reserved. In order to reserve an offer we must pay the reserving cost but the reservation expires finite k periods after. A major finding is that a reserved offer should not be recalled and accepted prior to its maturity of reservation, however, it may be done so on the maturity.

5.1 Model

Suppose that a person periodically searches for offers with the intention to accept one of them within t periods from time t to the deadline where the value of an offer w is a random variable following a known offer value distribution function $F(w)$ having the mean μ and producing $w \in [a,b]$. If the search cost $s > 0$ is paid, an offer w will be found, and it can be not only accepted or passed up but also reserved by paying the reserving cost $r(w) > 0$. The reserving cost depends on the offer value and any reservation is effective only for given finite k periods independently of the offer value. His objective is to maximize the total expected present discounted net profit.

5.2 Analysis

Without loss of generality, we can consider the value of an offer to be in $(-\infty, b]$. So, for expressional simplicity, let "for any w" and "for any x" mean "for any w with $w \leq b$" and "for any $x \in R^k$ with $\hat{x} \leq b$," respectively.

5.2.1 Notation

Let $\boldsymbol{x}$ denote a k-dimensional vector, that is,

$$\boldsymbol{x} = (x_1, x_2, \cdots, x_k) \in R^k. \tag{5.2.1}$$

Then, let $\boldsymbol{x}_i$ denote the vector defined by removing the i-th element x_i from $\boldsymbol{x}$, that is,

$$\boldsymbol{x}_i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k) \in R^{k-1}, \quad 1 \leq i \leq k. \tag{5.2.2}$$

Especially, let

$$\boldsymbol{y} = \boldsymbol{x}_k = (x_1, x_2, \cdots, x_{k-1}) \in R^{k-1}. \tag{5.2.3}$$

In addition, let

$$\boldsymbol{y}_i = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{k-1}) \in R^{k-2}, \quad 1 \leq i < k. \tag{5.2.4}$$

Let the maximum element of a vector be denoted by the character with hat ($\hat{}$), hence

$$\hat{x} = \max\{x_1, x_2, \cdots, x_k\}, \tag{5.2.5}$$

$$\hat{x}_i = \max\{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k\}, \quad 1 \leq i \leq k, \tag{5.2.6}$$

$$\hat{y} = \max\{x_1, x_2, \cdots, x_{k-1}\}, \tag{5.2.7}$$

$$\hat{y}_i = \max\{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{k-1}\}, \quad 1 \leq i < k. \tag{5.2.8}$$

Then, it follows from Eqs.(5.2.5) to (5.2.8) that

$$\hat{y} = \max\{\hat{y}_i,\ x_i\} \tag{5.2.9}$$

and

$$\hat{x} = \max\{\hat{y},\ x_k\} = \max\{\hat{y}_i,\ x_i,\ x_k\} = \max\{\hat{x}_i,\ x_i\}. \tag{5.2.10}$$

5.2.2 Optimal Equation

Suppose that we are at time t, let w_i be the offer found i periods ago, or at time $t+i$, and let

$$x_i = \begin{cases} w_i & \text{if } w_i \text{ was reserved,} \\ 0 & \text{if it was not reserved.} \end{cases} \tag{5.2.11}$$

Then, since each reserved offer is available for only finite k periods, the vector

$$\boldsymbol{x} = (x_1, x_2, \cdots, x_k) \in R^k \tag{5.2.12}$$

represents all reserved offers available at time t where $x_i = 0$ can be regarded as a fictitious reserved offer. We call the vector $\boldsymbol{x}$ the *reserved offer vector*. Consequently, the leading offer of time t is given by $\hat{x}$, that is,

$$\hat{x} = \max\{x_1, x_2, \cdots, x_k\}. \tag{5.2.13}$$

By $u_t(w, \boldsymbol{x})$ we denote the maximum total expected present discounted net profit attainable by starting the search for offers from time t with the current offer w and the reserved offer vector $\boldsymbol{x}$, and by $v_t(\boldsymbol{x})$ we denote the expectation of $u_t(w, \boldsymbol{x})$ with respect to w, that is,

$$v_t(\boldsymbol{x}) = \int_a^b u_t(w, \boldsymbol{x}) dF(w), \quad t \geq 0. \tag{5.2.14}$$

If taking decision AS or PS, we will quit the search process by accepting the offer w or $\hat{x}$, respectively. If taking decision RC or PC, we are to continue the search. Then, since the oldest available reserved offer x_k is on the maturity, the reserved offers which will be inherited from time t to time $t-1$ are expressed as

$$\boldsymbol{y} = \boldsymbol{x}_k = (x_1, x_2, \cdots, x_{k-1}) \in R^{k-1}. \tag{5.2.15}$$

Hence, if the current offer w is reserved at time t, the reserved offer vector of time $t-1$ becomes

$$(w, \boldsymbol{y}) = (w, x_1, x_2 \cdots, x_{k-1}) \in R^k, \tag{5.2.16}$$

and if not reserved, then

$$(0, \boldsymbol{y}) = (0, x_1, x_2, \cdots, x_{k-1}) \in R^k. \tag{5.2.17}$$

In general, the relationship among the current offer and the reserved offer vector at each of previous, present, and next time can be depicted as in Figure 5.2.1.

In view of the above, we find that $u_t(w, \boldsymbol{x})$ can be expressed as follows:

$$u_0(w, \boldsymbol{x}) = \max \left\{ \begin{array}{lcl} \mathtt{AS} & : & w, \\ \mathtt{PS} & : & \hat{x} \end{array} \right\}, \tag{5.2.18}$$

$$u_t(w, \boldsymbol{x}) = \max \left\{ \begin{array}{lcl} \mathtt{AS} & : & w, \\ \mathtt{RC} & : & -r(w) - s + \beta v_{t-1}(w, \boldsymbol{y}), \\ \mathtt{PS} & : & \hat{x}, \\ \mathtt{PC} & : & -s + \beta v_{t-1}(0, \boldsymbol{y}) \end{array} \right\}, \quad t \geq 1. \tag{5.2.19}$$

$$\begin{array}{lllllllllllll}
\text{(previous)} & & & & & & & & & & & & \\
\text{at time } t+1: & w, & \boldsymbol{x} = & (x_1, & x_2, & \cdots, & x_{i-1}, & x_i, & x_{i+1}, & \cdots, & x_{k-1}, & x_k) & \\
\text{(present)} & & & & & & & & & & & & \\
\text{at time } t: & w', & \boldsymbol{x}' = & (x'_1, & x'_2, & \cdots, & x'_{i-1}, & x'_i, & x'_{i+1}, & \cdots, & x'_{k-1}, & x'_k) & \text{expired} \\
\text{(next)} & & & & & & & & & & & & \\
\text{at time } t-1: & w'', & \boldsymbol{x}'' = & (x''_1, & x''_2, & \cdots, & x''_{i-1}, & x''_i, & x''_{i+1}, & \cdots, & x''_{k-1}, & x''_k) & \text{expired}
\end{array}$$

($x'_1 = 0$ and $x''_1 = 0$ if w and w' are passed up at times $t+1$ and t, respectively)

Figure 5.2.1: Reserved offer vector $\boldsymbol{x} \in R^k$

Due to Eqs. (5.2.14), (5.2.18), and (3.1.1), we have

$$v_0(\boldsymbol{x}) = \int_a^b u_0(w, \boldsymbol{x})dF(w) = \int_a^b \max\{w, \hat{x}\}dF(w) = S(\hat{x}). \tag{5.2.20}$$

Lemma 5.2.1

(a) $u_t(w, \boldsymbol{x})$ *is :*

1. *continuous in* w *and* $\boldsymbol{x}$*,*
2. *nondecreasing in* $\boldsymbol{x}$*,*
3. *convex in* $\boldsymbol{x}$*,*
4. *nondecreasing in* t*.*

(b) $v_t(\boldsymbol{x})$ *is :*

1. *continuous in* $\boldsymbol{x}$*,*
2. *nondecreasing in* $\boldsymbol{x}$*,*
3. *convex in* $\boldsymbol{x}$*,*
4. *nondecreasing in* t*.*

PROOF. In almost the same way as in the proofs of Lemma 4.2.1 (p.15) and Corollary 4.2.1 (p.17) in Chapter 4, we can easily confirm all the assertions. ∎

Lemma 5.2.2 *For* $t \geq 0$ *:*

(a) $\hat{x} < v_t(\boldsymbol{x})$ *for any* $\boldsymbol{x}$ *with* $\hat{x} < b$*.*

(b) $v_t(\boldsymbol{x}) = b$ *for any* $\boldsymbol{x}$ *with* $\hat{x} = b$*.*

(c) $\mu \leq v_t(\boldsymbol{x}) \leq b$ *for any* $\boldsymbol{x}$*.*

(d) $\beta v_t(0, \boldsymbol{y}) - x_i$ *is strictly decreasing in* x_i *for any* i *and* $\boldsymbol{x}_i \in R^{k-1}$*.*

(e) $\beta v_t(w, \boldsymbol{y}) - w$ *is strictly decreasing in* w *for any* $\boldsymbol{y} \in R^{k-1}$*.*

PROOF.

(a) If $\hat{x} < b$, then $\hat{x} < S(\hat{x}) = v_0(\boldsymbol{x}) \leq v_1(\boldsymbol{x}) \leq \cdots$ from Lemmas 3.2.1(a), 5.2.1(b4), and Eq. (5.2.20).

(b) If $\hat{x} = b$, then $v_0(\boldsymbol{x}) = S(b) = b$ by Eq. (5.2.20) and Lemma 3.2.1(a).

Assume the assertion to be true for $t-1$, and let $\boldsymbol{b} = (b, b, \cdots, b) \in R^{k-1}$. Since $\boldsymbol{y} \leq \boldsymbol{b}$, it follows from Lemma 5.2.1(b2) and the assumption that, for any $w \geq 0$,

$$v_{t-1}(0, \boldsymbol{y}) \leq v_{t-1}(w, \boldsymbol{y}) \leq v_{t-1}(w, \boldsymbol{b}) = b. \tag{5.2.21}$$

Therefore, due to Eqs. (5.2.19) and (5.2.21) we claim that, for any $w \geq 0$ and $\boldsymbol{x}$ with $\hat{x} = b$,

$$u_t(w,\boldsymbol{x}) = \max \left\{ \begin{array}{l} w, \\ -r(w) - s + \beta v_{t-1}(w,\boldsymbol{y}), \\ b, \\ -s + \beta v_{t-1}(0,\boldsymbol{y}) \end{array} \right\} = b. \tag{5.2.22}$$

Hence, for any $\boldsymbol{x}$ with $\hat{x} = b$, from Eqs. (5.2.20) and (5.2.22) we get

$$v_t(\boldsymbol{x}) = \int_a^b b \, dF(w) = b. \tag{5.2.23}$$

(c) For any $\boldsymbol{x}$, since $w \leq u_t(w,\boldsymbol{x})$ for any w by Eqs. (5.2.18) and (5.2.19), we obtain

$$\mu = \int_a^b w \, dF(w) \leq \int_a^b u_t(w,\boldsymbol{x}) dF(w) = v_t(\boldsymbol{x}).$$

We get $v_t(\boldsymbol{x}) \leq v_t(b,b,\cdots,b) = b$ due to Lemmas 5.2.1(b2) and assertion (b).

(d) First, suppose $i = k$. Then, since $\boldsymbol{y} \in R^{k-1}$ is independent of x_k, so also is $v_t(0,\boldsymbol{y})$, thus $\beta v_t(0,\boldsymbol{y}) - x_k$ is strictly decreasing in x_k for any $\boldsymbol{x}_k$ $(= \boldsymbol{y}) \in R^{k-1}$.

Next, suppose $i < k$ and $\hat{y}_i = b$. Then, due to Eq. (5.2.9) we have $\hat{y} = \max\{x_i, \hat{y}_i\} = b$ for any x_i, thus from (b) we get $v_t(0,\boldsymbol{y}) = b$ for any x_i. Therefore, $\beta v_t(0,\boldsymbol{y}) - x_i$ is strictly decreasing in x_i if $\hat{y}_i = b$.

Finally, for the case of $i < k$ and $\hat{y}_i < b$, choose k-dimensional vectors $\boldsymbol{x}^1$, $\boldsymbol{x}^2$, and $\boldsymbol{x}^b$ such that

$$x_i^1 < x_i^2 < x_i^b = b \tag{5.2.24}$$

and

$$x_j^1 = x_j^2 = x_j^b < b, \quad j \neq i,k. \tag{5.2.25}$$

Then, by Eqs. (5.2.8) and (5.2.25) we get

$$\hat{y}_i^1 = \hat{y}_i^2 = \hat{y}_i^b = \max\{x_1^1,\cdots,x_{i-1}^1,x_{i+1}^1,\cdots,x_{k-1}^1\} < b. \tag{5.2.26}$$

From Eqs. (5.2.24) and (5.2.26) we deduce

$$\max\{x_i^1,\hat{y}_i^1\} \leq \max\{x_i^2,\hat{y}_i^2\} < \max\{x_i^b,\hat{y}_i^b\} \ (= \max\{b,\hat{y}_i^b\}) \ = b,$$

that is, $\hat{y}^1 \leq \hat{y}^2 < \hat{y}^b = b$ by Eq. (5.2.9). Hence, by assertions (b) and (a) we get $v_t(0,\boldsymbol{y}^b) = b$ and $x_i^1 \leq \hat{y}^1 < v_t(0,\boldsymbol{y}^1)$, respectively. Accordingly, by noting that $v_t(0,\boldsymbol{y})$ is convex in x_i from Lemma 5.2.1(b3), we obtain

$$\beta\frac{v_t(0,\boldsymbol{y}^2) - v_t(0,\boldsymbol{y}^1)}{x_i^2 - x_i^1} \leq \beta\frac{v_t(0,\boldsymbol{y}^b) - v_t(0,\boldsymbol{y}^1)}{b - x_i^1} < \beta\frac{b - x_i^1}{b - x_i^1} \leq 1,$$

from which $\beta v_t(0,\boldsymbol{y}^1) - x_i^1 > \beta v_t(0,\boldsymbol{y}^2) - x_i^2$. Thus we get the assertion.

(e) If $\hat{y} = b$, then $v_t(w,\boldsymbol{y}) = b$ for any w from assertion (b), thus we find that $\beta v_t(w,\boldsymbol{y}) - w$ is strictly decreasing in w.

If $\hat{y} < b$, choose w^1 and w^2 so that $w^1 < w^2 < b$. Then $\max\{b,\hat{y}\} = b$, thus $v_t(b,\boldsymbol{y}) = b$

by assertion (b). Furthermore, $\max\{w^1, \hat{y}\} < b$, thus $w^1 \le \max\{w^1, \hat{y}\} < v_t(w^1, \boldsymbol{y})$ due to (a). Hence, from Lemma 5.2.1(b3) we get

$$\beta \frac{v_t(w^2, \boldsymbol{y}) - v_t(w^1, \boldsymbol{y})}{w^2 - w^1} \le \beta \frac{v_t(b, \boldsymbol{y}) - v_t(w^1, \boldsymbol{y})}{b - w^1} < \beta \frac{b - w^1}{b - w^1} \le 1,$$

implying $\beta v_t(w^1, \boldsymbol{y}) - w^1 > \beta v_t(w^2, \boldsymbol{y}) - w^2$. We have thus confirmed the assertion. ∎

5.2.3 Optimal Decision Rule

Let us define the two functions $z_t^o(\boldsymbol{x})$ and $z_t^r(w, \boldsymbol{y})$ as follows:

$$z_t^o(\boldsymbol{x}) = \max\{\hat{x},\ -s + \beta v_{t-1}(0, \boldsymbol{y})\}, \qquad t \ge 1, \tag{5.2.27}$$

$$z_t^r(w, \boldsymbol{y}) = \max\{w,\ -r(w) - s + \beta v_{t-1}(w, \boldsymbol{y})\}, \quad t \ge 1, \tag{5.2.28}$$

$z_0^o(\boldsymbol{x}) = \hat{x}$, and $z_0^r(w, \boldsymbol{y}) = w$. From Eqs.(5.2.18) and (5.2.19) we know that $z_t^o(\boldsymbol{x})$ and $z_t^r(w, \boldsymbol{y})$ stand for the maximum total expected present discounted net profits attainable after passing up or not passing up the current offer w, respectively, at time t.

Therefore, the set of current offers that should not be passed up can be denoted by

$$W_t(\boldsymbol{x}) = \{w \mid z_t^o(\boldsymbol{x}) \le z_t^r(w, \boldsymbol{y})\}, \quad t \ge 0. \tag{5.2.29}$$

By using these notations, we have

$$u_t(w, \boldsymbol{x}) = \max\{z_t^r(w, \boldsymbol{y}),\ z_t^o(\boldsymbol{x})\} \tag{5.2.30}$$

$$= \begin{cases} z_t^r(w, \boldsymbol{y}) & \text{if } \ w \in W_t(\boldsymbol{x}), \\ z_t^o(\boldsymbol{x}) & \text{if } \ w \notin W_t(\boldsymbol{x}), \end{cases} \quad t \ge 0, \tag{5.2.31}$$

from which

$$v_t(\boldsymbol{x}) = \int_a^b \max\{z_t^r(w, \boldsymbol{y}),\ z_t^o(\boldsymbol{x})\} dF(w) \tag{5.2.32}$$

$$= \int_{W_t(\boldsymbol{x})} z_t^r(w, \boldsymbol{y}) dF(w) + \int_{W_t(\boldsymbol{x})^c} z_t^o(\boldsymbol{x}) dF(w), \quad t \ge 0. \tag{5.2.33}$$

Theorem 5.2.1 *If $\alpha \le a$, then $u_t(w, \boldsymbol{x}) = \max\{w, \hat{x}\}$ for any $w \ge a$, $\boldsymbol{x} \ge \mathbf{0}$, and $t \ge 0$.*

PROOF. Suppose $\alpha \le a$, or $\alpha - a \le 0$. Clearly the assertion holds true for $t = 0$ by Eq.(4.2.12).

Assume $u_{t-1}(w, \boldsymbol{x}) = \max\{w, \hat{x}\}$ for any $w \ge a$ and $\boldsymbol{x} \ge \mathbf{0}$. Then, we get $v_{t-1}(\boldsymbol{x}) = S(\hat{x})$, thus $z_t^o(\boldsymbol{x}) = \max\{\hat{x}, -s + \beta S(\max\{0, \hat{y}\})\}$ and $z_t^r(w, \boldsymbol{y}) = \max\{w, -r(w) - s + \beta S(\max\{w, \hat{y}\})\}$.

If $\hat{y} \le a$, then $\max\{0, \hat{y}\} \le a$, thus it follows from Lemma 3.2.1(a) and Eq.(3.2.2) that $-s + \beta S(\max\{0, \hat{y}\}) = -s + \beta\mu = \alpha \le a$, from which we get

$$\max\{\hat{y}, -s + \beta S(\max\{0, \hat{y}\})\} \le a. \tag{5.2.34}$$

If $a \leq \hat{y}$, then $a \leq \max\{0, \hat{y}\} = \hat{y}$, thus it follows from Eq. (3.1.3) and Lemma 3.2.2(b,a) that $-s + \beta S(\max\{0, \hat{y}\}) - \hat{y} = K(\hat{y}) \leq K(a) = \alpha - a \leq 0$, from which we arrive at

$$\max\{\hat{y}, -s + \beta S(\max\{0, \hat{y}\})\} = \hat{y}. \tag{5.2.35}$$

Since $\hat{y} \leq \hat{x}$, we can consider the three cases: (a) $\hat{y} \leq \hat{x} \leq a$, (b) $\hat{y} \leq a \leq \hat{x}$, and (c) $a \leq \hat{y} \leq \hat{x}$. Then, from Eqs. (5.2.27), (5.2.34), and (5.2.35), we obtain

$$z_t^o(\boldsymbol{x}) = \max\{\hat{x}, \hat{y}, -s + \beta S(\max\{0, \hat{y}\})\} \begin{cases} \leq a & \text{if } \hat{x} \leq a, \\ = \hat{x} & \text{if } a \leq \hat{x}. \end{cases} \tag{5.2.36}$$

Let $a \leq w$. Then $a \leq \max\{w, \hat{y}\}$ for any $\boldsymbol{y}$, thus by Eq. (3.1.3) and Lemma 3.2.2(b,a) we get

$$\begin{aligned} -r(w) - s + \beta S(\max\{w, \hat{y}\}) - \max\{w, \hat{y}\} &< -s + \beta S(\max\{w, \hat{y}\}) - \max\{w, \hat{y}\} \\ &= K(\max\{w, \hat{y}\}) \\ &\leq K(a) = \alpha - a \leq 0, \end{aligned}$$

from which we conclude

$$-r(w) - s + \beta S(\max\{w, \hat{y}\}) < \max\{w, \hat{y}\} = \begin{cases} \hat{y} & \text{if } w < \hat{y}, \\ w & \text{if } \hat{y} \leq w, \end{cases}$$

producing

$$z_t^r(w, \boldsymbol{y}) = \max\{w, -r(w) - s + \beta S(\max\{w, \hat{y}\})\} \begin{cases} < \hat{y} & \text{if } w < \hat{y}, \\ = w & \text{if } \hat{y} \leq w. \end{cases} \tag{5.2.37}$$

Due to Eqs. (5.2.36) and (5.2.37) we obtain the following three relations:

(a) If $\hat{x} \leq a$ ($\leq w$), then $\hat{y} \leq a$, thus $\hat{y} \leq w$, hence

$$z_t^o(x) \leq a \leq w = z_t^r(w). \tag{5.2.38}$$

(b) If $a \leq \hat{x} \leq w$, then $\hat{y} \leq w$, hence,

$$z_t^o(x) = x \leq w = z_t^r(w). \tag{5.2.39}$$

(c) If ($a \leq$) $w < \hat{x}$, then

$$\begin{cases} z_t^r(w, \boldsymbol{y}) < \hat{y} \leq \hat{x} = z_t^o(\boldsymbol{x}) & \text{if } w < \hat{y} \ (\leq \hat{x}) \\ z_t^r(w, \boldsymbol{y}) = w < \hat{x} = z_t^o(\boldsymbol{x}) & \text{if } \hat{y} \leq w \ (< \hat{x}) \end{cases} \tag{5.2.40}$$

Here, it follows from Eqs. (5.2.38) to (5.2.40) that

$$u_t(w, \boldsymbol{x}) = \max\{z_t^r(w, \boldsymbol{y}), z_t^o(\boldsymbol{x})\} = \begin{cases} z_t^r(w, \boldsymbol{y}) = w & \text{if } \hat{x} \leq w, \\ z_t^o(\boldsymbol{x}) = \hat{x} & \text{if } w < \hat{x}. \end{cases}$$

We have thus confirmed the assertion. ∎

From Theorem 5.2.1, the optimal decision rule for the case $\alpha \leq a$ can be prescribed as follows:

◇ **Optimal Decision Rule:** In the case where $\alpha \leq a$, if $w \geq \hat{x}$, accept the current offer w and stop the search, or else accept the leading offer $\hat{x}$ and stop the search.

Below, let us postulate $a < \alpha$.

Lemma 5.2.3 *For any $\boldsymbol{x}$, let $\boldsymbol{x}^a \in R^k$ be the vector whose elements are*

$$x_i^a = \begin{cases} x_i & \text{if } a \leq x_i, \\ a & \text{if } x_i < a. \end{cases} \tag{5.2.41}$$

Then :

(a) *For $t \geq 1$, we have $z_t^o(\boldsymbol{x}) = z_t^o(\boldsymbol{x}^a)$ and $z_t^r(w, \boldsymbol{y}) = z_t^r(w, \boldsymbol{y}^a)$ for $w \geq a$.*

(b) *For $t \geq 0$, we have $v_t(\boldsymbol{x}) = v_t(\boldsymbol{x}^a)$.*

Proof. From Eq. (5.2.41) we know $\boldsymbol{x} \leq \boldsymbol{x}^a$, thus $\hat{x} \leq \hat{x}^a$ for any $\boldsymbol{x}$. Now, if $a \leq x_i$ for at least one i, then $\hat{x} = \hat{x}^a$. Hence, by contraposition, if $\hat{x} < \hat{x}^a$, then $x_i < a$ for all i, thus $\hat{x}^a = a$ due to Eq. (5.2.41). Consequently, the relation between $\hat{x}$ and $\hat{x}^a$ is given by

$$\begin{cases} a \leq \hat{x} = \hat{x}^a & \text{if } \hat{x} = \hat{x}^a, \\ \hat{x} < \hat{x}^a = a & \text{if } \hat{x} \neq \hat{x}^a. \end{cases} \tag{5.2.42}$$

Since $v_0(\boldsymbol{x}) = S(\hat{x})$ from Eq. (5.2.20), we get $v_0(\boldsymbol{x}) = v_0(\boldsymbol{x}^a)$ by Lemma 3.2.1(a) and Eq. (5.2.42).

Assume assertion (b) to be true for $t-1$, that is, $v_{t-1}(\boldsymbol{x}) = v_{t-1}(\boldsymbol{x}^a)$ for any $\boldsymbol{x}$. Then, since $(0, \boldsymbol{y})^a = (a, \boldsymbol{y}^a)$, we get $v_{t-1}(0, \boldsymbol{y}) = v_{t-1}(a, \boldsymbol{y}^a)$. Due to this and $v_{t-1}(0, \boldsymbol{y}) \leq v_{t-1}(0, \boldsymbol{y}^a) \leq v_{t-1}(a, \boldsymbol{y}^a)$, derived from Lemma 5.2.1(b2), we arrive at

$$v_{t-1}(0, \boldsymbol{y}) = v_{t-1}(0, \boldsymbol{y}^a). \tag{5.2.43}$$

Hence, if $\hat{x} = \hat{x}^a$, it follows from Eqs. (5.2.27) and (5.2.43) that

$$\begin{aligned} z_t^o(\boldsymbol{x}) &= \max\{\hat{x}, -s + \beta v_{t-1}(0, \boldsymbol{y})\} \\ &= \max\{\hat{x}^a, -s + \beta v_{t-1}(0, \boldsymbol{y}^a)\} = z_t^o(\boldsymbol{x}^a). \end{aligned} \tag{5.2.44}$$

If $\hat{x} < \hat{x}^a$, from Eq. (5.2.42), assumption $a < \alpha$, Lemma 5.2.2(c), and Eq. (5.2.43), we get

$$\hat{x} < \hat{x}^a = a < \alpha = -s + \beta\mu \leq -s + \beta v_{t-1}(0, \boldsymbol{y}) = -s + \beta v_{t-1}(0, \boldsymbol{y}^a). \tag{5.2.45}$$

Hence, it follows from Eqs. (5.2.27) and (5.2.45) that if $\hat{x} < \hat{x}^a$, then

$$z_t^o(\boldsymbol{x}) = -s + \beta v_{t-1}(0, \boldsymbol{y}) = -s + \beta v_{t-1}(0, \boldsymbol{y}^a) = z_t^o(\boldsymbol{x}^a). \tag{5.2.46}$$

Due to Eqs. (5.2.44) and (5.2.46) we get $z_t^o(\boldsymbol{x}) = z_t^o(\boldsymbol{x}^a)$ for any $\boldsymbol{x}$.

Next, for any $w \geq a$ we have $v_{t-1}(w, \boldsymbol{y}) = v_{t-1}(w, \boldsymbol{y}^a)$ due to $(w, \boldsymbol{y})^a = (w, \boldsymbol{y}^a)$ and the assumption. Hence, it follows from Eq. (5.2.28) that

$$z_t^r(w, \boldsymbol{y}) = \max\{w, -r(w) - s + \beta v_{t-1}(w, \boldsymbol{y})\}$$

$$= \max\{w, -r(w) - s + \beta v_{t-1}(w, \boldsymbol{y}^a)\} = z_t^r(w, \boldsymbol{y}^a), \tag{5.2.47}$$

which completes the proof of assertion (a).

By use of assertion (a) and Eq. (5.2.32), we can immediately prove the assertion (b). Hence, both assertions prove to be true by induction. ∎

Lemma 5.2.4

(a) $z_t^o(\boldsymbol{x})$ *is continuous, nondecreasing, and convex in* $\boldsymbol{x}$*, and nondecreasing in* t*.*

(b) $z_t^r(w, \boldsymbol{y})$ *is continuous in* w *and* $\boldsymbol{y}$*, nondecreasing and convex in* $\boldsymbol{y}$*, and nondecreasing in* t*.*

(c) $W_t(\boldsymbol{x})$ *is a closed set such that* $a \notin W_t(\boldsymbol{x})$ *and* $b \in W_t(\boldsymbol{x})$ *for any* $\boldsymbol{x}$ *and* $t \geq 1$*.*

PROOF.

(a,b) Both assertions clearly hold by virtue of Eqs. (5.2.27), (5.2.28), and Lemma 5.2.1(b).

(c) By applying almost the same manner as in the proof of Lemma 4.2.4(e) on p.20, we can verify that $W_t(\boldsymbol{x})$ is a closed set for any $\boldsymbol{x}$.

Due to the assumption $a < \alpha$ and Lemma 5.2.2(c) we have

$$a < \alpha = -s + \beta\mu \leq -s + \beta v_{t-1}(0, \boldsymbol{y}). \tag{5.2.48}$$

Let $\boldsymbol{y}^a \in R^{k-1}$ be a vector defined by Eq. (5.2.41). Then, $v_{t-1}(0, \boldsymbol{y}) \leq v_{t-1}(a, \boldsymbol{y}) \leq v_{t-1}(a, \boldsymbol{y}^a)$ by Lemma 5.2.1(b2), and $v_{t-1}(0, \boldsymbol{y}) = v_{t-1}(a, \boldsymbol{y}^a)$ from Lemma 5.2.3(b). Hence, we get $v_{t-1}(a, \boldsymbol{y}) = v_{t-1}(0, \boldsymbol{y})$, from which

$$-r(a) - s + \beta v_{t-1}(a, \boldsymbol{y}) < -s + \beta v_{t-1}(a, \boldsymbol{y}) = -s + \beta v_{t-1}(0, \boldsymbol{y}). \tag{5.2.49}$$

Therefore, due to Eqs. (5.2.49) and (5.2.48) we get

$$z_t^r(w, \boldsymbol{y}) = \max\{a, -r(a) - s + \beta v_{t-1}(a, \boldsymbol{y})\} < -s + \beta v_{t-1}(0, \boldsymbol{y}) \leq z_t^o(\boldsymbol{x}),$$

which indicates $a \notin W_t(\boldsymbol{x})$.

For any $\boldsymbol{x} \in R^k$, since $v_{t-1}(0, \boldsymbol{y}) \leq v_{t-1}(b, \boldsymbol{y}) = b$ by Lemmas 5.2.1(b2) and 5.2.2(b), we get

$$z_t^o(\boldsymbol{x}) \leq \max\{b, -s + \beta b\} = b = \max\{b, -r(b) - s + \beta b\} = z_t^r(b, \boldsymbol{y}),$$

which indicates $b \in W_t(\boldsymbol{x})$. ∎

Lemma 5.2.5

(a) *Given any* n *with* $0 \leq n \leq k$*, suppose that* $\boldsymbol{p} \in R^n$*,* $\boldsymbol{q}^1 \in R^{k-n}$ *and* $\boldsymbol{q}^2 \in R^{k-n}$ *satisfy*

$$\boldsymbol{q}^1 \leq \boldsymbol{q}^2 \tag{5.2.50}$$

and

$$v_t(\boldsymbol{p}, \boldsymbol{0}) = v_t(\boldsymbol{p}, \boldsymbol{q}^2). \tag{5.2.51}$$

Then, $v_t(\boldsymbol{p}, \boldsymbol{q}^1) = v_t(\boldsymbol{p}, \boldsymbol{q}^2)$ (see Figure 5.2.2).

(b) *Given any* n *with* $1 \le n \le k$*, suppose that* $\boldsymbol{x}^1 \in R^k$ *and* $\boldsymbol{x}^2 \in R^k$ *satisfy*

$$\hat{x}^1 = x_n^1 = x_n^2 = \hat{x}^2 \tag{5.2.52}$$

and

$$x_i^1 = x_i^2, \quad i < n. \tag{5.2.53}$$

Then, $v_t(\boldsymbol{x}^1) = v_t(\boldsymbol{x}^2)$ (see Figure 5.2.3).

(c) *Given any* $t < k$*, suppose that* $\boldsymbol{x}^1 \in R^k$ *and* $\boldsymbol{x}^2 \in R^k$ *satisfy*

$$\max\{x_i^1 \mid i \le k - t\} = \max\{x_i^2 \mid i \le k - t\} \tag{5.2.54}$$

and

$$x_i^1 = x_i^2, \quad k - t < i. \tag{5.2.55}$$

Then, $v_t(\boldsymbol{x}^1) = v_t(\boldsymbol{x}^2)$ (see Figure 5.2.4).

PROOF.

(a) Lemma 5.2.3(a) implies $v_t(\boldsymbol{x}) = v_t(\boldsymbol{0})$ $(= v_t(\boldsymbol{a}))$ for any $\boldsymbol{x}$ with $\boldsymbol{x} \le \boldsymbol{0}$ $(\le \boldsymbol{a})$ where $\boldsymbol{a} = (a, a, \cdots, a) \in R^k$. From this and Lemma 5.2.1(b2) we deduce that if $\boldsymbol{q}^1 \le \boldsymbol{q}^2$, then $v_t(\boldsymbol{p}, \boldsymbol{0}) \le v_t(\boldsymbol{p}, \boldsymbol{q}^1) \le v_t(\boldsymbol{p}, \boldsymbol{q}^2)$. Hence, if $v_t(\boldsymbol{p}, \boldsymbol{0}) = v_t(\boldsymbol{p}, \boldsymbol{q}^2)$, then $v_t(\boldsymbol{p}, \boldsymbol{0}) = v_t(\boldsymbol{p}, \boldsymbol{q}^1) = v_t(\boldsymbol{p}, \boldsymbol{q}^2)$.

(b) If $n = k$, then $\boldsymbol{x}^1 = \boldsymbol{x}^2$, thus the assertion holds true.

Let $n < k$ below. The assertion holds true for $t = 0$ from $\hat{x}^1 = \hat{x}^2$ and Eq. (5.2.20).

Assume the assertion to be true for $t - 1$. Choose $\boldsymbol{x}^1 \in R^k$ and $\boldsymbol{x}^2 \in R^k$ to satisfy Eqs. (5.2.52) and (5.2.53) with $n < k$, and let $\boldsymbol{p}^1 = (w, \boldsymbol{y}^1) \in R^k$ and $\boldsymbol{p}^2 = (w, \boldsymbol{y}^2) \in R^k$ with a $w \ge 0$. Then, by Eq. (5.2.16) we get

$$p_1^1 = w = p_1^2 \tag{5.2.56}$$

and

$$p_i^1 = x_{i-1}^1 = x_{i-1}^2 = p_i^2, \qquad 2 \le i \le n + 1. \tag{5.2.57}$$

From Eqs. (5.2.56) and (5.2.57) we get

$$\hat{p}^1 = \hat{p}^2 = \begin{cases} p_{n+1}^1 = p_{n+1}^2 \ \ (= x_n^1 = x_n^2) & \text{if } \ w \le x_n^1 = x_n^2, \\ p_1^1 = p_1^2 \qquad\quad\ (= w) & \text{if } \ x_n^1 = x_n^2 < w. \end{cases} \tag{5.2.58}$$

Eqs. (5.2.58) and (5.2.57) show that $\boldsymbol{p}^1$ and $\boldsymbol{p}^2$ satisfy Eqs. (5.2.52) and (5.2.53), respectively, with either $n + 1$ or 1. Hence, $v_{t-1}(w, \boldsymbol{y}^1) = v_{t-1}(w, \boldsymbol{y}^2)$ for any $w \ge 0$ by the assumption. Consequently, since $\hat{x}^1 = \hat{x}^2$, from Eq. (5.2.19) we get $u_t(w, \boldsymbol{x}^1) = u_t(w, \boldsymbol{x}^2)$ for any w, implying $v_t(\boldsymbol{x}^1) = v_t(\boldsymbol{x}^2)$ due to Eq. (5.2.14).

(c) Choose $\boldsymbol{x}^1 \in R^k$ and $\boldsymbol{x}^2 \in R^k$ to satisfy Eqs. (5.2.54) and (5.2.55) where clearly $\hat{x}^1 = \hat{x}^2$. The assertion holds true for $t = 0$ due to Eq. (5.2.20) and $\hat{x}^1 = \hat{x}^2$.

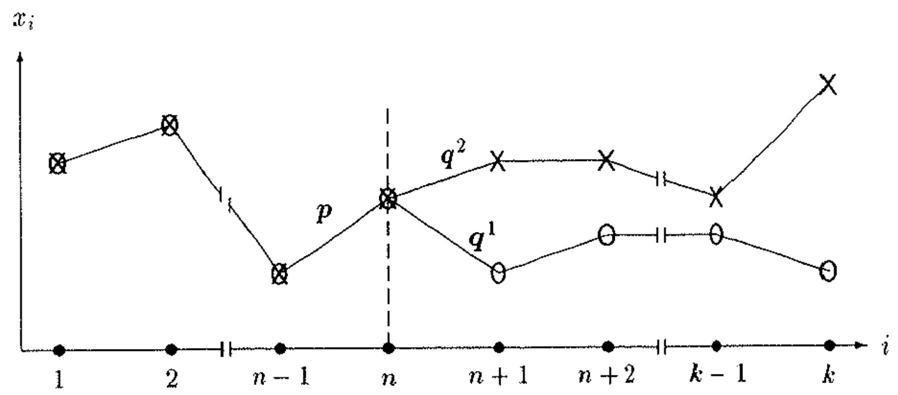


Figure 5.2.2: $(\boldsymbol{p}, \boldsymbol{q}^1) \in R^k$ ... O, and $(\boldsymbol{p}, \boldsymbol{q}^2) \in R^k$ ... X

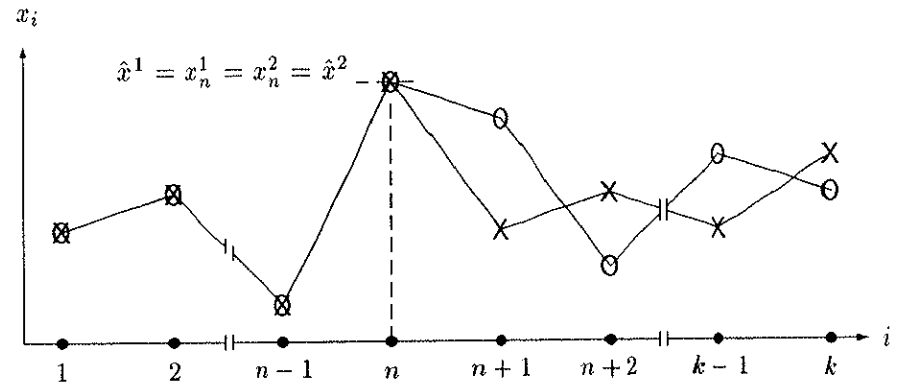


Figure 5.2.3: $\boldsymbol{x}^1 \in R^k$ ... O, and $\boldsymbol{x}^2 \in R^k$ ... X

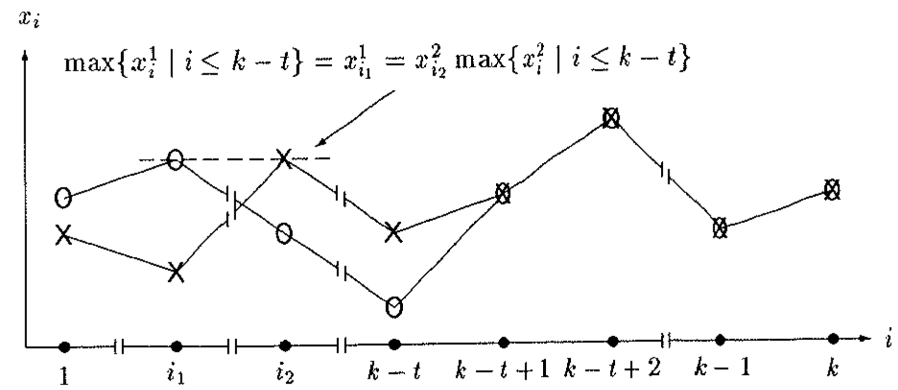


Figure 5.2.4: $\boldsymbol{x}^1 \in R^k$ ... O, and $\boldsymbol{x}^2 \in R^k$ ... X

Assume the assertion to be true for $t-1$, and let $\boldsymbol{p}^1 = (w, \boldsymbol{y}^1) \in R^k$ and $\boldsymbol{p}^2 = (w, \boldsymbol{y}^2) \in R^k$ with a $w \ge 0$. Then, $p_1^1 = w = p_1^2$, and $p_i^1 = x_{i-1}^1$ and $p_i^2 = x_{i-1}^2$ for $2 \le i \le k$. Hence,

$$\max\{p_i^1 \mid i \le k-t+1\} = \max\{w, x_1^1, \cdots, x_{k-t}^1\}$$

$$
\begin{aligned}
&= \max\{w,\ \max\{x_i^1 \mid i \le k-t\}\} \\
&= \max\{w,\ \max\{x_i^2 \mid i \le k-t\}\} \\
&= \max\{w, x_1^2, \cdots, x_{k-t}^2\} \\
&= \max\{p_i^2 \mid i \le k-t+1\} \qquad (5.2.59)
\end{aligned}
$$

and

$$p_i^1 = x_{i-1}^1 = x_{i-1}^2 = p_i^2, \qquad k-t+1 < i. \qquad (5.2.60)$$

Eqs. (5.2.59) and (5.2.60) show that $\boldsymbol{p}^1$ and $\boldsymbol{p}^2$ satisfy Eqs. (5.2.54) and (5.2.55), respectively, with $t-1$. Hence, $v_{t-1}(w, \boldsymbol{y}^1) = v_{t-1}(w, \boldsymbol{y}^2)$ for any $w \ge 0$ by the assumption. From $\hat{x}^1 = \hat{x}^2$ and Eq. (5.2.19) we get $u_t(w, \boldsymbol{x}^1) = u_t(w, \boldsymbol{x}^2)$ for any w, thus $v_t(\boldsymbol{x}^1) = v_t(\boldsymbol{x}^2)$ from Eq. (5.2.14). ∎

Let $g_t^i(x_i|\boldsymbol{x}_i)$ and $f_t(w|\boldsymbol{y})$ with $t \ge 1$ denote the following function of x_i for given $\boldsymbol{x}_i \in R^{k-1}$ and function of w for given $\boldsymbol{y} \in R^{k-1}$, respectively:

$$
\begin{aligned}
g_t^i(x_i|\boldsymbol{x}_i) &= -s + \beta v_{t-1}(0, \boldsymbol{y}) - x_i, && t \ge 1, \quad i \le k, && (5.2.61) \\
f_t(w|\boldsymbol{y}) &= -r(w) - s + \beta v_{t-1}(w, \boldsymbol{y}) - w, && t \ge 1. && (5.2.62)
\end{aligned}
$$

Corollary 5.2.1 *For $t \ge 1$:*

(a) *$g_t^i(x_i|\boldsymbol{x}_i)$ is continuous and strictly decreasing in x_i for any i and $\boldsymbol{x}_i \in R^{k-1}$.*

(b) *$f_t(w|\boldsymbol{y})$ is continuous and strictly decreasing in w for any $\boldsymbol{y} \in R^{k-1}$.*

PROOF.

(a) Evident due to Eq. (5.2.61), Lemmas 5.2.1(b1), and 5.2.2(d).

(b) Since $r(w)$ is continuous and nondecreasing in w, the assertion holds true by Eq. (5.2.62), Lemmas 5.2.1(b1), and 5.2.2(e). ∎

Let $\theta_t^i(\boldsymbol{x}_i)$ and $\lambda_t(\boldsymbol{y})$ with $t \ge 1$ denote the respective roots of $g_t^i(x_i|\boldsymbol{x}_i) = 0$ for given $\boldsymbol{x}_i \in R^{k-1}$ and $f_t(w|\boldsymbol{y}) = 0$ for given $\boldsymbol{y} \in R^{k-1}$, if they exist, that is,

$$
\begin{aligned}
g_t^i(\theta_t^i(\boldsymbol{x}_i)|\boldsymbol{x}_i) &= -s + \beta v_{t-1}(0, \boldsymbol{y}) - \theta_t^i(\boldsymbol{x}_i) = 0, && t \ge 1, && (5.2.63) \\
f_t(\lambda_t(\boldsymbol{y})|\boldsymbol{y}) &= -r(\lambda_t(\boldsymbol{y})) - s + \beta v_{t-1}(\lambda_t(\boldsymbol{y}), \boldsymbol{y}) - \lambda_t(\boldsymbol{y}) = 0, && t \ge 1. && (5.2.64)
\end{aligned}
$$

We know that $\theta_t^i(\boldsymbol{x}_i)$ represents an indifferent point in terms of x_i between accepting the reserved offer x_i and continuing the search under a given $\boldsymbol{x}_i \in R^{k-1}$, and $\lambda_t(\boldsymbol{y})$ an indifferent point in terms of w between accepting the current offer w and reserving it under a given $\boldsymbol{y} \in R^{k-1}$.

Lemma 5.2.6 *For $t \ge 1$:*

(a) *$\theta_t^i(\boldsymbol{x}_i)$ exists uniquely with $a \le \theta_t^i(\boldsymbol{x}_i) < b$ for any i and $\boldsymbol{x}_i \in R^{k-1}$.*

(b) $\lambda_t(\boldsymbol{y})$ *exists uniquely with* $\alpha - r(b) \leq \lambda_t(\boldsymbol{y}) < b$ *any* $\boldsymbol{y} \in R^{k-1}$.

PROOF. By using a method similar to those in the proofs of Lemma 4.2.5(a,b) on p.23, it follows that, for any i and $\boldsymbol{x}_i \in R^{k-1}$,

$$g_t^i(\alpha|\boldsymbol{x}_i) \geq 0 > g_t^i(b|\boldsymbol{x}_i) \tag{5.2.65}$$

and that, for any $\boldsymbol{y} \in R^{k-1}$,

$$f_t(\alpha - r(b)|\boldsymbol{y}) \geq 0 > f_t(b|\boldsymbol{y}). \tag{5.2.66}$$

From Corollary 5.2.1, Eqs. (5.2.65), and (5.2.66), the assertion proves to be true. ∎

Corollary 5.2.2

(a) *For any* $i \leq k$, $\boldsymbol{x}_i \in R^{k-1}$, *and* $t \geq 1$:

1. *If* $x_i < \theta_t^i(\boldsymbol{x}_i)$, *then* $x_i < -s + \beta v_{t-1}(0, \boldsymbol{y})$.
2. *If* $x_i = \theta_t^i(\boldsymbol{x}_i)$, *then* $x_i = -s + \beta v_{t-1}(0, \boldsymbol{y})$.
3. *If* $x_i > \theta_t^i(\boldsymbol{x}_i)$, *then* $x_i > -s + \beta v_{t-1}(0, \boldsymbol{y})$.

(b) *For* $\boldsymbol{y} \in R^{k-1}$ *and* $t \geq 1$:

1. *If* $w < \lambda_t(\boldsymbol{y})$, *then* $w < -r(w) - s + \beta v_{t-1}(w, \boldsymbol{y})$.
2. *If* $w = \lambda_t(\boldsymbol{y})$, *then* $w = -r(w) - s + \beta v_{t-1}(w, \boldsymbol{y})$.
3. *If* $w > \lambda_t(\boldsymbol{y})$, *then* $w > -r(w) - s + \beta v_{t-1}(w, \boldsymbol{y})$.

PROOF. Clear from Corollary 5.2.1 and Lemma 5.2.6(a,b). ∎

By use of Corollary 5.2.2(a) and Eq. (5.2.27) we get

$$\begin{aligned} \exists i : \theta_t^i(\boldsymbol{x}_i) < x_i \quad &\Longrightarrow \quad -s + \beta v_{t-1}(0, \boldsymbol{y}) < \max\{x_1, \cdots, x_k\} = \hat{x} \\ &\Longrightarrow \quad z_t^o(\boldsymbol{x}) = \hat{x} \end{aligned}$$

and

$$\begin{aligned} \forall i : x_i \leq \theta_t^i(x_i) \quad &\Longrightarrow \quad \hat{x} = \max\{x_1, \cdots, x_k\} \leq -s + \beta v_{t-1}(0, \boldsymbol{y}) \\ &\Longrightarrow \quad z_t^o(\boldsymbol{x}) = -s + \beta v_{t-1}(0, \boldsymbol{y}). \end{aligned}$$

As a result, we obtain

$$u_t(w, \boldsymbol{x}) = \begin{cases} z_t^r(w, \boldsymbol{y}) = \begin{cases} w & \text{if } w \in W_t(\boldsymbol{x}) \text{ and } \lambda_t(\boldsymbol{y}) < w, \\ -r(w) - s + \beta v_{t-1}(w, \boldsymbol{y}) & \text{if } w \in W_t(\boldsymbol{x}) \text{ and } w \leq \lambda_t(\boldsymbol{y}), \end{cases} \\ z_t^o(\boldsymbol{x}) \quad\;\; = \begin{cases} \hat{x} & \text{if } w \notin W_t(\boldsymbol{x}) \text{ and } \theta_t^i(\boldsymbol{x}_i) < x_i \text{ for an } i, \\ -s + \beta v_{t-1}(0, \boldsymbol{y}) & \text{if } w \notin W_t(\boldsymbol{x}) \text{ and } x_i \leq \theta_t^i(\boldsymbol{x}_i) \text{ for all } i. \end{cases} \end{cases} \tag{5.2.67}$$

Now, let $\theta_0^i(\boldsymbol{x}_i) = \lambda_0(\boldsymbol{y}) = -\infty$ for convenience. Then, in general, by Eq. (5.2.67) we can prescribe the optimal decision rule as follows:

$\Diamond$ **Optimal Decision Rule:** Suppose that you are at time t with the reserved offer vector $\boldsymbol{x}$ and have just drawn an offer w. Then, the choices are:

(a) If $w \in W_t(\boldsymbol{x})$, then:

1. If $\lambda_t(\boldsymbol{y}) < w$, then AS (accept the current offer w and stop the search).
2. If $w \leq \lambda_t(\boldsymbol{y})$, then RC (reserve the current offer w and continue the search).

(b) If $w \notin W_t(\boldsymbol{x})$, then:

1. If $\theta_t^i(\boldsymbol{x}_i) < x_i = \hat{x}$ for a certain i, then PS (pass up the current offer w and stop the search by accepting the leading offer $\hat{x}$).
2. If $x_i \leq \theta_t^i(\boldsymbol{x}_i)$ for all i, then PC (pass up the current offer w and continue the search).

Theorem 5.2.2 *For $t \geq 1$:*

(a) *For $i < k$, if $\boldsymbol{x}_i \in R^{k-1}$ is such that $\hat{y}_i \leq \theta$, then $\theta_t^i(\boldsymbol{x}_i) = \theta$.*

(b) *If $\boldsymbol{y} \in R^{k-1}$ is such that $\hat{y} \leq \theta$, then $\hat{y} \leq \theta_t^k(\boldsymbol{y}) = -s + \beta v_{t-1}(0, \boldsymbol{y}) \leq \theta$.*

(c) *If $\boldsymbol{y} \in R^{k-1}$ is such that $\hat{y} \leq \theta$, then $\lambda_t(\boldsymbol{y}) < \theta$.*

(d) *$\theta_t^k(\boldsymbol{y})$ and $\lambda_t(\boldsymbol{y})$ are continuous and nondecreasing in $\boldsymbol{y}$, and nondecreasing in t.*

Proof. Assertions (a-c) are shown together by induction. From Eq. (5.2.20) we have $v_0(\boldsymbol{x}) = S(\theta)$ for any $\boldsymbol{x}$ with $\hat{x} = \theta$. Assume $v_{t-1}(\boldsymbol{x}) = S(\theta)$ for any $\boldsymbol{x}$ with $\hat{x} = \theta$ throughout the proofs of assertions (a-c).

(a) Given any $i < k$, choose an $\boldsymbol{x} \in R^k$ so that $\hat{y}_i \leq \theta$, and let

$$\boldsymbol{x}' = (x_1, \cdots, x_{i-1}, \theta, x_{i+1}, \cdots, x_k) \in R^k.$$

Then, we get

$$\boldsymbol{x}_i' = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_k) = \boldsymbol{x}_i \in R^{k-1}, \tag{5.2.68}$$

$$\boldsymbol{y}' = (x_1, \cdots, x_{i-1}, \theta, x_{i+1}, \cdots, x_{k-1}) \in R^{k-1},$$

$$\hat{y}_i' = \max\{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{k-1}\} = \hat{y}_i \leq \theta. \tag{5.2.69}$$

Since $\hat{y}' = \max\{\hat{y}_i', \theta\} = \theta$ due to Eq. (5.2.69), we have $v_{t-1}(0, \boldsymbol{y}') = S(\theta)$ by the assumption, thus it follows from Eq. (5.2.68) that

$$g_t^i(\theta|\boldsymbol{x}_i) = g_t^i(\theta|\boldsymbol{x}_i') = -s + \beta v_{t-1}(0, \boldsymbol{y}') - \theta = -s + \beta S(\theta) - \theta = K(\theta) = 0. \tag{5.2.70}$$

From the uniqueness of $\theta_t^i(\boldsymbol{x}_i)$ and Eq. (5.2.70), we get $\theta_t^i(\boldsymbol{x}_i) = \theta$ for any $\boldsymbol{x}_i \in R^{k-1}$ with $\hat{y}_i \leq \theta$.

(b) Since $v_{t-1}(0, \boldsymbol{y})$ is independent of x_k, due to Eq. (5.2.63) we get

$$\theta_t^k(\boldsymbol{y}) = -s + \beta v_{t-1}(0, \boldsymbol{y}). \tag{5.2.71}$$

Note that $0 = K(\theta) \leq K(\hat{y})$ if $\hat{y} \leq \theta$ due to Lemma 3.2.2(b). Then, it follows from Eqs. (3.1.3), (5.2.20), (5.2.71), and Lemma 5.2.1(b4) that, for any $\boldsymbol{y} \in R^{k-1}$ with $\hat{y} \leq \theta$,

$$\hat{y} \leq K(\hat{y}) + \hat{y} = -s + \beta S(\hat{y}) = -s + \beta v_0(0, \boldsymbol{y}) \leq -s + \beta v_{t-1}(0, \boldsymbol{y}) = \theta_t^k(\boldsymbol{y}). \tag{5.2.72}$$

Since $\hat{y} \leq \theta$ implies $\boldsymbol{y} \leq (\theta, \theta, \cdots \theta) \in R^{k-1}$, we have $v_{t-1}(0, \boldsymbol{y}) \leq v_{t-1}(0, \theta, \cdots, \theta) = S(\theta)$ due to Lemma 5.2.1(b2) and the assumption, from which

$$\theta_t^k(\boldsymbol{y}) = -s + \beta v_{t-1}(0, \boldsymbol{y}) \leq -s + \beta S(\theta) = K(\theta) + \theta = \theta. \tag{5.2.73}$$

Due to Eqs. (5.2.71), (5.2.72), and (5.2.73), we complete the proof of the assertion.

(c) Since $v_{t-1}(\theta, \boldsymbol{y}) = S(\theta)$ for any $\boldsymbol{y} \in R^{k-1}$ with $\hat{y} \leq \theta$ by the assumption, we get

$$f_t(\theta|\boldsymbol{y}) = -r(\theta) - s + \beta S(\theta) - \theta = -r(\theta) + K(\theta) = -r(\theta) < 0. \tag{5.2.74}$$

Hence, by Corollary 5.2.1(b) we claim that $\lambda_t(\boldsymbol{y})$, the unique root of $f_t(w|\boldsymbol{y}) = 0$, must be less than θ for any $\boldsymbol{y} \in R^{k-1}$ with $\hat{y} \leq \theta$.

(a-c) To complete the proofs, we shall show $v_t(\boldsymbol{x}) = S(\theta)$ for any $\boldsymbol{x}$ with $\hat{x} = \theta$. Suppose $\hat{x} = \theta$, thus $\hat{y} \leq \theta$. Then, we have $-s + \beta v_{t-1}(0, \boldsymbol{y}) \leq \theta$ from Eq. (5.2.73), thus

$$z_t^o(\boldsymbol{x}) = \max\{\hat{x}, -s + \beta v_{t-1}(0, \boldsymbol{y})\} = \max\{\theta, -s + \beta v_{t-1}(0, \boldsymbol{y})\} = \theta. \tag{5.2.75}$$

Since $\hat{y} \leq \theta$, we have $v_{t-1}(w, \boldsymbol{y}) \leq v_{t-1}(\theta, \boldsymbol{y}) = S(\theta)$ for any w with $w < \theta$ by Lemma 5.2.1(b2) and the assumption, thus

$$-r(w) - s + \beta v_{t-1}(w, \boldsymbol{y}) < -s + \beta v_{t-1}(w, \boldsymbol{y}) \leq -s + \beta S(\theta) = K(\theta) + \theta = \theta. \tag{5.2.76}$$

Hence, if $w < \theta$, then $z_t^r(w, \boldsymbol{y}) < \max\{\theta, \theta\} = \theta$ from Eqs. (5.2.28) and (5.2.76). Conversely, if $\theta \leq w$, then $\lambda_t(\boldsymbol{y}) < w$ due to $\lambda_t(\boldsymbol{y}) < \theta$ as in the proof of (c), therefore $z_t^r(w, \boldsymbol{y}) = w$ from Corollary 5.2.2(b3). Consequently, if $\hat{y} \leq \theta$, then

$$z_t^r(w, \boldsymbol{y}) \begin{cases} < \theta & \text{if } w < \theta, \\ = w & \text{if } \theta \leq w. \end{cases} \tag{5.2.77}$$

Substituting Eqs. (5.2.75) and (5.2.77) into Eq. (5.2.32), we arrive at

$$\begin{aligned} v_t(\boldsymbol{x}) &= \int_a^b \max\{z_t^r(w, \boldsymbol{y}), \theta\} dF(w) \\ &= \int_a^\theta \max\{z_t^r(w, \boldsymbol{y}), \theta\} dF(w) + \int_\theta^b \max\{z_t^r(w, \boldsymbol{y}), \theta\} dF(w) \\ &= \int_a^\theta \theta \, dF(w) + \int_\theta^b w \, dF(w) \;=\; S(\theta), \end{aligned}$$

which guarantees the assertions (a-c) to be true.

(d) Since $\theta_t^k(\boldsymbol{y}) = -s + \beta v_{t-1}(0, \boldsymbol{y})$ due to assertion (b), we find that $\theta_t^k(\boldsymbol{y})$ is continuous and nondecreasing in $\boldsymbol{y}$ according to Lemma 5.2.1(b).

We know that $\lambda_t(\boldsymbol{y})$ is bounded for every t due to Lemma 5.2.6(b). Given any $t \geq 1$, choose

$y \in R^{k-1}$ so that

$$\inf\{w \mid f_t(w|\boldsymbol{y}) = 0\} < \lambda_t(\boldsymbol{y}) < \sup\{w \mid f_t(w|\boldsymbol{y}) = 0\}. \tag{5.2.78}$$

Given any sufficiently small $\epsilon > 0$, let $\lambda^- = \lambda_t(\boldsymbol{y}) - \epsilon$ and $\lambda^+ = \lambda_t(\boldsymbol{y}) + \epsilon$. Then, since $f_t(w|\boldsymbol{y})$ is strictly decreasing in w as in Corollary 5.2.1(b), we have

$$f_t(\lambda^-|\boldsymbol{y}) = -r(\lambda^-) - s + \beta v_{t-1}(\lambda^-, \boldsymbol{y}) - \lambda^- > 0 \tag{5.2.79}$$

and

$$f_t(\lambda^+|\boldsymbol{y}) = -r(\lambda^+) - s + \beta v_{t-1}(\lambda^+, \boldsymbol{y}) - \lambda^+ < 0. \tag{5.2.80}$$

Note that $v_{t-1}(w, \boldsymbol{y})$ is continuous and nondecreasing in $\boldsymbol{y}$ for any w due to Lemma 5.2.1(b2). Hence, from Eqs. (5.2.78) to (5.2.80) there exist two points $\boldsymbol{y}^- \in R^{k-1}$ and $\boldsymbol{y}^+ \in R^{k-1}$ such that

$$\boldsymbol{y}^- \leq \boldsymbol{y}, \quad \boldsymbol{y}^- \neq \boldsymbol{y}, \quad f_t(\lambda^-|\boldsymbol{y}^-) = 0$$

and

$$\boldsymbol{y} \leq \boldsymbol{y}^+, \quad \boldsymbol{y} \neq \boldsymbol{y}^+, \quad f_t(\lambda^+|\boldsymbol{y}^+) = 0.$$

Then, we can take $\boldsymbol{y}'$ such that

$$\boldsymbol{y}^- \leq \boldsymbol{y}' \leq \boldsymbol{y}^+, \quad \boldsymbol{y}' \neq \boldsymbol{y}. \tag{5.2.81}$$

Due to Lemma 5.2.1(b2) we know that $f_t(w|\boldsymbol{y})$ is nondecreasing in $\boldsymbol{y}$, thus it follows from Eq. (5.2.81) that $0 = f_t(\lambda^-|\boldsymbol{y}^-) \leq f_t(\lambda^-|\boldsymbol{y}')$ and $f_t(\lambda^+|\boldsymbol{y}') \leq f_t(\lambda^+|\boldsymbol{y}^+) = 0$. Hence, since $f_t(w|\boldsymbol{y}')$ is strictly decreasing in w by Corollary 5.2.1(b), we claim that $\lambda_t(\boldsymbol{y}')$, the unique root of $f_t(w|\boldsymbol{y}') = 0$, exists in $[\lambda^-, \lambda^+]$, that is, in either $[\lambda^-, \lambda_t(\boldsymbol{y})]$ or $[\lambda_t(\boldsymbol{y}), \lambda^+]$. By noting that the length of either interval is less than or equal to ϵ, we obtain $|\lambda_t(\boldsymbol{y}') - \lambda_t(\boldsymbol{y})| \leq \epsilon$, which shows the continuity of $\lambda_t(\boldsymbol{y})$ at $\boldsymbol{y}$.

Due to Lemma 5.2.1(b2,4), we deduce that $f_t(w|\boldsymbol{y})$ is nondecreasing in $\boldsymbol{y}$ and t, respectively. Hence, since it is strictly decreasing in w from Corollary 5.2.1(b), it follows that $\lambda_t(\boldsymbol{y})$ is nondecreasing in $\boldsymbol{y}$ and t, respectively. ∎

Lemma 5.2.7 *For $t \geq 0$:*

(a) *Let $\boldsymbol{x}$ be such that $\theta \leq \hat{x}$. Then :*

$$z_t^o(\boldsymbol{x}) = \hat{x}, \tag{5.2.82}$$

$$z_t^r(w, \boldsymbol{y}) \begin{cases} < \hat{x} & \text{if } w < \hat{x}, \\ = w & \text{if } \hat{x} \leq w, \end{cases} \tag{5.2.83}$$

$$v_t(\boldsymbol{x}) = S(\hat{x}). \tag{5.2.84}$$

(b) *Let $\boldsymbol{x}$ be such that $\theta < \hat{x}$. Then, $v_t(\boldsymbol{x}') < v_t(\boldsymbol{x})$ for any $\boldsymbol{x}' \in R^k$ such that $\hat{x}' < \hat{x}$.*

PROOF.

(a) Eqs. (5.2.82) to (5.2.84) are evident for $t = 0$ by definitions of $z_0^o(\boldsymbol{x})$, $z_0^r(w, \boldsymbol{y})$, and $v_0(\boldsymbol{x})$. Choose an $\boldsymbol{x} \in R^k$ with $\theta \leq \hat{x}$ and assume $v_{t-1}(\boldsymbol{x}) = S(\hat{x})$.

(i) Suppose $\hat{y} \leq \theta$. Then, $-s + \beta v_{t-1}(0, \boldsymbol{y}) \leq \theta \leq \hat{x}$ by Theorem 5.2.2(b). Hence, from Eq. (5.2.27) we obtain Eq. (5.2.82). Furthermore, from Eq. (5.2.77) we get Eq. (5.2.83) by considering three cases: $w < \theta, \theta \leq w < \hat{x}$, and $\hat{x} \leq w$.

(ii) Suppose $\theta < \hat{y}$. Then, for any w we have $\theta < \max\{w, \hat{y}\}$, thus $v_{t-1}(w, \boldsymbol{y}) = S(\max\{w, \hat{y}\})$ by the assumption. Hence, from Lemma 3.2.2(b) we have

$$-s + \beta v_{t-1}(w, \boldsymbol{y}) - \max\{w, \hat{y}\} = K(\max\{w, \hat{y}\}) < K(\theta) = 0. \tag{5.2.85}$$

Setting $w = 0$ in Eq. (5.2.85), we get $-s + \beta v_{t-1}(0, \boldsymbol{y}) < \hat{y} \leq \hat{x}$. Hence, we obtain Eq. (5.2.82) from Eq. (5.2.27). Due to Eq. (5.2.85) we get

$$-r(w) - s + \beta v_{t-1}(w, \boldsymbol{y}) < \max\{w, \hat{y}\} \leq \max\{w, \hat{x}\} = \begin{cases} \hat{x} & \text{if } \ w < \hat{x}, \\ w & \text{if } \ \hat{x} \leq w. \end{cases} \tag{5.2.86}$$

It follows from Eqs. (5.2.28) and (5.2.86) that if $w < \hat{x}$, then $z_t^r(w, \boldsymbol{y}) < \max\{\hat{x}, \hat{x}\} = \hat{x}$, or else $z_t^r(w, \boldsymbol{y}) = w$. Hence, we also get Eq. (5.2.83).

As seen in (i) and (ii), we have obtained Eqs. (5.2.82) and (5.2.83).

Now, it follows from Eqs. (5.2.82) and (5.2.83) that, for any $\boldsymbol{x}$ with $\theta \leq \hat{x}$,

$$v_t(\boldsymbol{x}) = \int_a^b \max\{z_t^r(w, \boldsymbol{y}), \hat{x}\} dF(w) = \int_a^{\hat{x}} \hat{x} \ dF(w) + \int_{\hat{x}}^b w \ dF(w) = S(\hat{x}),$$

thus the assertion holds true by induction.

(b) Suppose that $\boldsymbol{x}$ satisfies $\theta < \hat{x}$, and let $\boldsymbol{x}' \in R^k$ be such that $\hat{x}' < \hat{x}$. Now, choose $\boldsymbol{x}'' \in R^k$ so that $\boldsymbol{x}' \leq \boldsymbol{x}''$ and

$$\max\{\theta, \hat{x}'\} < \hat{x}'' < \hat{x}. \tag{5.2.87}$$

Since $a < \alpha \leq \theta$ by the assumption $a < \alpha$ and Lemma 3.2.3(b), we get $a < \theta < \hat{x}'' < \hat{x}$ due to Eq. (5.2.87), thus it follows from (a) and Lemma 3.2.1(c) that

$$v_t(\boldsymbol{x}'') = S(\hat{x}'') < S(\hat{x}) = v_t(\boldsymbol{x}). \tag{5.2.88}$$

Since $\boldsymbol{x}' \leq \boldsymbol{x}''$ is assumed, from Lemma 5.2.1(b2) we claim

$$v_t(\boldsymbol{x}') \leq v_t(\boldsymbol{x}''). \tag{5.2.89}$$

Hence, if $v_t(\boldsymbol{x}) \leq v_t(\boldsymbol{x}')$, it follows from Eqs. (5.2.88) and (5.2.89) that $v_t(\boldsymbol{x}'') < v_t(\boldsymbol{x}'')$, which is a contradiction. Therefore, $v_t(\boldsymbol{x}') < v_t(\boldsymbol{x})$ must hold. ∎

Corollary 5.2.3 *If $\boldsymbol{x}$ is such that $\theta \leq \hat{x}$, then $u_t(w, \boldsymbol{x}) = \max\{w, \hat{x}\}$.*

Proof. From Lemma 5.2.7(a), if $w < \hat{x}$, then $z_t^r(w, \boldsymbol{y}) < \hat{x} = z_t^o(\boldsymbol{x})$, or else $z_t^o(\boldsymbol{x}) = \hat{x} \leq w = z_t^r(w, \boldsymbol{y})$. This relation and Eq. (5.2.30) complete the proof. ∎

From Corollary 5.2.3, the optimal decision rule for the case $\theta \le \hat{x}$ can be prescribed as follows:

$\Diamond$ **Optimal Decision Rule:** In the case where $\boldsymbol{x} \in R^k$ is such that $\theta \le \hat{x}$, if $w \ge \hat{x}$, accept the current offer w and stop the search, or else accept the leading offer $\hat{x}$ and stop the search.

Here, let us define the following vectors:

$$\boldsymbol{x}_i^L = (x_1, \cdots, x_i, 0, \cdots, 0) \in R^k, \qquad i \le k, \tag{5.2.90}$$

$$\boldsymbol{y}_i^L = (x_1, \cdots, x_i, 0, \cdots, 0) \in R^{k-1}, \qquad i < k, \tag{5.2.91}$$

$$\boldsymbol{x}_i^R = (0, \cdots, 0, x_{i+1}, \cdots, x_k) \in R^k, \qquad i \le k, \tag{5.2.92}$$

$$\boldsymbol{y}_i^R = (0, \cdots, 0, x_{i+1}, \cdots, x_{k-1}) \in R^{k-1}, \qquad i < k. \tag{5.2.93}$$

Then, clearly,

$$\hat{x} = \max\{\hat{x}_i^L,\ \hat{x}_i^R\}, \quad i \le k, \tag{5.2.94}$$

$$\hat{y} = \max\{\hat{y}_i^L,\ \hat{y}_i^R\}, \quad i < k. \tag{5.2.95}$$

Furthermore, we define a set of x_i under a given $\boldsymbol{x}_i \in R^{k-1}$ as follows:

$$X_t^i(\boldsymbol{x}_i) = \{x_i \mid v_t(\boldsymbol{x}) = v_t(\boldsymbol{x}_i^R)\}, \quad i \le k, \quad t \ge 0. \tag{5.2.96}$$

Lemma 5.2.8 *Given any $i \le k$, $\boldsymbol{x}_i \in R^{k-1}$, and $t \ge 1$, suppose*

$$x_i \notin X_{t-1}^i(\boldsymbol{x}_i) \implies v_{t-1}(\boldsymbol{x}) = v_{t-1}(\boldsymbol{x}_i^L) \tag{5.2.97}$$

where

$$X_{t-1}^i(\boldsymbol{x}_i) = \{x_i \mid v_{t-1}(\boldsymbol{x}) = v_{t-1}(\boldsymbol{x}_i^R)\}. \tag{5.2.98}$$

Then :

(a) *Given any $\boldsymbol{y} \in R^{k-1}$,*

$$w \notin X_{t-1}^1(\boldsymbol{y}) \implies v_{t-1}(w, \boldsymbol{y}) = v_{t-1}(w, 0) \tag{5.2.99}$$

where

$$X_{t-1}^1(\boldsymbol{y}) = \{w \mid v_{t-1}(w, \boldsymbol{y}) = v_{t-1}(0, \boldsymbol{y})\}. \tag{5.2.100}$$

(b) *Given any $i < k$ and $\boldsymbol{y}_i \in R^{k-2}$,*

$$x_i \notin X_{t-1}^{i+1}(0, \boldsymbol{y}_i) \implies v_{t-1}(0, \boldsymbol{y}) = v_{t-1}(0, \boldsymbol{y}_i^L) \tag{5.2.101}$$

where

$$X_{t-1}^{i+1}(0, \boldsymbol{y}_i) = \{x_i \mid v_{t-1}(0, \boldsymbol{y}) = v_{t-1}(0, \boldsymbol{y}_i^R)\}. \tag{5.2.102}$$

(c) *For any $\boldsymbol{x}^1 \in R^k$ and $\boldsymbol{x}^2 \in R^k$ such that $\boldsymbol{x}^1 \le \boldsymbol{x}^2$,*

$$v_t(\boldsymbol{x}^1) = v_t(\boldsymbol{x}^2) \iff z_t^o(\boldsymbol{x}^1) = z_t^o(\boldsymbol{x}^2). \tag{5.2.103}$$

PROOF.

(a,b) Consider the vector

$$p = (w, x_1, x_2, \cdots, x_{k-1}) = (w, \boldsymbol{y}) \in R^k$$

whose elements are $p_1 = 2$ and $p_j = x_{j-1}$ for $2 \le j \le k$. Then, from Eq. (5.2.96) we get

$$X^j_{t-1}(\boldsymbol{p}_j) = \{p_j \mid v_{t-1}(\boldsymbol{p}) = v_{t-1}(\boldsymbol{p}^R_j)\}. \qquad (5.2.104)$$

Furthermore, Eq. (5.2.97) can be rewritten as

$$p_j \notin X^j_{t-1}(\boldsymbol{p}_j) \implies v_{t-1}(\boldsymbol{p}) = v_{t-1}(\boldsymbol{p}^L_j). \qquad (5.2.105)$$

Since we have

$$\begin{aligned}
\boldsymbol{p}_1 &= (x_1, x_2, \cdots, x_{k-1}) &&= \boldsymbol{y} &&\in R^{k-1},\\
\boldsymbol{p}^L_1 &= (w,\ 0,\ 0, \cdots,\ 0) &&= (w, \boldsymbol{0}) &&\in R^k,\\
\boldsymbol{p}^R_1 &= (0, x_1, x_2, \cdots, x_{k-1}) &&= (0, \boldsymbol{y}) &&\in R^k,
\end{aligned}$$

assertion (a) is verified from Eqs. (5.2.105) and (5.2.104).

For $2 \le j$, it follows that

$$\begin{aligned}
\boldsymbol{y}_{j-1} &= (x_1, \cdots, x_{j-2}, x_j, \cdots, x_{k-1}) \in R^{k-2},\\
\boldsymbol{y}^L_{j-1} &= (x_1, \cdots, x_{j-1},\ 0, \cdots,\ 0) \in R^{k-1},\\
\boldsymbol{y}^R_{j-1} &= (0, \cdots,\ 0, x_j, \cdots, x_{k-1}) \in R^{k-1},
\end{aligned}$$

from which we get

$$\begin{aligned}
\boldsymbol{p}_j &= (w, x_1, \cdots, x_{j-2}, x_j, \cdots, x_{k-1}) &&= (w, \boldsymbol{y}_{j-1}) \in R^{k-1},\\
\boldsymbol{p}^L_j &= (w, x_1, \cdots, x_{j-1},\ 0, \cdots,\ 0) &&= (w, \boldsymbol{y}^L_{j-1}) \in R^k,\\
\boldsymbol{p}^R_j &= (0,\ 0, \cdots,\ 0, \cdots, x_j, \cdots, x_{k-1}) &&= (0, \boldsymbol{y}^R_{j-1}) \in R^k
\end{aligned}$$

Hence, by noting $p_j = x_{j-1}$ for $2 \le j \le k$, it follows from Eqs. (5.2.104) and (5.2.105) that

$$X^j_{t-1}(w, \boldsymbol{y}_{j-1}) = \{x_{j-1} \mid v_{t-1}(w, \boldsymbol{y}) = v_{t-1}(0, \boldsymbol{y}^R_{j-1})\}, \quad 2 \le j \le k, \qquad (5.2.106)$$

and

$$x_{j-1} \notin X^j_{t-1}(w, \boldsymbol{y}_{j-1}) \implies v_{t-1}(w, \boldsymbol{y}) = v_{t-1}(w, \boldsymbol{y}^L_{j-1}), \quad 2 \le j \le k. \qquad (5.2.107)$$

By setting $w = 0$ and $j = i + 1$ in Eqs. (5.2.106) and (5.2.107), assertion (b) proves true.

(c) Choose $\boldsymbol{x}^1 \in R^k$ and $\boldsymbol{x}^2 \in R^k$ so that $\boldsymbol{x}^1 \le \boldsymbol{x}^2$, thus $\boldsymbol{y}^1 \le \boldsymbol{y}^2$.

(1) First, suppose $\hat{y}^2 \le w$ or $w \notin X^1_{t-1}(\boldsymbol{y}^2)$. If $\hat{y}^2 \le w$, then $\hat{y}^1 \le w$, thus $v_{t-1}(w, \boldsymbol{y}^1) = v_{t-1}(w, \boldsymbol{y}^2)$ by Lemma 5.2.5(b) with $n = 1$. If $w \notin X^1_{t-1}(\boldsymbol{y}^2)$, then $v_{t-1}(w, \boldsymbol{0}) = v_{t-1}(w, \boldsymbol{y}^2)$ due to assertion (a), thus we also get $v_{t-1}(w, \boldsymbol{y}^1) = v_{t-1}(w, \boldsymbol{y}^2)$ from Lemma 5.2.5(a) because of

$\boldsymbol{y}^1 \leq \boldsymbol{y}^2$. From these two results and Eq. (5.2.28) we get

$$\begin{aligned} \hat{y}^2 \leq w \quad \text{or} \quad w \notin X_{t-1}^1(\boldsymbol{y}^2) \quad &\Longrightarrow \quad v_{t-1}(w, \boldsymbol{y}^1) = v_{t-1}(w, \boldsymbol{y}^2) \\ &\Longrightarrow \quad z_t^r(w, \boldsymbol{y}^1) = z_t^r(w, \boldsymbol{y}^2). \end{aligned} \tag{5.2.108}$$

(2) Conversely, suppose $w < \hat{y}^2$ and $w \in X_{t-1}^1(\boldsymbol{y}^2)$. If $w < \hat{y}^2$, then $w < \hat{x}^2$. If $w \in X_{t-1}^1(\boldsymbol{y}^2)$, then $v_{t-1}(w, \boldsymbol{y}^2) = v_{t-1}(0, \boldsymbol{y}^2)$ due to Eq. (5.2.100), thus $-r(w) - s + \beta v_{t-1}(w, \boldsymbol{y}^2) < -s + \beta v_{t-1}(0, \boldsymbol{y}^2)$. Therefore, by Eqs. (5.2.27) and (5.2.28),

$$\begin{aligned} w < \hat{y}^2 \quad \text{and} \quad w \in X_{t-1}^1(\boldsymbol{y}^2) \quad \Longrightarrow \quad z_t^r(w, \boldsymbol{y}^2) &= \max\{w, -r(w) - s + \beta v_{t-1}(w, \boldsymbol{y}^2)\} \\ &< \max\{\hat{x}^2, -s + \beta v_{t-1}(0, \boldsymbol{y}^2)\} \\ &= z_t^o(\boldsymbol{x}^2). \end{aligned} \tag{5.2.109}$$

(3) Now, it follows from Eqs. (5.2.108) and (5.2.109) that either $z_t^r(w, \boldsymbol{y}^1) = z_t^r(w, \boldsymbol{y}^2)$ or $z_t^r(w, \boldsymbol{y}^2) < z_t^o(\boldsymbol{x}^2)$ for any w. Hence, since $z_t^r(w, \boldsymbol{y}^1) \leq z_t^r(w, \boldsymbol{y}^2)$ by Lemma 5.2.4(b), we get

$$z_t^r(w, \boldsymbol{y}^1) \neq z_t^r(w, \boldsymbol{y}^2) \quad \Longrightarrow \quad z_t^r(w, \boldsymbol{y}^1) < z_t^r(w, \boldsymbol{y}^2) < z_t^o(\boldsymbol{x}^2). \tag{5.2.110}$$

(4) Here, suppose $z_t^o(\boldsymbol{x}^1) = z_t^o(\boldsymbol{x}^2)$. Then, if $z_t^r(w, \boldsymbol{y}^1) = z_t^r(w, \boldsymbol{y}^2)$, we immediately get

$$\max\{z_t^r(w, \boldsymbol{y}^1), z_t^o(\boldsymbol{x}^1)\} = \max\{z_t^r(w, \boldsymbol{y}^2), z_t^o(\boldsymbol{x}^2)\}. \tag{5.2.111}$$

Even if $z_t^o(\boldsymbol{x}^1) \neq z_t^o(\boldsymbol{x}^2)$, from Eq. (5.2.110) we get

$$\max\{z_t^r(w, \boldsymbol{y}^1), z_t^o(\boldsymbol{x}^1)\} = z_t^o(\boldsymbol{x}^1) = z_t^o(\boldsymbol{x}^2) = \max\{z_t^r(w, \boldsymbol{y}^2), z_t^o(\boldsymbol{x}^2)\}.$$

Hence, Eq. (5.2.111) holds true for any w. Thereby, due to Eq. (5.2.32) we obtain

$$z_t^o(\boldsymbol{x}^1) = z_t^o(\boldsymbol{x}^2) \quad \Longrightarrow \quad v_t(\boldsymbol{x}^1) = v_t(\boldsymbol{x}^2). \tag{5.2.112}$$

(5) Conversely, suppose $z_t^o(\boldsymbol{x}^1) < z_t^o(\boldsymbol{x}^2)$. Then, for any $w \notin W_t(\boldsymbol{x}^2)$ it follows from Eq. (5.2.29) and Lemma 5.2.4(b) that $z_t^r(w, \boldsymbol{y}^1) \leq z_t^r(w, \boldsymbol{y}^2) < z_t^o(\boldsymbol{x}^2)$, from which

$$\max\{z_t^r(w, \boldsymbol{y}^1), z_t^o(\boldsymbol{x}^1)\} < z_t^o(\boldsymbol{x}^2) = \max\{z_t^r(w, \boldsymbol{y}^2), z_t^o(\boldsymbol{x}^2)\}.$$

Thereby, since $W_t(\boldsymbol{x}^2)^c \neq \phi$ by Lemma 5.2.6(c), we arrive at

$$z_t^o(\boldsymbol{x}^1) < z_t^o(\boldsymbol{x}^2) \quad \Longrightarrow \quad v_t(\boldsymbol{x}^1) < v_t(\boldsymbol{x}^2). \tag{5.2.113}$$

(6) Thereupon, the assertion proves to be true from Eqs. (5.2.112) and (5.2.113). ∎

Lemma 5.2.9 *Given any $i \leq k$, $x_i \in R^{k-1}$, and $t \geq 0$,*

$$x_i \notin X_t^i(\boldsymbol{x}_i) \quad \Longrightarrow \quad v_t(\boldsymbol{x}) = v_t(\boldsymbol{x}_i^L). \tag{5.2.114}$$

Proof. Consider the following three cases: (a) $i = k$, (b) $\hat{x} > \theta$, and (c) $i < k$ and $\hat{x} \leq \theta$.

(a) If $i = k$, then $\boldsymbol{x} = \boldsymbol{x}_k^L$ by Eq. (5.2.91), thus we get the assertion for any $X_t^k(\boldsymbol{y})$.

(b) Choose an $\boldsymbol{x} \in R^k$ so that $\hat{x} > \theta$. If $\hat{x} = \hat{x}_i^R$, then $v_t(\boldsymbol{x}) = S(\hat{x}) = S(\hat{x}_i^R) = v_t(\boldsymbol{x}_i^R)$ by Lemma 5.2.7(a), or else $\hat{x} > \hat{x}_i^R$ by Eq. (5.2.94), thus $v_t(\boldsymbol{x}) > v_t(\boldsymbol{x}_i^R)$ due to Lemma 5.2.7(b).

From above, $v_t(\boldsymbol{x}) = v_t(\boldsymbol{x}_i^R)$ if and only if $\hat{x} = \hat{x}_i^R$, thus $X_t^i(\boldsymbol{x}_i) = \{x_i \mid \hat{x} = \hat{x}_i^R\}$. Hence, if $x_i \notin X_t^i(\boldsymbol{x}_i)$, by Eq. (5.2.94) we have $\hat{x} > \hat{x}_i^R$, so $\hat{x} = \hat{x}_i^L > \theta$, thus $v_t(\boldsymbol{x}) = S(\hat{x}) = S(\hat{x}_i^L) = v_t(\boldsymbol{x}_i^L)$ due to Lemma 5.2.7(a). The proof for (b) has been completed.

(c) Below, suppose $i < k$, and let $\boldsymbol{x} \in R^k$ be such that $\hat{x} \le \theta$. The proof for the case is made by induction on t as follows:

(c-1) If $\hat{x} = \hat{x}_i^R$, then $v_0(\boldsymbol{x}) = S(\hat{x}) = S(\hat{x}_i^R) = v_0(\boldsymbol{x}_i^R)$ by Eq. (5.2.20).

Suppose $\hat{x} > \hat{x}_i^R$. If $a \ge \hat{x}$, then $a \ge \hat{x} > \hat{x}_i^R$, thus $v_0(\boldsymbol{x}) = S(\hat{x}) = S(\hat{x}_i^R) = v_0(\boldsymbol{x}_i^R)$ $(= \mu)$ by Lemma 3.2.1(a), or else $\hat{x} > \max\{a, \hat{x}_i^R\}$, thus $v_0(\boldsymbol{x}) = S(\hat{x}) > S(\hat{x}_i^R) = v_0(\boldsymbol{x}_i^R)$ due to Lemma 3.2.1(c).

From above, $X_0^i(\boldsymbol{x}_i) = \{x_i \mid \hat{x} = \hat{x}_i^R \quad \text{or} \quad a \ge \hat{x}\}$. If $x_i \notin X_0^i(\boldsymbol{x}_i)$, then $\hat{x} > \hat{x}_i^R$, thus $\hat{x} = \hat{x}_i^L$, from which $v_0(\boldsymbol{x}) = v_0(\boldsymbol{x}_i^L)$. Consequently, the assertion holds true for $t = 0$.

(c-2) Assume the assertion holds true for $t-1$, that is,

$$x_i \notin X_{t-1}^i(\boldsymbol{x}_i) = \{x_i \mid v_{t-1}(\boldsymbol{x}) = v_{t-1}(\boldsymbol{x}_i^R)\} \implies v_{t-1}(\boldsymbol{x}) = v_{t-1}(\boldsymbol{x}_i^L), \tag{5.2.115}$$

which is the premise of Lemma 5.2.8.

(c-2-1) To begin with, we shall show

$$X_t^i(\boldsymbol{x}_i) = \{x_i \mid \theta_t^k(\boldsymbol{y}) \le x_k \quad \text{or} \quad x_i \in X_{t-1}^{i+1}(0, \boldsymbol{y}_i)\}, \quad i < k. \tag{5.2.116}$$

Since $\hat{x} \le \theta$ is assumed here, we have $\hat{y} \le \theta$, $\hat{y}_i^L \le \theta$ and $\hat{y}_i^R \le \theta$. Hence, due to Eq. (5.2.10) and Theorem 5.2.2(b) we have

$$\begin{aligned}
z_t^o(\boldsymbol{x}) &= \max\{\ \hat{x},\ \ \theta_t^k(\boldsymbol{y})\ \ \} = \max\{\ \hat{y},\ \ x_k,\ \theta_t^k(\boldsymbol{y})\ \ \} = \max\{\ x_k,\ \theta_t^k(\boldsymbol{y})\ \ \}, && (5.2.117)\\
z_t^o(\boldsymbol{x}_i^L) &= \max\{\ \hat{x}_i^L,\ \theta_t^k(\boldsymbol{y}_i^L)\ \} = \max\{\ \hat{y}_i^L,\ 0,\ \ \theta_t^k(\boldsymbol{y}_i^L)\ \} = \theta_t^k(\boldsymbol{y}_i^L), && (5.2.118)\\
z_t^o(\boldsymbol{x}_i^R) &= \max\{\ \hat{x}_i^R,\ \theta_t^k(\boldsymbol{y}_i^R)\ \} = \max\{\ \hat{y}_i^R,\ x_k,\ \theta_t^k(\boldsymbol{y}_i^R)\ \} = \max\{\ x_k,\ \theta_t^k(\boldsymbol{y}_i^R)\ \}. && (5.2.119)
\end{aligned}$$

(i) First, suppose $\theta_t^k(\boldsymbol{y}) \le x_k$ or $x_i \in X_{t-1}^{i+1}(0, \boldsymbol{y}_i)$. If $\theta_t^k(\boldsymbol{y}) \le x_k$, since $\theta_t^k(\boldsymbol{y}_i^R) \le \theta_t^k(\boldsymbol{y})$ from Theorem 5.2.2(d), it follows from Eqs. (5.2.117) and (5.2.119) that

$$z_t^o(\boldsymbol{x}) = z_t^o(\boldsymbol{x}_i^R) = x_k. \tag{5.2.120}$$

If $x_i \in X_{t-1}^{i+1}(0, \boldsymbol{y}_i)$, then $v_{t-1}(0, \boldsymbol{y}) = v_{t-1}(0, \boldsymbol{y}_i^R)$ by Eq. (5.2.102), thus $\theta_t^k(\boldsymbol{y}) = \theta_t^k(\boldsymbol{y}_i^R)$ due to Theorem 5.2.2(b). Hence, by Eqs. (5.2.117) and (5.2.119) we get

$$z_t^o(\boldsymbol{x}) = \max\{x_k, \theta_t^k(\boldsymbol{y})\} = \max\{x_k, \theta_t^k(\boldsymbol{y}_i^R)\} = z_t^o(\boldsymbol{x}_i^R). \tag{5.2.121}$$

Therefore, due to Eqs. (5.2.120) and (5.2.121) we obtain

$$\theta_t^k(\boldsymbol{y}) \leq x_k \quad \text{or} \quad x_i \in X_{t-1}^{i+1}(0, y_i) \implies z_t^o(\boldsymbol{x}) = z_t^o(\boldsymbol{x}_i^R). \tag{5.2.122}$$

(ii) Conversely, suppose $x_k < \theta_t^k(\boldsymbol{y})$ and $x_i \notin X_{t-1}^{i+1}(0, \boldsymbol{y}_i)$. If $x_i \notin X_{t-1}^{i+1}(0, \boldsymbol{y}_i)$, then $v_{t-1}(0, \boldsymbol{y}) > v_{t-1}(0, \boldsymbol{y}_i^R)$ from Eq. (5.2.102), thus $\theta_t^k(\boldsymbol{y}) > \theta_t^k(\boldsymbol{y}_i^R)$ by Theorem 5.2.2(b). Therefore, from Eqs. (5.2.117) and (5.2.119) we have

$$\begin{aligned} x_k < \theta_t^k(\boldsymbol{y}) \quad \text{and} \quad x_i \notin X_{t-1}^{i+1}(0, \boldsymbol{y}_i) \implies z_t^o(\boldsymbol{x}) &= \theta_t^k(\boldsymbol{y}) \\ &> \max\{x_k, \theta_t^k(\boldsymbol{y}_i^R)\} \\ &= z_t^o(\boldsymbol{x}_i^R). \end{aligned} \tag{5.2.123}$$

(iii) Now, it follows from Eqs. (5.2.122), (5.2.123), and Lemma 5.2.8(c) that

$$\begin{aligned} \theta_t^k(\boldsymbol{y}) \leq x_k \quad \text{or} \quad x_i \in X_{t-1}^{i+1}(0, \boldsymbol{y}_i) &\iff z_t^o(\boldsymbol{x}) = z_t^o(\boldsymbol{x}_i^R) \\ &\iff v_t(\boldsymbol{x}) = v_t(\boldsymbol{x}_i^R), \end{aligned} \tag{5.2.124}$$

which guarantees Eq. (5.2.116) to be true.

(c-2-2) Finally, we shall show Eq. (5.2.114). If $x_i \notin X_{t-1}^{i+1}(0, \boldsymbol{y}_i)$, then $v_{t-1}(0, \boldsymbol{y}) = v_{t-1}(0, \boldsymbol{y}_i^L)$ by Eq. (5.2.101), thus $\theta_t^k(\boldsymbol{y}) = \theta_t^k(\boldsymbol{y}_i^L)$ by Theorem 5.2.2(b). Hence, by using Eqs. (5.2.116), (5.2.117), (5.2.118), and Lemma 5.2.8(c), we arrive at

$$\begin{aligned} x_i \notin X_t^i(\boldsymbol{x}_i) &\iff x_k < \theta_t^k(\boldsymbol{y}) \quad \text{and} \quad x_i \notin X_{t-1}^{i+1}(0, \boldsymbol{y}_i) \\ &\implies z_t^o(\boldsymbol{x}) = \theta_t^k(\boldsymbol{y}) = \theta_t^k(\boldsymbol{y}_i^L) = z_t^o(\boldsymbol{x}_i^L) \\ &\implies v_t(\boldsymbol{x}) = v_t(\boldsymbol{x}_i^L). \end{aligned} \tag{5.2.125}$$

Since the proof for the case (c) is completed, we have confirmed the assertion. ∎

Theorem 5.2.3 *For $t \geq 0$:*

(a) *If $\boldsymbol{x}^1 \leq \boldsymbol{x}^2$, then $W_t(\boldsymbol{x}^1) \supseteq W_t(\boldsymbol{x}^2)$.*

(b) *If $t < k$ and $w < \max\{x_i \mid i \leq k-t\}$, then $w \notin W_t(\boldsymbol{x})$.*

PROOF.

(a) We shall let $\boldsymbol{x}^1 \in R^k$ and $\boldsymbol{x}^2 \in R^k$ be such that $\boldsymbol{x}^1 \leq \boldsymbol{x}^2$, and show that $w \in W_t(\boldsymbol{x}^2)$ also belongs to $W_t(\boldsymbol{x}^1)$. In the case of $t = 0$ we have $z_0^o(\boldsymbol{x}^i) = \hat{x}^i$ and $z_0^r(w, \boldsymbol{y}^i) = w$ for $i = 1$ and 2 by definition. From this and $\boldsymbol{x}^1 \leq \boldsymbol{x}^2$, thus $\hat{x}^1 \leq \hat{x}^2$, we obtain

$$W_0(\boldsymbol{x}^1) = \{w \mid \hat{x}^1 \leq w\} \supseteq \{w \mid \hat{x}^2 \leq w\} = W_0(\boldsymbol{x}^2).$$

Suppose $t > 0$ and choose a w so that $w \in W_t(\boldsymbol{x}^2)$ and $w < \lambda_t(\boldsymbol{y}^2)$. Then, by Eq. (5.2.29) and Corollary 5.2.2(b1) we get

$$-s + \beta v_{t-1}(0, \boldsymbol{y}^2) \leq z_t^o(\boldsymbol{x}^2) \leq z_t^r(w, \boldsymbol{y}^2) = -r(w) - s + \beta v_{t-1}(w, \boldsymbol{y}^2),$$

from which

$$(0 <)\ r(w) \le \beta\Big(v_{t-1}(w, \boldsymbol{y}^2) - v_{t-1}(0, \boldsymbol{y}^2)\Big),$$

implying $v_{t-1}(w, \boldsymbol{y}^2) > v_{t-1}(0, \boldsymbol{y}^2)$. Hence, from Lemmas 5.2.9 and 5.2.8(a) we know $w \notin X^1_{t-1}(\boldsymbol{y}^2)$, thus $v_{t-1}(w, \boldsymbol{y}^2) = v_{t-1}(w, \boldsymbol{0})$, from which $v_{t-1}(w, \boldsymbol{y}^2) = v_{t-1}(w, \boldsymbol{y}^1)$ due to $\boldsymbol{y}^2 \ge \boldsymbol{y}^1$ and Lemma 5.2.5(a). Therefore, from Eq. (5.2.28) we get

$$\begin{aligned} w \in W_t(\boldsymbol{x}^2) \quad \text{and} \quad w < \lambda_t(\boldsymbol{y}^2) \quad &\Longrightarrow \quad v_{t-1}(w, \boldsymbol{y}^2) = v_{t-1}(w, \boldsymbol{y}^1) \\ &\Longrightarrow \quad z^r_t(w, \boldsymbol{y}^2) = z^r_t(w, \boldsymbol{y}^1). \end{aligned} \tag{5.2.126}$$

If w is such that $\lambda_t(\boldsymbol{y}^2) \le w$, then $\lambda_t(\boldsymbol{y}^1) \le w$ since $\lambda_t(\boldsymbol{y}^1) \le \lambda_t(\boldsymbol{y}^2)$ from Theorem 5.2.2(d). Hence, due to Corollary 5.2.2(b2,b3) we obtain

$$\lambda_t(\boldsymbol{y}^2) \le w \implies z^r_t(w, \boldsymbol{y}^2) = z^r_t(w, \boldsymbol{y}^1)\ (= w). \tag{5.2.127}$$

Note that $z^o_t(\boldsymbol{x}^i) \le z^r_t(w, \boldsymbol{y}^i)$ for any $w \in W_t(\boldsymbol{x}^i)$ with $i = 1$ and 2 from Eq. (5.2.29). Then, due to Lemma 5.2.4(a) and Eqs. (5.2.126) and (5.2.127) we get

$$w \in W_t(\boldsymbol{x}^2) \implies z^o_t(\boldsymbol{x}^1) \le z^o_t(\boldsymbol{x}^2) \le z^r_t(w, \boldsymbol{y}^2) = z^r_t(w, \boldsymbol{y}^1) \implies w \in W_t(\boldsymbol{x}^1),$$

which completes the proof of the assertion.

(b) If $t = 0$, then $\max\{x_i \mid i \le k - 0\} = \hat{x}$ and $W_0(\boldsymbol{x}) = \{w \mid \hat{x} \le w\}$ by Eq. (5.2.29). Hence, if $w < \max\{x_i \mid i \le k - 0\} = \hat{x}$, then $w \notin W_0(\boldsymbol{x})$. So, the assertion holds for $t = 0$.

Suppose $0 < t < k$ and $w < \max\{x_i \mid i \le k - t\}$, thus $w < \hat{x}$, and let $\boldsymbol{p}^1 = (w, \boldsymbol{y})$ and $\boldsymbol{p}^2 = (0, \boldsymbol{y})$. Then, $p^1_1 = w$, $p^2_1 = 0$, and $p^1_i = x_{i-1} = p^2_i$ for $2 \le k \le k$. Hence, we obtain

$$\begin{aligned} \max\{p^1_i \mid i \le k - t + 1\} &= \max\{w, x_1, \cdots, x_{k-t}\} \\ &= \max\{x_1, \cdots, x_{k-t}\} \\ &= \max\{0, x_1, \cdots, x_{k-t}\} \\ &= \max\{p^2_i \mid i \le k - t + 1\} \end{aligned}$$

and $p^1_i = x_{i-1} = p^2_i$ for $k - t + 1 < i$. Therefore, $\boldsymbol{p}^1$ and $\boldsymbol{p}^2$ satisfy Eqs. (5.2.54) and (5.2.55) with $t - 1$, thus $v_{t-1}(w, \boldsymbol{y}) = v_{t-1}(0, \boldsymbol{y})$ from Lemma 5.2.5(c). Thereby, we conclude $z^r_t(w, \boldsymbol{y}) < \max\{\hat{x}, -r(w) - s + \beta v_{t-1}(0, \boldsymbol{y})\} \le z^o_t(\boldsymbol{x})$, that is, $w \notin W_t(\boldsymbol{x})$. ∎

Lemma 5.2.10

(a) *If $\hat{y} \le \lambda_t(\boldsymbol{0})$, then $\lambda_t(\boldsymbol{y}) = \lambda_t(\boldsymbol{0})$, and if $\lambda_t(\boldsymbol{0}) < \hat{y}$, then $\lambda_t(\boldsymbol{y}) \notin W_t(\boldsymbol{x})$.*

(b) *If $w \in W_t(\boldsymbol{x})$, then either $w \le \lambda_t(\boldsymbol{0})$ or $\lambda_t(\boldsymbol{y}) < w$.*

PROOF.

(a) If $\hat{y} \le \lambda_t(\boldsymbol{0})$, it follows from Lemma 5.2.5(b) with $n = 1$ that $v_{t-1}(\lambda_t(\boldsymbol{0}), \boldsymbol{y}) = v_{t-1}(\lambda_t(\boldsymbol{0}), \boldsymbol{0})$.

Hence, due to Eqs. (5.2.62) and (5.2.64) we get

$$\begin{aligned} f_t(\lambda_t(\mathbf{0})|\boldsymbol{y}) &= -r(\lambda_t(\mathbf{0})) - s + \beta v_{t-1}(\lambda_t(\mathbf{0}), \boldsymbol{y}) - \lambda_t(\mathbf{0}) \\ &= -r(\lambda_t(\mathbf{0})) - s + \beta v_{t-1}(\lambda_t(\mathbf{0}), \mathbf{0}) - \lambda_t(\mathbf{0}) \\ &= f_t(\lambda_t(\mathbf{0})|\mathbf{0}) \;=\; 0. \end{aligned} \tag{5.2.128}$$

Therefore, if $\hat{y} \leq \lambda_t(\mathbf{0})$, then $\lambda_t(\boldsymbol{y})$, the unique root of $f_t(w|\boldsymbol{y}) = 0$, must be equal to $\lambda_t(\mathbf{0})$.

The latter part is proven by contraposition. Suppose $\lambda_t(\boldsymbol{y}) \in W_t(\boldsymbol{x})$. Then,

$$\hat{y} \leq \hat{x} \leq z_t^o(\boldsymbol{x}) \leq z_t^r(\lambda_t(\boldsymbol{y}), \boldsymbol{y}) = \lambda_t(\boldsymbol{y}) \tag{5.2.129}$$

from Eqs. (5.2.27), (5.2.29), and Corollary 5.2.2(b2). If $\hat{y} \leq \lambda_t(\boldsymbol{y})$, then $\lambda_t(\boldsymbol{y}) = \lambda_t(\mathbf{0})$ because we obtain $f_t(\lambda_t(\boldsymbol{y})|\boldsymbol{y}) = f_t(\lambda_t(\boldsymbol{y})|\mathbf{0}) = 0$ in the same way as in Eq. (5.2.128). We have thus checked that if $\lambda_t(\boldsymbol{y}) \in W_t(\boldsymbol{x})$, then $\hat{y} \leq \lambda_t(\boldsymbol{y}) = \lambda_t(\mathbf{0})$. Hence, the latter part proves true.

(b) Since any w satisfies either $w \leq \lambda_t(\mathbf{0})$ or $\lambda_t(\mathbf{0}) < w$, the assertion holds for $\hat{y} \leq \lambda_t(\mathbf{0})$ from the former part of assertion (a).

The proof for $\lambda_t(\mathbf{0}) < \hat{y}$ is by contradiction. Choose an $\boldsymbol{x}^2$ with $\lambda_t(\mathbf{0}) < \hat{y}^2$ and suppose that a certain $w \in W_t(\boldsymbol{x}^2)$ satisfies $\lambda_t(\mathbf{0}) < w \leq \lambda_t(\boldsymbol{y}^2)$. Then, since $\lambda_t(\boldsymbol{y})$ is continuous and nondecreasing in $\boldsymbol{y}$ from Theorem 5.2.2(d), there is an $\boldsymbol{x}^1$ such that $\mathbf{0} \leq \boldsymbol{x}^1 \leq \boldsymbol{x}^2$ and $w = \lambda_t(\boldsymbol{y}^1)$ according to the intermediate value theorem. Hence, it follows that

$$\lambda_t(\mathbf{0}) < w = \lambda_t(\boldsymbol{y}^1) \leq \lambda_t(\boldsymbol{y}^2). \tag{5.2.130}$$

From $\lambda_t(\mathbf{0}) < \lambda_t(\boldsymbol{y}^1)$ in Eq. (5.2.130) and the contraposition of the former part of (a) we get $\lambda_t(\mathbf{0}) < \hat{y}^1$. Due to this and the latter part of (a) we obtain $(w =)\ \lambda_t(\boldsymbol{y}^1) \notin W_t(\boldsymbol{x}^1)$.

To sum up the above, if there is a $w \in W_t(\boldsymbol{x}^2)$ such that $\lambda_t(\mathbf{0}) < w \leq \lambda_t(\boldsymbol{y}^2)$, then $w \notin W_t(\boldsymbol{x}^1)$ for a certain $\boldsymbol{x}^1$ with $\boldsymbol{x}^1 \leq \boldsymbol{x}^2$. However, this is a contradiction because $w \in W_t(\boldsymbol{x}^1)$ must hold for any $\boldsymbol{x}^1$ with $\boldsymbol{x}^1 \leq \boldsymbol{x}^2$ according to Theorem 5.2.3(a). Therefore, no $w \in W_t(\boldsymbol{x})$ satisfies $\lambda_t(\mathbf{0}) < w \leq \lambda_t(\boldsymbol{y})$, that is, any $w \in W_t(\boldsymbol{x})$ satisfies either $w \leq \lambda_t(\mathbf{0})$ or $\lambda_t(\boldsymbol{y}) < w$. ∎

Corollary 5.2.4 *For any $w \in W_t(\boldsymbol{x})$ we have $w \leq \lambda_t(\boldsymbol{y})$ if and only if $w \leq \lambda_t(\mathbf{0})$.*

PROOF. Any $w \in W_t(\boldsymbol{x})$ such that $w \leq \lambda_t(\boldsymbol{y})$ satisfies $w \leq \lambda_t(\mathbf{0})$ since it is impossible for $w \in W_t(\boldsymbol{x})$ to satisfy $\lambda_t(\mathbf{0}) < w \leq \lambda_t(\boldsymbol{y})$ according to Lemma 5.2.10(b).

Any $w \in W_t(\boldsymbol{x})$ such that $\lambda_t(\boldsymbol{y}) < w$ satisfies $\lambda_t(\mathbf{0}) < w$ because $\lambda_t(\mathbf{0}) \leq \lambda_t(\boldsymbol{y})$ by Theorem 5.2.2(d). ∎

Theorem 5.2.4 *If $r(w)$ is concave, $W_t(\boldsymbol{x})$ is a connected set for any $\boldsymbol{x}$.*

PROOF. In the same way as in the proof of Theorem 4.2.6 (p.29), we can show that if $r(w)$ is concave, $W_t(\boldsymbol{x})$ is connected for each x_i under given any $\boldsymbol{x}_i \in R^{k-1}$. ∎

5.2.4 Infinite Planning Horizon

Theorem 5.2.5

(a) $\bar{u}_t(w) \le u_t(w, \boldsymbol{x}) \le \tilde{u}_t(w, \hat{x})$ *for any* w *and* $\boldsymbol{x} \in R^k$.

(b) $u_t(w, \boldsymbol{x})$ *converges to* $u(w, \boldsymbol{x}) = \max\{w, \hat{x}, \theta\}$ *as* $t \to \infty$.

PROOF. See Lemma 4.2.9 (p.31) and Theorem 4.2.8 (p.32), respectively. ∎

This theorem yields the optimal decision rule for an infinite planning horizon:

◇ **Optimal Decision Rule:** In the case of an infinite planning horizon, if $\theta \le \max\{w, \hat{x}\}$, accept the more lucrative between the current offer w and the leading offer $\hat{x}$, or else continue the search.

5.3 Numerical Example

In the case of $k = 2$, all offers to be considered for each time are reserved offers x_1 $(= \boldsymbol{y})$, x_2 $(= x_k)$, and current offer w. Hence, in this case, the optimal decision rule can be schematized in 3-dimensional diagrams like Figure 5.3.1, which illustrates the optimal decision rule for $t = 1$ with provision that $F(w)$ is the uniform distribution on $[1, 2]$ (, so $a = 1$ and $b = 2$), $\beta = 0.95$, $s = 0.005$, and $r(w) = 0.002w$.

Figure 5.3.2(a) is the cross section of Figure 5.3.1 with $x_2 = 1.3$, and (b) is the one with $x_1 = 1.4$. The thick curved lines in (a) and (b) represent $\{(x_1, x_2, w) \mid z_1^o(x_1, x_2) = z_1^r(w, x_1),\ x_2 = 1.3\}$ and $\{(x_1, x_2, w) \mid z_1^o(x_1, x_2) = z_1^r(w, x_1),\ x_1 = 1.4\}$, respectively. Hence the areas on the left sides of the thick curved lines shows $W_1(x_1, 1.3)$ and $W_1(1.4, x_2)$, respectively. Either of the diagrams indicates that if we have the reserved offer vector $(x_1, x_2) = (1.4, 1.3)$ at $t = 1$, the best choice for a current offer w is:

If $1.570 \le w \le 2.000$, AS (accept the current offer w and stop the search).
If $1.410 \le w \le 1.570$, RC (reserve the current offer w and continue the search).
If $1.000 \le w < 1.410$, PC (pass up the current offer w and continue the search).

Figure 5.3.3 shows the optimal decision rules for $t = 1$ to 4.

Next, we shall show two cases of the search process.

Case 1: Table 5.1 shows a result calculated on the condition that $F(w)$ is a discrete uniform distribution function with

$$\Pr\{w = 0.25\} = \Pr\{w = 0.50\} = \Pr\{w = 0.75\} = \Pr\{w = 1.00\} = 0.25, \tag{5.3.1}$$

and that $s = 0.1$, $\beta = 0.9$, $r(w) = 0.001w$, and $k = 2$ (, or $\boldsymbol{y} = x_1$ and $x_k = x_2$). Since $w \in \{0.25, 0.50, 0.75, 1.00\}$ for each time due to Eq. (5.3.1), it follows that $\theta_t^1(x_2)$, $\theta_t^2(x_1)$, and

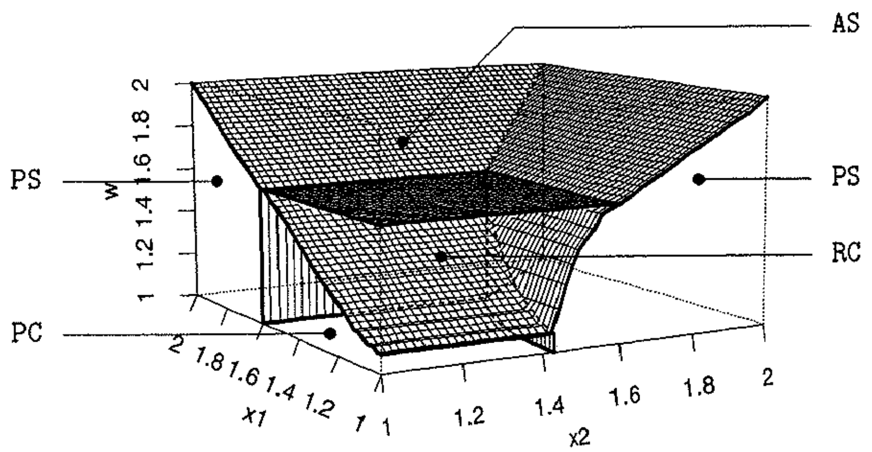
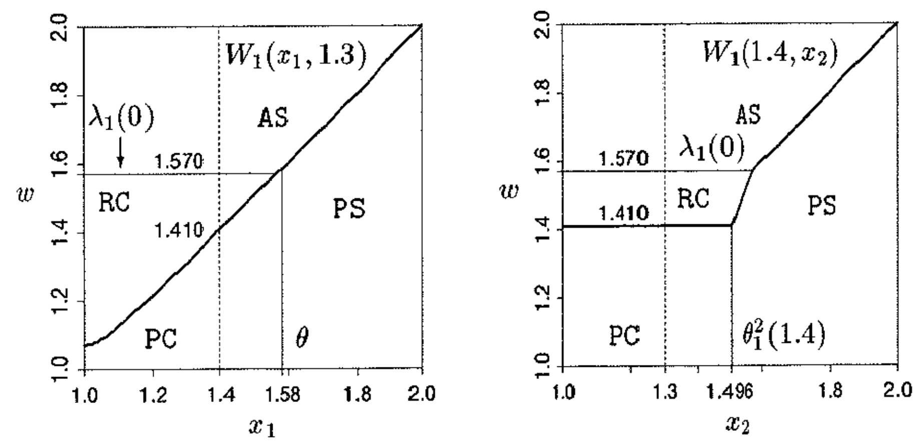


Figure 5.3.1: Optimal Decision Rule for $t = 1$ with $k = 2$



(a) Cross section with $x_2 = 1.3$ (b) Cross section with $x_1 = 1.4$

Figure 5.3.2: Cross sections of Figure 5.3.1

$\lambda_t(x_1)$ can be substituted, respectively, by

$$\theta_t^1(x_2)' = \max\{x_1 \mid x_1 \le \theta_t^1(x_2),\ x_1 \in \{0.25, 0.50, 0.75, 1.00\}\},$$

$$\theta_t^2(x_1)' = \max\{x_2 \mid x_2 \le \theta_t^2(x_1),\ x_2 \in \{0.25, 0.50, 0.75, 1.00\}\},$$

$$\lambda_t(x_1)' = \max\{w \mid w \le \lambda_t(x_1),\ w \in \{0.25, 0.50, 0.75, 1.00\}\}.$$

The way to see Table 5.1 is as follows: Suppose that we are at $t = 3$ with no reserved offers, or $(x_1, x_2) = (0.00, 0.00)$, and find an offer $w = 0.50$. Then, since $0.50 \in W_3(0.00, 0.00)$ and $0.50 \le \lambda_3(0.00)'$, the offer $w = 0.50$ should be reserved due to the optimal decision rule (a2) on p.51. Hence, $\boldsymbol{x} = (0.50, 0.00)$ at $t = 2$. If $w = 0.25 \notin W_2(0.50, 0.00)$ appears at $t = 2$, it should be passed up, and since $x_1 = 0.50 \le \theta_2^1(0.00)'$ and $x_2 = 0.00 \le \theta_2^2(0.50)'$, the search should be continued from (b2) of the rule. Therefore, we reach $t = 1$ with $\boldsymbol{x} = (0.00, 0.50)$. If $w = 0.25 \notin W_1(0.00, 0.50)$ appears, it should be passed up, and since $x_1 = 0.00 \le \theta_1^1(0.50)'$ but

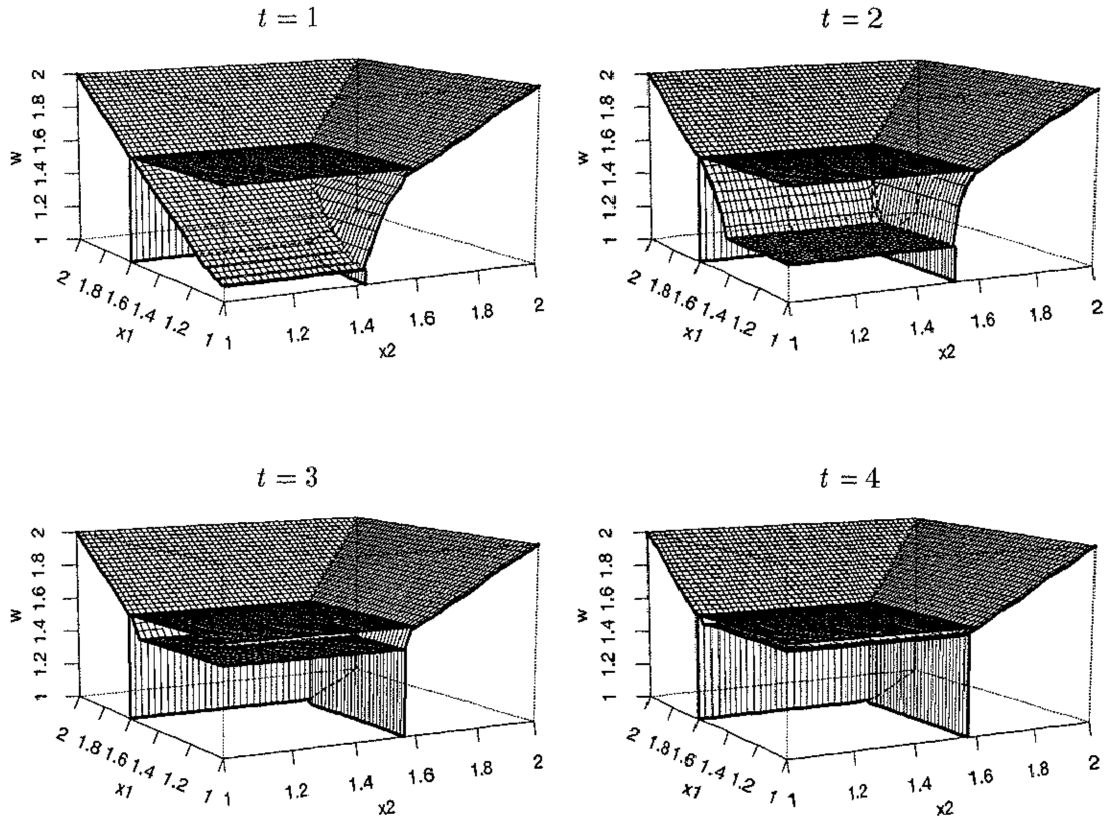


Figure 5.3.3: Optimal Decision Rule for $t = 1, 2, 3, 4$ with $k = 2$

$x_2 = 0.50 > \theta_1^2(0.00)'$, the search should be stopped by accepting the leading offer $x_2 = 0.50$ by virtue of (b1) of the rule.

From the above, we find that the leading offer can be accepted at neither the starting time nor the deadline. Note that such a thing does not happen at all in the model with finite-period reservation (Property C on p.35).

Table 5.1: Case 1 of the search process

t	(x_1, x_2)	θ'	$\theta_t^2(x_1)'$	$W_t(x_1, x_2)$	$\lambda_t(x_1)'$	w	Decision	
3	(0.00, 0.00)	0.50	0.50	{0.50, 0.75, 1.00}	0.50	0.50	RC	(Reserve $w = 0.50$)
2	(0.50, 0.00)	0.50	0.50	{0.50, 0.75, 1.00}	0.50	0.25	PC	
1	(0.00, 0.50)	0.50	0.25	{0.50, 0.75, 1.00}	0.50	0.25	PS	(Accept $x_2 = 0.50$)

Case 2: The condition for Table 5.2 is the same as that used for Figure 5.3.1. From Table 5.2 we shall confirm that, at $t = 3$, the offer $w = 1.54$, which is inferior to the available reserved offer $x_1 = 1.55$, is reserved. This is different from Property D of Model 1 on p.36.

Table 5.2: Case 2 of the search process

t	$(x_1 \quad ,x_2 \quad)$	θ	$\theta_t^2(x_1)$	$W_t(x_1, x_2)$	$\lambda_t(0)$	w	Decision	
4	(0.00 ,0.00)	1.580	1.569	$1.545 \leq w$	1.575	1.55	RC	(Reserve $w = 1.55$)
3	(1.55 ,0.00)	1.580	1.566	$1.535 \leq w$	1.575	1.54	RC	(Reserve $w = 1.54$)
2	(1.54 ,1.55)	1.580	1.560	$1.515 \leq w$	1.575	1.50	PC	
1	(0.00 ,1.54)	1.580	1.422	$1.510 \leq w$	1.570	1.50	PS	(Accept $x_2 = 1.54$)

5.4 Properties of the Optimal Decision Rule

A. *If $\alpha \leq a$, then the continuation of the search is not optimal at all.*

This is a restatement of the optimal decision rule on p.45, which is exactly the same as Property A of Model 1 on p.35.

The following properties are obtained for the case of $a < \alpha$.

B. *For each time t with any $\boldsymbol{x}$, an offer $w \in W_t(\boldsymbol{x})$ must be reserved if and only if $w \leq \lambda_t(\mathbf{0})$.*

The result is immediate from Corollary 5.2.4 and the optimal decision rule (a2) on p.51. Note that, whatever $\boldsymbol{x}$ we have, if $w \in W_t(\boldsymbol{x})$, we only have to compare it with $\lambda_t(\mathbf{0})$ in order to decide whether or not to reserve it.

C. *If the reserved offer vector $\boldsymbol{x}$ is such that $\theta \leq \hat{x}$, then accept the more lucrative between the leading offer $\hat{x}$ and the current offer w.*

This is already stated in the optimal decision rule on p.55. Hence, if an offer vector $\boldsymbol{x}$ with $\theta \leq \hat{x}$ is given before entering the search, we should immediately stop the search by accepting the leading offer or the offer found from the first search.

D. *An offer reserved during the search process must not be accepted prior to its maturity of reservation, however, it may be accepted on the maturity even before the deadline.*

From the assumption $0 \leq a < \alpha$ and Lemma 3.2.3(b) on p.12 we have $0 \leq a < \alpha \leq \theta$, thus $\hat{0} < \theta$, yielding $\lambda_t(\mathbf{0}) < \theta$ for every t by Theorem 5.2.2(c). Consequently, from Property B, all offers to be reserved throughout the search process have less value than θ. Hence, if the search starts with the reserved offer vector $\boldsymbol{x}$ such that $\hat{x} < \theta$, the inequality holds forever, or $x_i < \theta$ for all i for every t. So, $\hat{y} < \theta$ and $\hat{y}_i < \theta$ for all $i < k$ and for all t. In this case, $\theta_t^i(\boldsymbol{x}_i) = \theta$ for $i < k$ by Theorem 5.2.2(a), thus $\theta_t^i(\boldsymbol{x}_i) < x_i$ never happens for $i < k$. However, $\theta_t^k(\boldsymbol{y}) < x_k$ is possible (see Tables 5.1 and 5.2 in Section 5.3).

The above facts suggest that no reserved offer x_i must be accepted if it remains available at the next time, or $i < k$, and that only the offer x_k, which is at maturity, has a chance to be accepted. This can be interpreted as follows: Since any offer once reserved is assumed not to deteriorate in its value over the search process, it seems a waste to accept an offer while its effective periods still remain.

This is one of points different from Property C of Model 1 (p.35) which dissuades us from accepting any reserved offer prior to the deadline. In Model 1, however, since the length of the effective periods is assumed to be infinite, any reserved offer available at time t remains available at the next time $t-1$, so no offer reaches the maturity. For the reason, Property C of Model 1 does not contradict Property D of this model.

What is to be emphasized here is that, although offers are reserved only to prevent the risk at the deadline in the model with infinite-period reservation, in the model with finite-period reservation we reserve offers so as not only to prevent that risk but also to facilitate stopping the search when we see no reason to pursue it further.

E. *If the reserved offer vector is better, the range of offers to be passed up should be wider.*
This is clear from Theorem 5.2.3(a).

F. *Although every offer should be passed up if it is inferior to any of the reserved offers which will be still available at the deadline, it may prove wise to reserve even an offer inferior to some of the reserved offers which will expire prior to the deadline.*
Theorem 5.2.3(b) indicates the result. In fact, we have a case that, with a reserved offer x_i, an offer w is to be reserved despite $w < x_i$

In Model 1 an offer to be reserved must be superior to the leading offer at each time due to Property D. This fact is different from Property F in the model but can be taken as consistent with it from the viewpoint that each reserved offer is assumed to be available at the deadline in Model 1.

We should notice that, although it seems better to recall and accept a reserved offer x_i than to reserve an offer w with $w < x_i$, it can be optimal to reserve such an offer w. Undoubtedly such an offer w will not be recalled and accepted while the offer x_i is available, but it is to be reserved as further insurance against any unfortunate situation awaiting after the expiration of the offer x_i.

G. *If $r(w)$ is concave, the indifferent point between reserving an offer and passing up an offer is determined at one critical point.*
This is immediately derived from Theorem 5.2.4, which is exactly the same as Property F of Model 1 on p.36.

H. *If the planning horizon is infinite, we should continue the search with reserving no offers until an offer exceeding θ is found.*
This is a restatement of the optimal decision rule on p.62, which is exactly the same as Property G of Model 1 on p.36.

From all the stated above, the optimal decision rule can be summarized as follows:

◇ **Optimal Decision Rule:** Suppose that you are at time t with the reserved offer vector $\boldsymbol{x}$ and have just drawn an offer w. Let $\boldsymbol{x}^0 \in R^k$ and $\hat{x}^0$ be the initial offer vector and the initial leading offer, respectively, thus $\boldsymbol{x} = \boldsymbol{x}^0$ and $\hat{x} = \hat{x}^0$ if time t is the start point of the search process. Then, the choices are:

(a) If $\alpha \leq a$ or $\theta \leq \hat{x}^0$, then:

1. AS if the offer w found at the start is such that $\hat{x}^0 \le w$ (accept it and stop the search).
2. PS otherwise (accept the initial leading offer $\hat{x}^0$ and stop the search).

(b) If $a < \alpha$ and $\hat{x}^0 < \theta$, then:

1. If $t = 0$ (deadline), then:
   - i AS if $\hat{x} \le w$ (accept the current offer w and stop the search).
   - ii PS otherwise (accept the leading offer $\hat{x}$ and stop the search).
2. If $t \ge 1$, then:
   - i AS if $w \in W_t(\boldsymbol{x})$ and $\lambda_t(\mathbf{0}) < w$ (accept the current offer w and stop the search).
   - ii RC if $w \in W_t(\boldsymbol{x})$ and $w \le \lambda_t(\mathbf{0})$ (reserve the current offer w and continue the search).
   - iii PS if $w \notin W_t(\boldsymbol{x})$ and $\theta_t^k(\boldsymbol{y}) < x_k = \hat{x}$ (pass up the current offer w and stop the search by accepting the leading offer x_k).
   - iv PC if $w \notin W_t(\boldsymbol{x})$ and $x_k \le \theta_t^k(\boldsymbol{y})$ (pass up the current offer w and continue the search).
3. If $t = \infty$ (infinite planning horizon), then:
   - i AS if $\theta \le w$ (accept the current offer w and stop the search).
   - ii PC otherwise (pass up the current offer w and continue the search).

Chapter 6

Model 3: Remaining Time Value

This chapter is devoted to the discrete-time optimal stopping problem where any of offers appearing subsequently can be reserved by paying the reserving cost and any reserved offer is allowed to be recalled and accepted at any time in the future. Furthermore, at the time of acceptance we can receive not only the value of the accepted offer but also the remaining time value, which increases as the number of the remaining periods increases. Two types of the remaining time value are considered: convex type and β-additive type. A major finding is that no reserved offer should be recalled and accepted prior to the deadline of the search process.

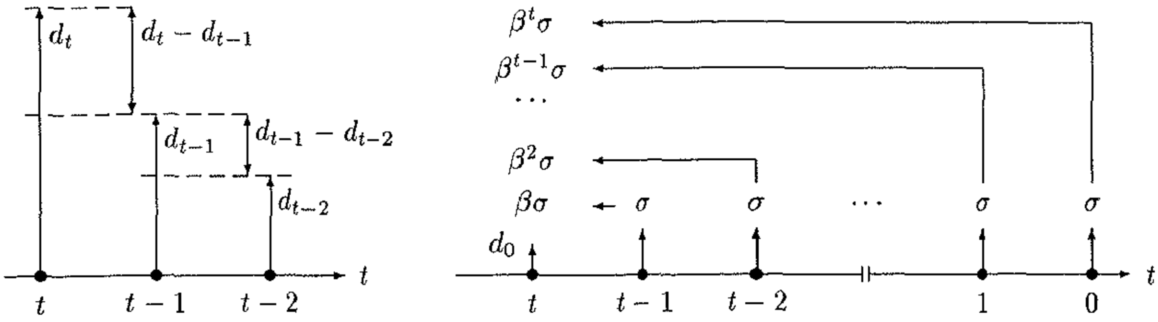
6.1 Model

Suppose that a person periodically searches for offers with the intention to accept one of them over the t periods from time t to the deadline $t = 0$ where the value of an offer w belongs to $[a, b]$ and follows a known offer value distribution function $F(w)$ with the mean μ. The search cost $s > 0$ is incurred to find an offer, and if the reserving cost $r(w) > 0$ is paid, any offer w can be reserved eternally as well as accepted or passed up. His objective is to maximize the total expected present discounted net profit.

After stopping the search, he is assumed to use the extra time to get some money by engaging in another economic activity instead of the search activity: More precisely, by stopping the search at time t, he gains the *remaining time value* d_t, that will be obtained over the remaining t periods up to the deadline $t = 0$. The idea of the remaining time value is presented in Sato [51]. Let us postulate that d_t is nonnegative and strictly increasing in t, that is,

$$0 \leq d_0 < d_1 < d_2 < \cdots < d_t < \cdots. \tag{6.1.1}$$

Furthermore, d_t is supposed to be either convex or concave in t where, in the thesis, "d_t is convex (concave) in t" means that $d_t - d_{t-1}$ is nondecreasing (nonincreasing) in t. In the case where d_t



(a) d_t is convex in t.

(b) d_t is β-additive in t

Figure 6.1.1: Two types of d_t.

is convex in t, that is,

$$(0 <)\ d_1 - d_0 \leq d_2 - d_1 \leq \cdots \leq d_t - d_{t-1} \leq \cdots \tag{6.1.2}$$

(see Figure 6.1.1(a)), it indicates the case where the remaining time value rises steeply as the remaining period becomes longer, and it is possible to find distinctive properties of the optimal decision rule. The case where d_t is concave in t, however, is yet to be investigated. For this reason, as a special case of a concave remaining time value, we shall treat d_t such that $d_0 \geq 0$ and

$$d_t = d_0 + (\beta + \beta^2 + \cdots + \beta^t)\sigma > 0, \quad \sigma > 0, \quad t \geq 1, \tag{6.1.3}$$

which represents the case where $\sigma > 0$ can be received per period throughout the t periods remaining up to the deadline (see Figure 6.1.1(b)). For expressional simplicity, let "d_t is β-additive in t" mean that d_t is expressed as in Eq.(6.1.3).

6.2 Analysis

Without loss of generality, we can consider the value of an offer to be in $(-\infty, b]$. So, for expressional simplicity, let "for any w" and "for any x" mean "for any w with $w \leq b$" and "for any x with $x \leq b$," respectively.

6.2.1 Optimal Equation

Since each reserved offer is assumed to be available at any time in the future, we can ignore all reserved offers except for the leading offer.

Let $u_t(w, x)$ denote the maximum total expected present discounted net profit attainable by starting the search for offers from time t with the current offer w and the leading offer x, and let $v_t(x)$ denote the expectation of $u_t(w, x)$ with respect to w, that is,

$$v_t(x) = \int_a^b u_t(w, x)dF(w), \quad t \geq 0. \tag{6.2.1}$$

If we stop the search at time t by accepting either the current offer or the leading offer, we can receive not only the value of the offer accepted but also the remaining time value d_t. On these grounds, $u_t(w, x)$ can be expressed as follows:

$$u_0(w, x) = \max\left\{\begin{array}{lcl} \texttt{AS} & : & w + d_0, \\ \texttt{PS} & : & x + d_0 \end{array}\right\}, \tag{6.2.2}$$

$$u_t(w, x) = \max\left\{\begin{array}{lcl} \texttt{AS} & : & w + d_t, \\ \texttt{RC} & : & -r(w) - s + \beta v_{t-1}(\max\{w, x\}), \\ \texttt{PS} & : & x + d_t, \\ \texttt{PC} & : & -s + \beta v_{t-1}(x) \end{array}\right\}, \quad t \geq 1. \tag{6.2.3}$$

Due to (6.2.2), Eqs. (6.2.1), and (3.1.1), we get

$$v_0(x) = \int_a^b u_0(w, x)dF(w) = \int_a^b \Big(\max\{w, x\} + d_0\Big)dF(w) = S(x) + d_0. \tag{6.2.4}$$

Lemma 6.2.1

(a) $u_t(w, x)$ *is :*

1. *continuous in w and x,*
2. *nondecreasing in x,*
3. *convex in x,*
4. *nondecreasing in t.*

(b) $v_t(x)$ *is :*

1. *continuous in x,*
2. *nondecreasing in x,*
3. *convex in x,*
4. *nondecreasing in t.*

PROOF. By noting that d_t is independent of x and w, and that d_t is strictly increasing in t, we can prove all the assertions in almost the same way as in the proofs of Lemma 4.2.1 (p.15) and Corollary 4.2.1 (p.17). ∎

Lemma 6.2.2 *For $t \geq 0$:*

(a) $x + d_t < v_t(x)$ *for* $x < b$.

(b) $v_t(b) = b + d_t$.

(c) $\mu + d_t \leq v_t(x) \leq b + d_t$ *for any* x.

(d) $\beta v_t(x) - x$ *is strictly decreasing in* x.

PROOF.

(a) Given any $x < b$, choose an x^1 so that $x < x^1 < b$, and let $x^2 = \max\{a, x^1\}$. Then $a \leq x^2 < b$, and thus $1 - F(x^2) > 0$. Further, $w - x \geq x^1 - x > 0$ for $w \geq x^2$ ($\geq x^1$). Hence, it follows from Eqs. (6.2.2) and (6.2.3) that

$$\begin{aligned} v_t(x) - x - d_t &= \int_a^{x^2} \Big(u_t(w,x) - x - d_t\Big) dF(w) + \int_{x^2}^b \Big(u_t(w,x) - x - d_t\Big) dF(w) \\ &\geq \int_a^{x^2} \Big(x + d_t - x - d_t\Big) dF(w) + \int_{x^2}^b \Big(w + d_t - x - d_t\Big) dF(w) \\ &\geq 0 + \int_{x^2}^b (x^1 - x) dF(w) \ = \ (x^1 - x)\Big(1 - F(x^2)\Big) \ > \ 0, \end{aligned}$$

from which we conclude $x + d_t < v_t(x)$ for $x < b$.

(b-d) Each of the assertions can be proven by applying almost the same way as in the proof of Lemma 4.2.2(b-d) on p.17, respectively. ∎

Lemma 6.2.3 *For any w, x, and $t \geq 1$,*

$$\max\left\{\begin{array}{l} w + d_t, \\ -r(w) - s + \beta v_{t-1}(\max\{w,x\}), \\ x + d_t, \\ -s + \beta v_{t-1}(x) \end{array}\right\} = \max\left\{\begin{array}{l} w + d_t, \\ -r(w) - s + \beta v_{t-1}(w), \\ x + d_t, \\ -s + \beta v_{t-1}(x) \end{array}\right\}.$$

PROOF. Easy by using almost the same fashion as in the proof of Lemma 4.2.3 (p.18). ∎

Lemma 6.2.3 enables us to rewrite $u_t(w,x)$, defined by Eq. (6.2.3), as follows:

$$u_t(w,x) = \max\left\{\begin{array}{lcl} \mathsf{AS} & : & w + d_t, \\ \mathsf{RC} & : & -r(w) - s + \beta v_{t-1}(w), \\ \mathsf{PS} & : & x + d_t, \\ \mathsf{PC} & : & -s + \beta v_{t-1}(x) \end{array}\right\}, \quad t \geq 1. \tag{6.2.5}$$

6.2.2 Optimal Decision Rule

6.2.2.1 Common Discussion

Let us define the two functions $z_t^o(x)$ and $z_t^r(w)$ as follows:

$$z_t^o(x) = \max\{x + d_t, \ -s + \beta v_{t-1}(x)\}, \qquad t \geq 1, \tag{6.2.6}$$

$$z_t^r(w) = \max\{w + d_t, \ -r(w) - s + \beta v_{t-1}(w)\}, \quad t \geq 1, \tag{6.2.7}$$

$z_0^o(x) = x$, and $z_0^r(w) = w$. Furthermore, we define the following set:

$$W_t(x) = \{w \mid z_t^o(x) \leq z_t^r(w)\}, \quad t \geq 0. \tag{6.2.8}$$

The meanings of $z_t^o(x)$, $z_t^r(w)$, and $W_t(x)$ are exactly the same as those stated in p.19. Then,

$$u_t(w,x) = \max\{z_t^r(w),\ z_t^o(x)\} \tag{6.2.9}$$

$$= \begin{cases} z_t^r(w) & \text{if } w \in W_t(x), \\ z_t^o(x) & \text{if } w \notin W_t(x), \end{cases} \quad t \geq 0, \tag{6.2.10}$$

from which

$$v_t(x) = \int_a^b \max\{z_t^r(w),\ z_t^o(x)\}dF(w) \tag{6.2.11}$$

$$= \int_{W_t(x)} z_t^r(w)dF(w) + \int_{W_t(x)^c} z_t^o(x)dF(w), \quad t \geq 0. \tag{6.2.12}$$

Lemma 6.2.4 *If $\alpha + \beta d_0 \leq a + d_1$, then $u_t(w,x) = \max\{w,x\} + d_t$ for any $w \geq a$, $x \geq 0$, and $t \geq 0$.*

PROOF. Suppose $\alpha + \beta d_0 \leq a + d_1$, or $\alpha - a \leq d_1 - \beta d_0$.

If d_t is convex in t, from Eq. (6.1.2) we have $\beta(d_t - \beta d_{t-1}) \leq d_t - d_{t-1} \leq d_{t+1} - d_t$ for any $t \geq 1$, thus

$$d_t - \beta d_{t-1} \leq d_{t+1} - \beta d_t, \qquad t \geq 1. \tag{6.2.13}$$

If d_t is β-additive in t, from Eq. (6.1.3) we get

$$d_t - \beta d_{t-1} = \beta\sigma = d_{t+1} - \beta d_t, \qquad t \geq 1. \tag{6.2.14}$$

Due to Eqs. (6.2.13) and (6.2.14), in either type of d_t we get

$$\alpha - a \leq d_1 - \beta d_0 \leq d_2 - \beta d_1 \leq \cdots. \tag{6.2.15}$$

By noting Eq. (6.2.15) and applying almost the same fashion as in the proof of Lemma 4.2.1 (p.20), we can prove the assertion. ∎

Due to the above lemma, we get the following optimal decision rule:

◇ **Optimal Decision Rule:** In the case where $\alpha + \beta d_0 \leq a + d_1$, if $w \geq x$, accept the current offer w and stop the search, or else accept the leading offer x and stop the search.

It should be noted that, in the case where d_t is convex in t, even if $\alpha + \beta d_{t-1} \leq a + d_t$ holds at a certain $t \geq 1$, we may have $\alpha + \beta d_0 > a + d_1$. In the case where d_t is β-additive in t, however, if $\alpha + \beta d_{t-1} \leq a + d_t$ holds at a certain $t \geq 1$, we get $\alpha + \beta d_0 \leq a + d_1$.

Lemma 6.2.5

(a) *$z_t^o(x)$ is continuous, nondecreasing, and convex in x, and nondecreasing in t.*

(b) $z_t^r(w)$ *is continuous in* w, *and nondecreasing in* t.

(c) $W_t(x)$ *is a closed set such that* $b \in W_t(x)$ *for any* x *and* $t \geq 0$.

PROOF. Assertions (a) and (b) can be easily confirmed due to Eqs.(6.2.6), (6.2.7), and Lemma 6.2.1(b). Assertion (c) can be proven by using almost the method as in the proof of Lemma 4.2.5(e) on p.21. ∎

Here, let us define the two functions $g_t(x)$ and $f_t(w)$ with $t \geq 1$ as follows:

$$g_t(x) = -s + \beta v_{t-1}(x) - x - d_t, \qquad t \geq 1, \tag{6.2.16}$$

$$f_t(w) = -r(w) - s + \beta v_{t-1}(w) - w - d_t, \qquad t \geq 1. \tag{6.2.17}$$

Corollary 6.2.1 *For* $t \geq 1$:

(a) $g_t(x)$ *is continuous and strictly decreasing in* x.

(b) $f_t(w)$ *is continuous and strictly decreasing in* w.

PROOF. By noting that $r(w)$ is continuous and nondecreasing in w, and using Eqs.(6.2.16), (6.2.17), Lemmas 6.2.1(b1), and 6.2.2(d), we will get both assertions. ∎

Now, we define θ_t and λ_t with $t \geq 1$ as the respective roots of $g_t(x) = 0$ and $f_t(w) = 0$, if they exist, that is,

$$g_t(\theta_t) = -s + \beta v_{t-1}(\theta_t) - \theta_t - d_t = 0, \qquad t \geq 1, \tag{6.2.18}$$

$$f_t(\lambda_t) = -r(\lambda_t) - s + \beta v_{t-1}(\lambda_t) - \lambda_t - d_t = 0, \qquad t \geq 1. \tag{6.2.19}$$

Obviously, θ_t is a point of indifference between PS and PC, and λ_t is that between AS and RC.

Lemma 6.2.6 *For* $t \geq 1$:

(a) θ_t *exists uniquely with* $a - (d_t - \beta d_{t-1}) \leq \theta_t < b - (d_t - \beta d_{t-1})$.

(b) λ_t *exists uniquely with* $a - r(b) - (d_t - \beta d_{t-1}) \leq \lambda_t < \theta_t$.

PROOF. Lemma 6.2.2(c) implies $g_t(a - d_t + \beta d_{t-1}) \geq 0 > g_t(b - d_t + \beta d_{t-1})$. From this and Corollary 6.2.1(a) we get (a). Assertion (b) can be proven in a like manner. ∎

Corollary 6.2.2

(a) *For* $t \geq 1$:

1. *If* $x < \theta_t$, *then* $x + d_t < -s + \beta v_{t-1}(x)$.
2. *If* $x = \theta_t$, *then* $x + d_t = -s + \beta v_{t-1}(x)$.
3. *If* $x > \theta_t$, *then* $x + d_t > -s + \beta v_{t-1}(x)$.

(b) *For* $t \geq 1$:

1. *If* $w < \lambda_t$, *then* $w + d_t < -r(w) - s + \beta v_{t-1}(w)$.
2. *If* $w = \lambda_t$, *then* $w + d_t = -r(w) - s + \beta v_{t-1}(w)$.
3. *If* $w > \lambda_t$, *then* $w + d_t > -r(w) - s + \beta v_{t-1}(w)$.

PROOF. Both assertions are evident from Corollary 6.2.1 and Lemma 6.2.6. ∎

It follows from Eq. (6.2.10) and Corollary 6.2.2 that, for $t \geq 1$,

$$u_t(w,x) = \begin{cases} z_t^r(w) = \begin{cases} w + d_t & \text{if } w \in W_t(x) \text{ and } \lambda_t < w, \\ -r(w) - s + \beta v_{t-1}(w) & \text{if } w \in W_t(x) \text{ and } w \leq \lambda_t, \end{cases} \\ z_t^o(x) = \begin{cases} x + d_t & \text{if } w \notin W_t(x) \text{ and } \theta_t < x, \\ -s + \beta v_{t-1}(x) & \text{if } w \notin W_t(x) \text{ and } x \leq \theta_t. \end{cases} \end{cases} \quad (6.2.20)$$

Here, define $\theta_0 = \lambda_0 = -\infty$ for convenience. Then, in general, due to Eq. (6.2.20) we can prescribe the optimal decision rule as follows:

◇ **Optimal Decision Rule:** Suppose that you are at time t with the leading offer x and have just drawn an offer w. Then, the choices are:

(a) If $w \in W_t(x)$, then:

1. If $\lambda_t < w$, then AS (accept the current offer w and stop the search).
2. If $w \leq \lambda_t$, then RC (reserve the current offer w and continue the search).

(b) If $w \notin W_t(x)$, then:

1. If $\theta_t < x$, then PS (pass up the current offer w and stop the search by accepting the leading offer x).
2. If $x \leq \theta_t$, then PC (pass up the current offer w and continue the search).

Lemma 6.2.7 *For* $t \geq 1$ *we have*

$$z_t^r(w) \begin{cases} < \theta_t + d_t & \text{if } w < \theta_t, \\ = w + d_t & \text{if } \theta_t \leq w. \end{cases}$$

PROOF. See the proof of Lemma 4.2.6 (p.24). ∎

Theorem 6.2.1 *For* $t \geq 0$:

(a) *For any* x, *if* $w \in W_t(x)$, *then* $x \leq w$.

(b) *If* $x^1 < x^2$, *then* $W_t(x^1) \supseteq W_t(x^2)$.

PROOF. See the proof of Theorem 4.2.3 (p.26). ∎

Theorem 6.2.2 *If* $r(w)$ *is concave,* $W_t(x)$ *is a connected set for any* x *and* $t \geq 0$.

PROOF. Given any x and $t \geq 0$, it follows from the contraposition of Theorem 6.2.1(a) that if $w < x$, then $w \notin W_t(x)$. By using such a w instead of a in the the proof of Theorem 4.2.6 (p.29), we can get the assertion. ∎

Theorem 6.2.3 *Let $\theta_t \leq x$. Then :*

(a) $W_t(x) = \max\{w \mid x \leq w\}$.
(b) $u_t(w, x) = \max\{w, x\} + d_t$.
(c) $v_t(x) = S(x) + d_t$.

PROOF. See the proof of Theorem 4.2.4 (p.26), replacing θ with θ_t. ∎

Theorem 6.2.3(b) indicates the optimal decision rule for the case $\theta_t \leq x$ as follows:

◊ **Optimal Decision Rule:** In the case where $\theta_t \leq x$, if $w \geq x$, accept the current offer w and stop the search, or else accept the leading offer x and stop the search.

Lemma 6.2.8 $v_t(x) - v_{t-1}(x)$ *is nonincreasing in $x \leq \theta_t$ for any $t \geq 1$.*

PROOF. See the proof of Lemma 4.2.8(a) on p.27, replacing θ with θ_t. ∎

Lemma 6.2.9

(a) *For $t \geq 1$ we have*

$$\begin{aligned}\theta_t > [=]\ \theta_{t+1} &\iff \beta\Big(v_t(\theta_t) - v_{t-1}(\theta_t)\Big) < [=]\ d_{t+1} - d_t \\ &\iff \beta\Big(v_t(\theta_{t+1}) - v_{t-1}(\theta_{t+1})\Big) < [=]\ d_{t+1} - d_t.\end{aligned}$$

(b) *For $t \geq 1$ we obtain*

$$\begin{aligned}\lambda_t > [=]\ \lambda_{t+1} &\iff \beta\Big(v_t(\lambda_t) - v_{t-1}(\lambda_t)\Big) < [=]\ d_{t+1} - d_t \\ &\iff \beta\Big(v_t(\lambda_{t+1}) - v_{t-1}(\lambda_{t+1})\Big) < [=]\ d_{t+1} - d_t.\end{aligned}$$

PROOF. In the case of $\theta_t > \theta_{t+1}$, two functions $g_t(x)$ and $g_{t+1}(x)$ can be depicted as in Figure 6.2.1 from Corollary 6.2.1(a). The figure shows

$$g_{t+1}(\theta_t) < 0 \iff \theta_t > \theta_{t+1} \iff g_t(\theta_{t+1}) > 0. \tag{6.2.21}$$

Since $g_t(\theta_t) = g_{t+1}(\theta_{t+1}) = 0$ from Eq.(6.2.18), we obtain

$$\begin{aligned}g_{t+1}(\theta_t) &= g_{t+1}(\theta_t) - g_t(\theta_t) \\ &= -s + \beta v_t(\theta_t) - \theta_t - d_{t+1} + s + \beta v_{t-1}(\theta_t) + \theta_t + d_t \\ &= \beta\Big(v_t(\theta_t) - v_{t-1}(\theta_t)\Big) - d_{t+1} + d_t\end{aligned} \tag{6.2.22}$$

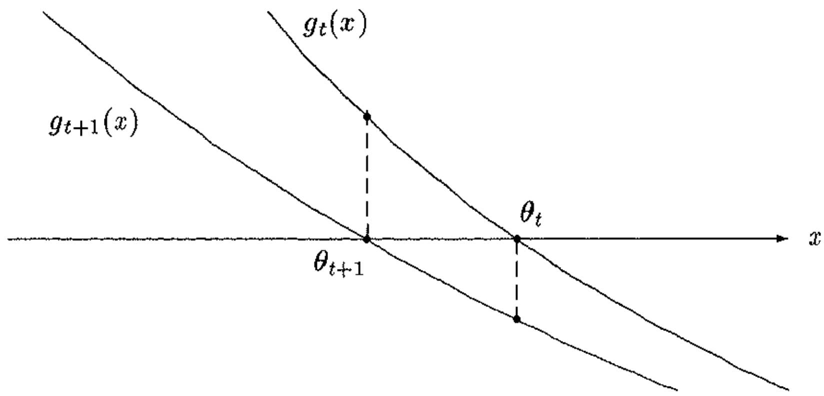


Figure 6.2.1: $g_t(x)$ and $g_{t+1}(x)$ in the case of $\theta_t > \theta_{t+1}$

and

$$\begin{aligned} g_t(\theta_{t+1}) &= g_t(\theta_{t+1}) - g_{t+1}(\theta_{t+1}) \\ &= -s + \beta v_{t-1}(\theta_{t+1}) - \theta_{t+1} - d_t + s + \beta v_t(\theta_{t+1}) + \theta_{t+1} + d_{t+1} \\ &= \beta\Big(v_{t-1}(\theta_{t+1}) - v_t(\theta_{t+1})\Big) + d_{t+1} - d_t. \end{aligned} \tag{6.2.23}$$

Therefore, it follows from Eqs.(6.2.22), (6.2.23), and Corollary 6.2.1(a) that

$$g_{t+1}(\theta_t) < 0 \iff \beta\Big(v_t(\theta_t) - v_{t-1}(\theta_t)\Big) < d_{t+1} - d_t \tag{6.2.24}$$

and

$$g_t(\theta_{t+1}) > 0 \iff \beta\Big(v_t(\theta_{t+1}) - v_{t-1}(\theta_{t+1})\Big) < d_{t+1} - d_t. \tag{6.2.25}$$

From Eqs. (6.2.21), (6.2.24), and (6.2.25), we have confirmed the assertion as for inequalities. The assertion as for equalities can be proven similarly.

(b) Similarly to the proof of (a), we can prove the assertion. ∎

Lemma 6.2.10 *Let X_t with $t \geq 1$ denote the following set :*

$$X_t = \{x \mid W_t(x) \not\supseteq W_{t+1}(x),\ x \leq \theta_t\}, \quad t \geq 1. \tag{6.2.26}$$

Then :

(a) *If $\theta_{t+1} < \theta_t$, then $X_t \neq \phi$.*

(b) *If $\theta_{t+1} = \theta_t$, then $X_t = \phi$.*

PROOF.

(a) Suppose $\theta_{t+1} < \theta_t$. Then, from Lemma 6.2.6(b) we have an x such that

$$\max\{\theta_{t+1}, \lambda_t\} \leq x < \theta_t. \tag{6.2.27}$$

Due to this, Corollaries 6.2.2(a,b2,b3), 6.2.1(a), and Eq. (6.2.18), we deduce

$$z_t^o(x) - z_t^r(x) = -s + \beta v_{t-1}(x) - x - d_t = g_t(x) > g_t(\theta_t) = 0,$$

from which we claim $x \notin W_t(x)$. However, $x \in W_{t+1}(x) = \{w \mid x \le w\}$ due to Eq. (6.2.27) and Theorem 6.2.3(a). Hence, if $\theta_{t+1} < \theta_t$, there exists an x having at least one w such that $w \notin W_t(x)$ and $w \in W_{t+1}(x)$. Therefore, $X_t \ne \phi$.

(b) Suppose $\theta_{t+1} = \theta_t$. This case is proven by contradiction. Suppose $X_t \ne \phi$ and choose an $x \in X_t$. Then, it follows from Eq. (6.2.26) that

$$x \le \theta_{t+1} = \theta_t \tag{6.2.28}$$

and that there exists at least one w satisfying $w \notin W_t(x)$ and $w \in W_{t+1}(x)$, that is, $w \in W_t(x)^c \cap W_{t+1}(x)$.

Due to Eq. (6.2.28) and Corollary 6.2.2(a1,a2) we have $z_t^o(x) = -s + \beta v_{t-1}(x)$ and $z_{t+1}^o(x) = -s + \beta v_t(x)$. From this and Lemma 6.2.1(b2) we find that, for any $w \in W_t(x)^c \cap W_{t+1}(x)$,

$$z_t^r(w) < z_t^o(x) = -s + \beta v_{t-1}(x) \le -s + \beta v_t(x) = z_{t+1}^o(x) \le z_{t+1}^r(w),$$

from which

$$\beta\Big(v_t(x) - v_{t-1}(x)\Big) < z_{t+1}^r(w) - z_t^r(w). \tag{6.2.29}$$

By using Lemmas 6.2.9(a), 6.2.8, and Eq. (6.2.28), we get

$$d_{t+1} - d_t = \beta\Big(v_t(\theta_t) - v_{t-1}(\theta_t)\Big) \le \beta\Big(v_t(x) - v_{t-1}(x)\Big). \tag{6.2.30}$$

(i) Suppose that a certain w with $w \ge \theta_t$ $(= \theta_{t+1})$ belongs to $W_t(x)^c \cap W_{t+1}(x)$. Then, $\max\{\lambda_{t+1}, \lambda_t\} < w$ by Lemma 6.2.6(b), thus from Corollaries 6.2.2(b3) and Eq. (6.2.30) we get

$$z_{t+1}^r(w) - z_t^r(w) = w + d_{t+1} - w - d_t = d_{t+1} - d_t \le \beta\Big(v_t(x) - v_{t-1}(x)\Big),$$

which contradicts Eq. (6.2.29). Hence, we conclude that if $\theta_t \le w$, then $w \notin W_t(x)^c \cap W_{t+1}(x)$, that is, $w \notin W_t(x)$ and $w \in W_{t+1}(x)$.

(ii) Suppose that a certain w with $w < \theta_t$ $(= \theta_{t+1})$ belongs to $W_t(x)^c \cap W_{t+1}(x)$. Then, since $w \in W_{t+1}(x)$ is assumed, we get $x \le w$ due to Theorem 6.2.1(a), thus $x \le w < \theta_t$ holds by the assumption. On account of this and Lemma 6.2.8 we get

$$\beta\Big(v_t(w) - v_{t-1}(w)\Big) \le \beta\Big(v_t(x) - v_{t-1}(x)\Big). \tag{6.2.31}$$

Hence, it follows from Eqs. (6.2.31) and (6.2.30) that

$$\begin{aligned} z_{t+1}^r(w) - z_t^r(w) &= \max\{w + d_{t+1}, -r(w) - s + \beta v_t(w)\} - \max\{w + d_t, -r(w) - s + \beta v_{t-1}(w)\} \\ &\le \max\Big\{d_{t+1} - d_t, \beta\Big(v_t(w) - v_{t-1}(w)\Big)\Big\} \\ &\le \max\Big\{d_{t+1} - d_t, \beta\Big(v_t(x) - v_{t-1}(x)\Big)\Big\} \end{aligned}$$

$$= \beta\Big(v_t(x) - v_{t-1}(x)\Big),$$

which contradicts Eq. (6.2.29). Therefore, we deduce that if $w < \theta_t$, then $w \notin W_t(x)^c \cap W_{t+1}(x)$, that is, $w \notin W_t(x)$ and $w \in W_{t+1}(x)$.

Now, from (i) and (ii) we claim that there exists no w satisfying $w \notin W_t(x)$ and $w \in W_{t+1}(x)$, thus $W_t(x)^c \cap W_{t+1}(x) = \phi$, that is, $W_t(x) \supseteq W_{t+1}(x)$. As this contradicts $x \in X_t \neq \phi$, we conclude that if $\theta_{t+1} = \theta_t$, then $X_t = \phi$. ∎

6.2.2.2 The Case where d_t is Convex in t

Lemma 6.2.11 *Let d_t be convex in t. Then:*

(a) *For $t \geq 1$, we have*

$$\beta\Big(v_t(\theta_t) - v_{t-1}(\theta_t)\Big) \begin{cases} < d_{t+1} - d_t & \text{if } \beta < 1, \\ \leq d_{t+1} - d_t & \text{if } \beta = 1. \end{cases} \tag{6.2.32}$$

(b) *For $t \geq 2$, if $\beta(v_{t-1}(\lambda_t) - v_{t-2}(\lambda_t)) \leq d_t - d_{t-1}$, then*

$$\beta\Big(v_t(\lambda_t) - v_{t-1}(\lambda_t)\Big) \begin{cases} < d_{t+1} - d_t & \text{if } \beta < 1, \\ \leq d_{t+1} - d_t & \text{if } \beta = 1. \end{cases} \tag{6.2.33}$$

Proof.

(a) Due to Eqs. (6.2.2) and (6.2.3) we claim that, for any w and $t \geq 1$,

$$u_{t-1}(w, \theta_t) \geq \max\{w + d_{t-1},\ \theta_t + d_{t-1}\}. \tag{6.2.34}$$

Hence, it follows from Theorem 6.2.3(b), Eqs. (6.2.34), and (6.1.2) that

$$\begin{aligned} & u_t(w,\theta_t) - u_{t-1}(w,\theta_t) \\ &\quad = \max\{w + d_t,\ \theta_t + d_t\} - u_{t-1}(w,\theta_t) \\ &\quad \leq \max\{w + d_t,\ \theta_t + d_t\} - \max\{w + d_{t-1},\ \theta_t + d_{t-1}\} \\ &\quad = d_t - d_{t-1} \ \leq \ d_{t+1} - d_t, \end{aligned}$$

implying

$$\begin{aligned} v_t(\theta_t) - v_{t-1}(\theta_t) &= \int_a^b \Big(u_t(w,\theta_t) - u_{t-1}(w,\theta_t)\Big) dF(w) \\ &\leq \int_a^b (d_{t+1} - d_t) dF(w) \ = \ d_{t+1} - d_t. \end{aligned}$$

which immediately indicates Eq. (6.2.32).

(b) For $t \geq 2$, it follows from Eq. (6.2.9) and Corollary 6.2.2(b) that

$$u_t(w, \lambda_t) - u_{t-1}(w, \lambda_t)$$

$$= \begin{cases} \max\{-r(w) - s + \beta v_{t-1}(w),\ z_t^o(\lambda_t)\} - u_{t-1}(w,\lambda_t) & \text{if } \ w \le \lambda_t, \\ \max\{w + d_t, \qquad\qquad\qquad\quad z_t^o(\lambda_t)\} - u_{t-1}(w,\lambda_t) & \text{if } \ \lambda_t \le w. \end{cases} \tag{6.2.35}$$

Due to Eq. (6.2.3) we get, for any w,

$$u_{t-1}(w,\lambda_t) \ge \max\{w + d_{t-1},\ -s + \beta v_{t-2}(\lambda_t)\}. \tag{6.2.36}$$

First, let $w \le \lambda_t$. Then, $v_{t-1}(w) \le v_{t-1}(\lambda_t)$ by Lemma 6.2.1(b2). From this, Eqs. (6.2.36), (6.1.2), and the inequality assumed in the assertion, we get

$$\begin{aligned} & -r(w) - s + \beta v_{t-1}(w) - u_{t-1}(w,\lambda_t) \\ & \le\ -r(w) - s + \beta v_{t-1}(\lambda_t) - u_{t-1}(w,\lambda_t) \\ & \le\ -r(w) - s + \beta v_{t-1}(\lambda_t) + s - \beta v_{t-2}(\lambda_t) \\ & <\ \beta\Big(v_{t-1}(\lambda_t) - v_{t-2}(\lambda_t)\Big)\ \le\ d_t - d_{t-1} \le\ d_{t+1} - d_t. \end{aligned} \tag{6.2.37}$$

Next, let $\lambda_t \le w$. Then, from Eq. (6.2.36) we obtain

$$\begin{aligned} & w + d_t - u_{t-1}(w,\lambda_t) \\ & \le\ w + d_t - w - d_{t-1}\ =\ d_t - d_{t-1} \le\ d_{t+1} - d_t. \end{aligned} \tag{6.2.38}$$

Finally, for any w, it follows from Lemma 6.2.6(b), Corollary 6.2.2(a1), Eqs. (6.2.36), (6.1.2), and the inequality assumed in the assertion that

$$\begin{aligned} & z_t^o(\lambda_t) - u_{t-1}(w,\lambda_t) \\ & =\ -s + \beta v_{t-1}(\lambda_t) - u_{t-1}(w,\lambda_t) \\ & \le\ -s + \beta v_{t-1}(\lambda_t) + s - \beta v_{t-2}(\lambda_t) \\ & =\ \beta\Big(v_{t-1}(\lambda_t) - v_{t-2}(\lambda_t)\Big)\ \le\ d_t - d_{t-1} \le\ d_{t+1} - d_t. \end{aligned} \tag{6.2.39}$$

From Eqs. (6.2.35), (6.2.37) to (6.2.39), we deduce that if $\beta(v_{t-1}(\lambda_t) - v_{t-2}(\lambda_t)) \le d_t - d_{t-1}$, then $u_t(w,\lambda_t) - u_{t-1}(w,\lambda_t) \le d_{t+1} - d_t$ for any w, thus we get $v_t(\lambda_t) - v_{t-1}(\lambda_t) \le d_{t+1} - d_t$, which immediately implies Eq. (6.2.33). ∎

Theorem 6.2.4 *Let d_t be convex in t. Then, for $t \ge 1$:*

(a) *If $\beta < 1$, then :*

1. *θ_t is strictly decreasing in t with $\theta_t < \theta$, and diverges to $-\infty$ as $t \to \infty$.*
2. *Suppose $\lambda_{t^*-1} \ge \lambda_{t^*}$ for a certain t^*. Then, λ_t is strictly decreasing in $t > t^*$, and diverges to $-\infty$ as $t \to \infty$.*
3. *There exists at least one t^* such that $\lambda_{t^*-1} \ge \lambda_{t^*}$.*

(b) *If $\beta = 1$, then :*

1. *θ_t is nonincreasing in t with $\theta_t < \theta$.*
2. *Suppose $\lambda_{t^*-1} \ge \lambda_{t^*}$ for a certain t^*. Then, λ_t is nonincreasing in $t > t^*$.*

PROOF. For $\beta \leq 1$, it follows from Eqs. (6.2.4) and (3.2.3) that

$$\begin{aligned} g_1(\theta) &= -s + \beta v_0(\theta) - \theta - d_1 \\ &= -s + \beta S(\theta) - \theta - d_1 + \beta d_0 \\ &= -d_1 + \beta d_0 \ < \ 0 \ = \ g_1(\theta_1), \end{aligned}$$

from which and Corollary 6.2.1(a) we conclude $\theta_1 < \theta$.

(a1) Suppose $\beta < 1$. It follows from Lemmas 6.2.9(a) and 6.2.11(a) that θ_t is strictly decreasing in t, thus $\theta_t \leq \theta_1 < \theta$ for each $t \geq 1$. Due to Eqs. (6.1.1) and (6.1.2) we find $d_t \to \infty$ as $t \to \infty$. From this and Eq. (6.1.2) we get $d_t - \beta d_{t-1}$ $(= \beta(d_t - d_{t-1}) + (1-\beta)d_t) \to \infty$ as $t \to \infty$. Hence, $\theta_t \to -\infty$ as $t \to \infty$ due to Lemma 6.2.6(a).

(a2) From Lemmas 6.2.9(b) and 6.2.11(b) we get $\lambda_{t^*} > \lambda_{t^*+1} > \cdots$ if $\lambda_{t^*-1} \geq \lambda_{t^*}$. Further, $\lambda_t \to -\infty$ as $t \to \infty$ due to (a1) and Lemma 6.2.6(b).

(a3) Since $\lambda_t \to -\infty$ as $t \to \infty$ from (a2), it follows that there exists a number t^* such that $\lambda_{t^*-1} \geq \lambda_{t^*}$.

(b1) Immediate due to Lemmas 6.2.9(a) and 6.2.11(a).

(b2) Easy from Lemmas 6.2.9(b) and 6.2.11(b). ∎

Corollary 6.2.3 *Let d_t be convex in t. Then :*

(a) *Let $\beta < 1$. Then, there exists an x such that $W_t(x) \not\supseteq W_{t+1}(x)$.*

(b) *Let $\beta = 1$. Then, if $\theta_{t+1} < \theta_t$, there exists an x such that $W_t(x) \not\supseteq W_{t+1}(x)$, or else $W_t(x) \supseteq W_{t+1}(x)$ for any x.*

PROOF.

(a) Since $\theta_{t+1} < \theta_t$ for any $t \geq 1$ by Theorem 6.2.4(a1), it follows from Lemma 6.2.10(a) that there exists an x satisfying $W_t(x) \not\supseteq W_{t+1}(x)$.

(b) The case $\theta_{t+1} < \theta_t$ can be proven in exactly the same way as above.

If $\theta_{t+1} = \theta_t$, then $W_t(x) \supseteq W_{t+1}(x)$ for any x with $x \leq \theta_t$ from Lemma 6.2.10(b). For $x \geq \theta_t$ $(= \theta_{t+1})$ we have $W_t(x) = W_{t+1}(x) = \{w \mid x \leq w\}$ due to Theorem 6.2.3(a).

Consequently, we get the assertion. ∎

6.2.2.3 The Case where d_t is β-additive in t

Lemma 6.2.12 *Let d_t be β-additive in t. Then, for $t \geq 1$:*

(a) $\beta(v_t(\theta_t) - v_{t-1}(\theta_t)) = d_{t+1} - d_t$.

(b) $\beta(v_t(\lambda_t) - v_{t-1}(\lambda_t)) \geq d_{t+1} - d_t$.

PROOF. Suppose $t \geq 1$. Then, from Eq. (6.1.3) we have

$$d_{t+1} - d_t = \beta^{t+1}\sigma. \tag{6.2.40}$$

(a) Due to Theorem 6.2.3(b), Eqs. (6.2.2), and (6.2.40), we get, for any w,

$$\begin{aligned} &u_1(w,\theta_1) - u_0(w,\theta_1) \\ &\quad = \max\{w + d_1,\ \theta_1 + d_1\} - \max\{w + d_0,\ \theta_1 + d_0\} \\ &\quad = d_1 - d_0 \ = \ \beta\sigma. \end{aligned} \tag{6.2.41}$$

Hence, it follows from Eqs. (6.2.41) and (6.2.40) that

$$\beta\Big(v_1(\theta_1) - v_0(\theta_1)\Big) = \beta^2\sigma = d_2 - d_1. \tag{6.2.42}$$

Suppose $\beta(v_{t-1}(\theta_{t-1}) - v_{t-2}(\theta_{t-1})) = d_t - d_{t-1}$ $(= \beta^t\sigma)$, from which and Lemma 6.2.9(a) we obtain $\theta_{t-1} = \theta_t$. Then, due to Theorem 6.2.3(b) we get

$$\begin{aligned} &u_t(w,\theta_t) - u_{t-1}(w,\theta_t) \\ &\quad = u_t(w,\theta_t) - u_{t-1}(w,\theta_{t-1}) \\ &\quad = \max\{w + d_t,\ \theta_t + d_t\} - \max\{w + d_{t-1},\ \theta_{t-1} + d_{t-1}\} \\ &\quad = \max\{w + d_t,\ \theta_t + d_t\} - \max\{w + d_{t-1},\ \theta_t + d_{t-1}\} \\ &\quad = d_t - d_{t-1} \ = \ \beta^t\sigma, \end{aligned}$$

from which

$$\beta\Big(v_t(\theta_t) - v_{t-1}(\theta_t)\Big) = \beta^{t+1}\sigma = d_{t+1} - d_t. \tag{6.2.43}$$

Due to Eqs. (6.2.42) and (6.2.43) we have confirmed the assertion by induction.

(b) It follows from Lemmas 6.2.6(b), 6.2.8, and assertion (a) that

$$\beta\Big(v_t(\lambda_t) - v_{t-1}(\lambda_t)\Big) \geq \beta\Big(v_t(\theta_t) - v_{t-1}(\theta_t)\Big) = d_{t+1} - d_t,$$

which completes the proof. ∎

Theorem 6.2.5 *Let d_t be β-additive in t.*

(a) *θ_t for each $t \geq 1$ becomes equal to the solution of $K(x) = \beta\sigma$, which is less than θ.*

(b) *λ_t is nondecreasing in t.*

PROOF.

(a) From Lemmas 6.2.9(a) and 6.2.12(a) we get $\theta_t = \theta_{t+1}$ for each $t \geq 1$.

Due to Eq. (6.1.3) we get $d_1 - \beta d_0 = \beta\sigma$. From this, Eqs. (6.2.4), and (3.2.3), we obtain

$$g_1(x) = -s + \beta S(x) - x - \beta\sigma = K(x) - \beta\sigma,$$

which indicates that θ_1 becomes equal to the solution of $K(x) = \beta\sigma$. Further, since $0 = K(\theta) > K(\theta) - \beta\sigma = K(\theta_1) - \beta\sigma$, it follows from Lemma 3.2.2(b) that $\theta_1 < \theta$. Therefore, the assertion proves to be true.

(b) Immediate from Lemmas 6.2.9(b) and 6.2.12(b). ∎

It should be noted that the results are almost the same as those stated in Theorem 4.2.2 (p.25) except for the value of θ_t. In Theorem 4.2.2 we have concluded that θ_t is the root of $K(x) = 0$. In Theorem 6.2.5 we have deduced that θ_t is the root of $K(x) = \beta\sigma$.

Corollary 6.2.4 *Let d_t be β-additive in t. Then, $W_t(x) \supseteq W_{t+1}(x)$ for any x and $t \geq 0$.*

PROOF. We easily get $W_0(x) = \max\{w \mid x \leq w\}$ for any x, thus from Theorem 6.2.1(a) we obtain $W_0(x) \supseteq W_1(x)$. For $t \geq 1$, since $\theta_{t+1} = \theta_t$ from Theorem 6.2.5(a), we claim $W_t(x) \supseteq W_{t+1}(x)$ for any $x \leq \theta_t$ due to Lemma 6.2.10(b). From Theorem 6.2.3(a) we get $W_t(x) = W_{t+1}(x) = \{w \mid x \leq w\}$ for any $x \geq \theta_t$. Therefore, we have confirmed the assertion. ∎

6.2.3 Infinite Planning Horizon

Lemma 6.2.13 *Suppose that θ_t converges to a certain number $\ddot{\theta} \geq 0$ as $t \to \infty$. Then, $v_t(\theta_t) - v_t(0)$ is strictly decreasing in t and converges to 0 as $t \to \infty$.*

PROOF. Consider the case where θ_t converges to a certain number $\ddot{\theta} \geq 0$ as $t \to \infty$. Then, since θ_t is nonincreasing in t due to Theorems 6.2.4(a1,b1) and 6.2.5(a), and $\theta_1 < b - d_1 + \beta d_0$ $(< b)$ due to Lemma 6.2.6(a), we get

$$0 \leq \ddot{\theta} \leq \cdots \leq \theta_t \leq \theta_{t-1} \leq \cdots \leq \theta_1 < b, \tag{6.2.44}$$

from which

$$0 \leq F(\ddot{\theta}) \leq \cdots \leq F(\theta_t) \leq F(\theta_{t-1}) \leq \cdots \leq F(\theta_1) < 1. \tag{6.2.45}$$

Let $W_1 = W_t(\theta_t)$ and $W_2 = W_t(0)$. Then, $W_1 \subseteq W_2$ and $W_1 = \{w \mid \theta_t \leq w\}$ due to Theorems 6.2.1(b) and 6.2.3(a), respectively. Hence, $W_2 = W_1 \cup (W_1^c \cap W_2)$ and $W_1^c = W_2^c \cup (W_1^c \cap W_2) = \{w \mid w < \theta_t\}$. Accordingly, from Eq. (6.2.12) we get

$$v_t(\theta_t) = \int_{W_1} z_t^r(w)dF(w) + \int_{W_1^c} z_t^o(\theta_t)dF(w). \tag{6.2.46}$$

By noting $z_t^o(0) \leq z_t^r(w)$ for any $w \in W_1^c \cap W_2$ $(\subseteq W_2)$ and using Eq. (6.2.12), we get

$$v_t(0) = \int_{W_2} z_t^r(w)dF(w) + \int_{W_2^c} z_t^o(0)dF(w)$$

$$
\begin{aligned}
&= \int_{W_1} z_t^r(w)dF(w) + \int_{W_1^c\cap W_2} z_t^r(w)dF(w) + \int_{W_2^c} z_t^o(0)dF(w) \\
&\geq \int_{W_1} z_t^r(w)dF(w) + \int_{W_1^c\cap W_2} z_t^o(0)dF(w) + \int_{W_2^c} z_t^o(0)dF(w) \\
&= \int_{W_1} z_t^r(w)dF(w) + \int_{W_1^c} z_t^o(0)dF(w). \qquad (6.2.47)
\end{aligned}
$$

Since $z_t^o(\theta_t) = -s + \beta v_{t-1}(\theta_t)$ and $z_t^o(0) = -s + \beta v_{t-1}(0)$

due to Corollary 6.2.2(a1,a2) and Eq. (6.2.44), it follows from Eqs. (6.2.46) and (6.2.47) that

$$
\begin{aligned}
v_t(\theta_t) - v_t(0) &\leq \int_{W_1^c} \Big(z_t^o(\theta_t) - z_t^o(0)\Big)dF(w) \\
&= \int_{W_1^c} \beta\Big(v_{t-1}(\theta_t) - v_{t-1}(0)\Big)dF(w) \\
&= \int_{-\infty}^{\theta_t} \beta\Big(v_{t-1}(\theta_t) - v_{t-1}(0)\Big)dF(w) \\
&= \beta F(\theta_t)\Big(v_{t-1}(\theta_t) - v_{t-1}(0)\Big). \qquad (6.2.48)
\end{aligned}
$$

Here, we have $v_{t-1}(0) \leq v_{t-1}(\theta_t) \leq v_{t-1}(\theta_{t-1})$ by Eq. (6.2.44) and Lemma 6.2.1(b2), and $0 \leq \beta F(\theta_t) \leq \beta F(\theta_1)$ due to Eq. (6.2.45). Hence, it follows from Eq. (6.2.48) that

$$
0 \leq v_t(\theta_t) - v_t(0) \leq \beta F(\theta_1)\Big(v_{t-1}(\theta_{t-1}) - v_{t-1}(0)\Big). \qquad (6.2.49)
$$

Repeating the above argument yields

$$
\begin{aligned}
0 \leq v_t(\theta_t) - v_t(0) &\leq \beta F(\theta_1)\Big(v_{t-1}(\theta_{t-1}) - v_{t-1}(0)\Big) \\
&\leq \Big(\beta F(\theta_1)\Big)^2\Big(v_{t-2}(\theta_{t-2}) - v_{t-2}(0)\Big) \\
&\vdots \\
&\leq \Big(\beta F(\theta_1)\Big)^{t-1}\Big(v_1(\theta_1) - v_1(0)\Big). \qquad (6.2.50)
\end{aligned}
$$

Since $0 \leq v_1(\theta_1) - v_1(0)$ due to Eq. (6.2.44) and Lemma 6.2.1(b2), and $0 \leq \beta F(\theta_1) < \beta \leq 1$ from Eq. (6.2.45), we lead to

$$
\lim_{t\to\infty} \Big(\beta F(\theta_1)\Big)^{t-1}\Big(v_1(\theta_1) - v_1(0)\Big) = 0. \qquad (6.2.51)
$$

Now, due to Eqs. (6.2.49) and (6.2.45) we get

$$
v_t(\theta_t) - v_t(0) \leq \beta F(\theta_1)\Big(v_{t-1}(\theta_{t-1}) - v_{t-1}(0)\Big) < v_{t-1}(\theta_{t-1}) - v_{t-1}(0),
$$

which shows that $v_t(\theta_t) - v_t(0)$ is strictly decreasing in t. Furthermore, it follows from Eqs. (6.2.50) and (6.2.51) that $v_t(\theta_t) - v_t(0)$ converges to 0 as $t \to \infty$. Hence, the lemma proves to be true. ∎

Lemma 6.2.14 *There exists a number t^* such that $-r(w) - s + \beta v_{t-1}(w) < u_t(w, x)$ for any $w \geq a$, $x \geq 0$, and $t > t^*$.*

PROOF. Let $\ddot{\theta}$ and $\ddot{\lambda}$ denote the limits of θ_t and λ_t, respectively, if they exist.

(a) First, consider the case where $\theta_t \to -\infty$ or $\theta_t \to \ddot{\theta} < 0$. Then, $\lambda_t \to -\infty$ or $\lambda_t \to \ddot{\lambda} < 0$ due to Lemma 6.2.6(b). Thus, from Corollary 6.2.2(b3) and Eq. (6.2.9) we can get a number t^* such that, for any $w \geq a$, $x \geq 0$, and $t > t^*$,

$$-r(w) - s + \beta v_{t-1}(w) < w + d_t \leq z_t^r(w) \leq \max\{z_t^r(w), z_t^o(x)\} = u_t(w, x).$$

(b) Next, consider the case where $\theta_t \to \ddot{\theta} \geq 0$. Then, it follows from Lemma 6.2.13 and $0 < r(a)/\beta$ that there exists a number t^* satisfying

$$0 \leq v_{t^*}(\theta_{t^*}) - v_{t^*}(0) < r(a)/\beta. \tag{6.2.52}$$

Choose any t with $t^* < t$, thus $t^* \leq t-1$. Then, due to Lemma 6.2.13 and Eq. (6.2.52) we have

$$0 \leq \beta v_{t-1}(\theta_{t-1}) - \beta v_{t-1}(0) < r(a). \tag{6.2.53}$$

(i) Suppose $\theta_{t-1} \leq x$. Then, since $\theta_t \leq \theta_{t-1}$ due to Theorems 6.2.4(a1,b1) and 6.2.5(a), we get $\theta_t \leq x$. Hence, if $w \leq \lambda_t$, then $w \leq \lambda_t < \theta_t \leq x$ due to Lemma 6.2.6(b), thus it follows from Lemma 6.2.1(b2) and $r(w) > 0$ that

$$-r(w) - s + \beta v_{t-1}(w) < -s + \beta v_{t-1}(x) \leq z_t^o(x) \leq u_t(w, x). \tag{6.2.54}$$

Contrarily, if $\lambda_t < w$, from Corollary 6.2.2(b3) we get

$$-r(w) - s + \beta v_{t-1}(w) < w + d_t \leq z_t^r(w) \leq u_t(w, x). \tag{6.2.55}$$

Hence, in the case of $\theta_{t-1} \leq x$, the assertion holds true.

(ii) Suppose $0 \leq x \leq \theta_{t-1}$.

First, if $w \leq x$, by the same way as in Eq. (6.2.54) we get $-r(w) - s + \beta v_{t-1}(w) < u_t(w, x)$.

Next, if $x \leq w \leq \theta_{t-1}$, Lemma 6.2.1(b2) implies

$$\beta v_{t-1}(0) \leq \beta v_{t-1}(x) \leq \beta v_{t-1}(w) \leq \beta v_{t-1}(\theta_{t-1}). \tag{6.2.56}$$

From Eqs. (6.2.53), (6.2.56), and $r(a) \leq r(w)$, we obtain

$$0 \leq \beta v_{t-1}(w) - \beta v_{t-1}(x) \leq \beta v_{t-1}(\theta_{t-1}) - \beta v_{t-1}(0) < r(a) \leq r(w),$$

thus

$$-r(w) - s + \beta v_{t-1}(w) < -s + \beta v_{t-1}(x) \leq z_t^o(x) \leq u_t(w, x). \tag{6.2.57}$$

Finally, if $\theta_{t-1} \leq w$, then $\lambda_t < \theta_t \leq \theta_{t-1} \leq w$ from $\lambda_t < \theta_t$ and $\theta_t \leq \theta_{t-1}$, thus we also get $-r(w) - s + \beta v_{t-1}(w) < u_t(w, x)$ by the same method as in Eq. (6.2.55).

From the above, we conclude that the lemma holds true. ∎

Corollary 6.2.5 *Let $\ddot{\theta}$ denote the limit of θ_t, including the case $\ddot{\theta} = -\infty$. Then, $u_t(w, x) - (\max\{w, x, \ddot{\theta}\} + d_t) \to 0$ for any $w \geq a$ and $x \geq 0$ as $t \to \infty$.*

PROOF. Easy from Lemmas 6.2.13 and 6.2.14. ∎

This corollary yields the optimal decision rule for an infinite planning horizon:

◇ **Optimal Decision Rule:** Let $\ddot{\theta}$ denote the limit of θ_t, including the case $\ddot{\theta} = -\infty$. Then, in the case of an infinite planning horizon, if $\ddot{\theta} \leq \max\{w, x\}$, accept the more lucrative between the current offer w and the leading offer x, or else continue the search.

6.3 Numerical Example

Here, we will pay attention to the movement of θ_t and λ_t with respect to t, which is schematized in the diagrams on the third and fourth columns of Figures 6.3.1 to 6.3.4. The condition of the calculations are as follows: $F(w)$ is the uniform distribution on $[0,1]$ (, so $a = 0$ and $b = 1$), $s = 0.005$, and $\beta = 0.97$ [1.00] for the diagrams on the third [fourth] column; $r(w)$ and d_t are depicted in the diagrams on the first and second columns, respectively. Figures 6.3.1 to 6.3.3 show the results of the case where d_t is convex in t. Figure 6.3.4 presents the result of the case where d_t is β-additive in t.

The remaining time value d_t used in Figure 6.3.1 satisfies $d_t - \beta d_{t-1} \to \infty$ as $t \to \infty$ in both $\beta < 1$ and $\beta = 1$, which produces $\theta_t \to -\infty$ and $\lambda_t \to -\infty$ as $t \to \infty$ according to Lemma 6.2.6. We can see the above result in the figure. Furthermore, it is checked that θ_t is strictly decreasing in t, and λ_t keeps decreasing after its increase in t (Theorem 6.2.4).

Exactly the same thing results in Figure 6.3.2 with $\beta = 0.97$. In the case of $\beta = 1$, the d_t can be rewritten as $d_t = 0.05 + 0.02(\beta + \beta^2 + \cdots + \beta^t)$. That is, the case of $\beta = 1$ is deduced to the case where d_t is β-additive in t. Hence, in the figure we see the properties as in Theorem 6.2.5 (see also Figure 6.3.4).

See Figure 6.3.3. If $\beta < 1$, then $d_t - \beta d_{t-1} \to \infty$, or else $d_t - \beta d_{t-1} \to 0.02$. That is, if $\beta < 1$, then $b - (d_t - \beta d_{t-1}) \to -\infty$, or else $a - (d_t - \beta d_{t-1})$ converges to 0.475. Hence, if $\beta = 0.97$, then θ_t and λ_t diverge to $-\infty$, and if $\beta = 1.00$, then θ_t and λ_t converge to certain finite numbers as follows: $\theta_t \to 0.776$ (top and bottom), $\lambda_t \to 0.593$ (top), and 0.706 (bottom).

In the case where d_t is β-additive in t, we can see the properties as in Theorem 6.2.5 by Figure 6.3.4.

Finally, we will see the optimal decision rules in Figure 6.3.5. The two diagrams in the top row show the rules for $t = 1$. From the second to bottom row includes the rules for $t = 31, 61$, and 91, respectively. The four diagrams on the left and right columns are calculated under the same conditions as in the tops of Figures 6.3.1 and 6.3.4, respectively, with $\beta = 0.97$.

As seen in Figure 6.3.1, we have $\theta_t \to -\infty$. In the left column of Figure 6.3.5, we will confirm that the area PC becomes smaller and finally disappears as the remaining period gets longer. The left bottom diagrams show that if we start the search at time 91, it is optimal to stop the

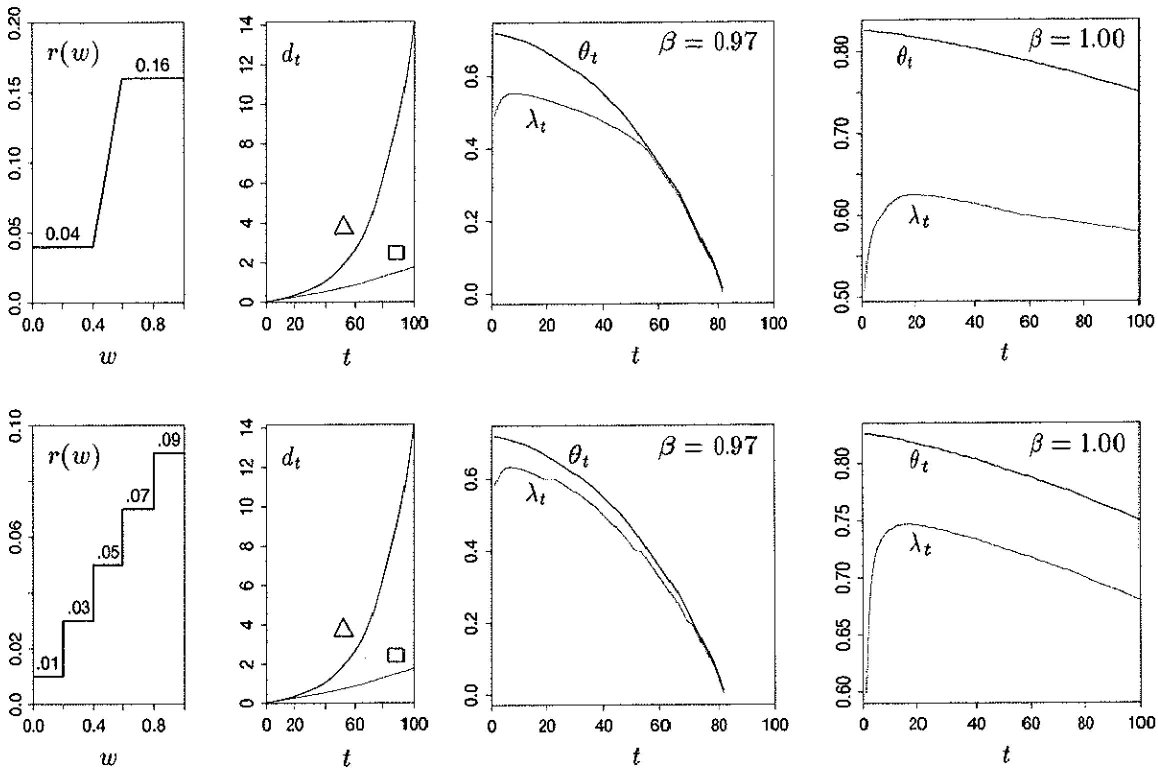


Figure 6.3.1: $d_t = 0.05 + 0.01(\gamma + \gamma^2 + \cdots + \gamma^t)$, $\gamma = 1.01/\beta$, $\beta = 0.97 \ldots \triangle$, $1.00 \ldots \square$

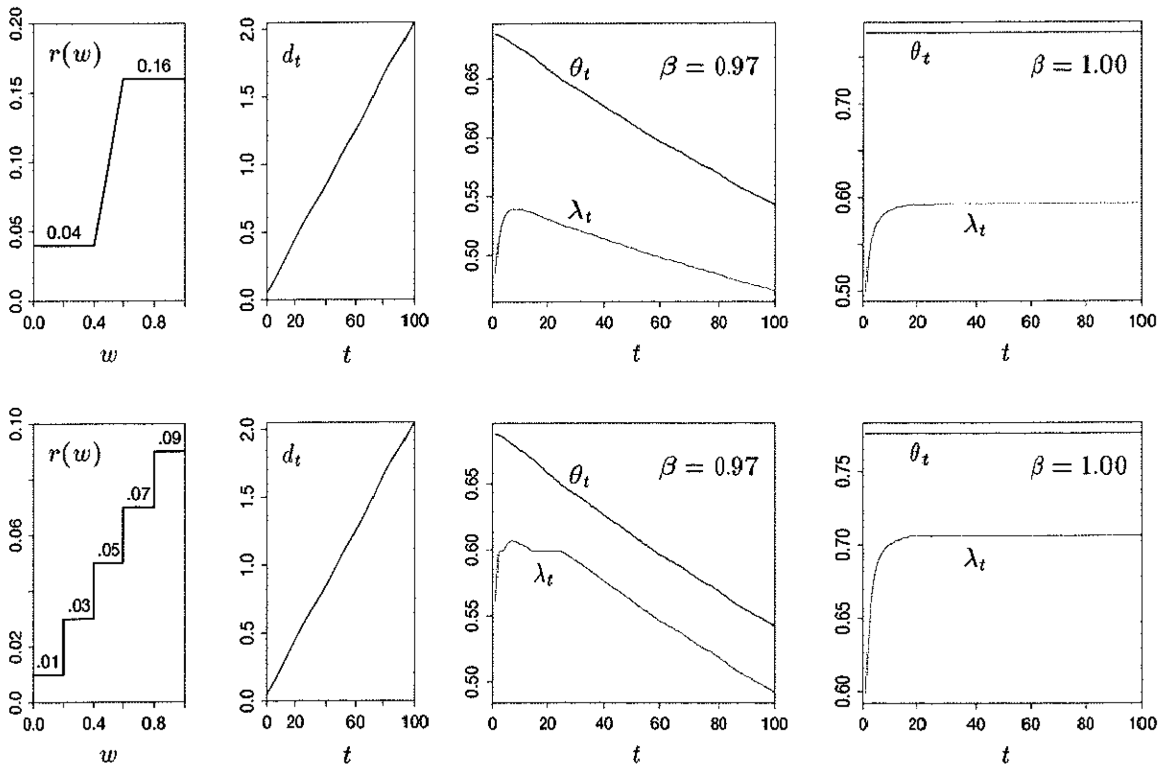


Figure 6.3.2: $d_t = 0.05 + 0.02t$, $\beta = 0.97,\ 1.00$

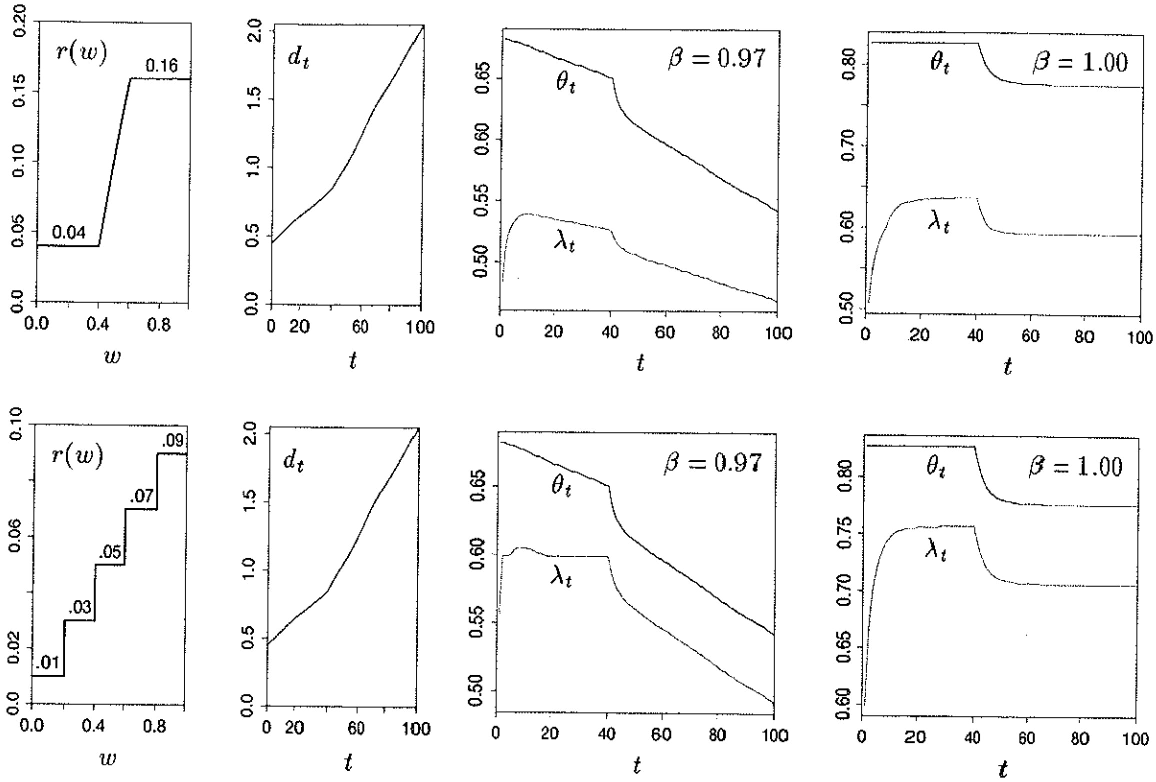


Figure 6.3.3: $d_t = 0.45 + 0.01t$ $(t < 40), 0.05 + 0.02t$ $(40 \le t)$, $\beta = 0.97,\ 1.00$

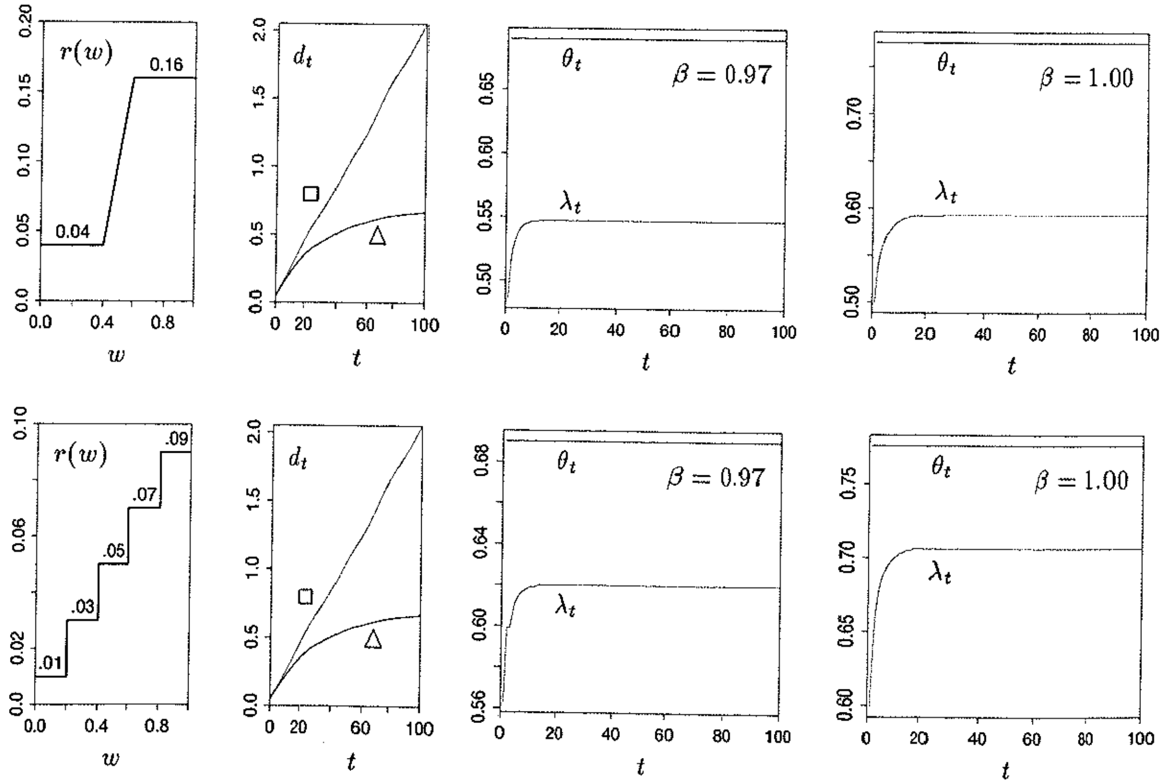
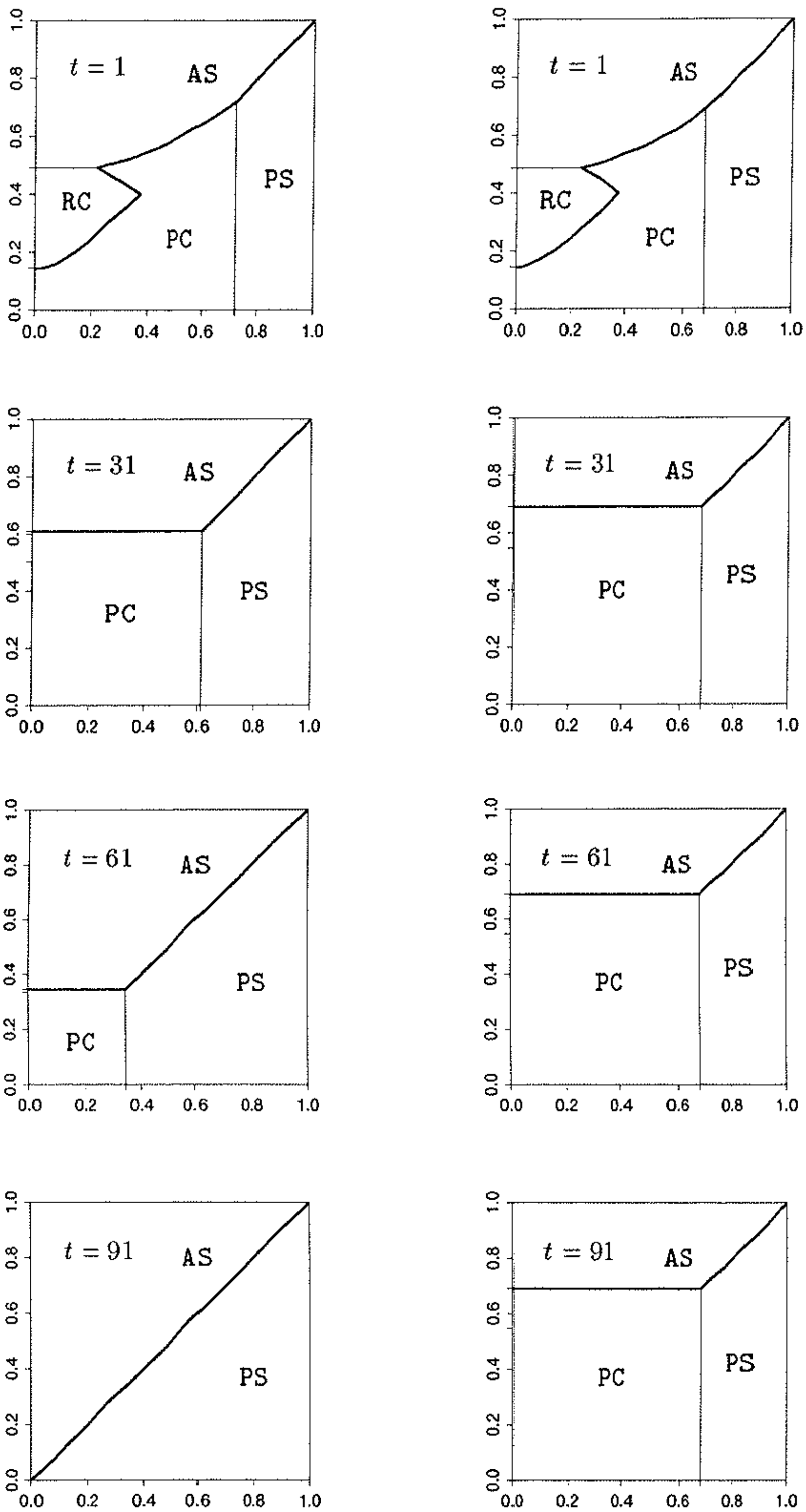


Figure 6.3.4: $d_t = 0.05 + 0.02(\beta + \beta^2 + \cdots + \beta^t)$, $\beta = 0.97 \ldots \triangle,\ 1.00 \ldots \square$



Left: $d_t = 0.05 + 0.01(\gamma + \gamma^2 + \cdots + \gamma^t)$ where $\gamma = 1.01/0.97$
Right: $d_t = 0.05 + 0.02(\beta + \beta^2 + \cdots + \beta^t)$ where $\beta = 0.97$

Figure 6.3.5: Optimal decision rules for $t = 1, 31, 61, 91$

search immediately by accepting an offer even if the value is 0. In the right column, the decision rules are almost the same from time 31 to 91. That is, it is possible that the continuation of the search become the optimal decision even if an infinite planning horizon is given.

6.4 Properties of Optimal Decision Rule

6.4.1 The Case where d_t is Convex in t

A. *If $\alpha + \beta d_0 \leq a + d_1$, then the continuation of the search is not optimal at all.*

This is a restatement of the optimal decision rule on p.72 and corresponds to Property A of Model 1 (p.35) and Property A of Model 2 (p.65).

B. *If the leading offer x is such that $\theta_t \leq x$ at a certain t, then accept the more lucrative between the leading offer x and the current offer w.*

This is already stated in the optimal decision rule on p.75, and it corresponds to Property B of Model 1 (p.35) and Property C of Model 2 (p.65).

C. *No offer reserved during the search process should be recalled and accepted except at the deadline.*

Let the leading offer x of time t be the offer reserved at t^* ($> t$). Then, $x \leq \lambda_{t^*}$ because an offer w which should be reserved at time t^* satisfies $w \leq \lambda_{t^*}$ according to the optimal decision rule (a2) on p.74. From this, Lemma 6.2.6(b), and Theorem 6.2.4(a1,b1), we get $x \leq \lambda_{t^*} < \theta_{t^*} \leq \theta_{t^*-1} \leq \cdots \leq \theta_t \leq \cdots \leq \theta_1$, which shows that the leading offer x satisfies $x < \theta_t$ if $t^* > t \geq 1$. However, the leading offer x which should be recalled and accepted at time t must satisfy $\theta_t < x$ due to the optimal decision rule (b1). Therefore, we get the property.

In the model, the earlier we stop the search, the larger the remaining time value becomes. So, it is more attractive for us to stop the search early in the process than in the situation of Model 1. However, we obtain the property which is exactly the same as Property C of Model 1.

Note that exactly the same property has already been obtained in Model 1 (Property C on p.35).

D. *At each time, any offer inferior to the leading offer should be passed up, and the range of offers to be passed up should be spread as the leading offer becomes better.*

We get the property from the optimal decision rule (b) on p.74 and Theorem 6.2.1(a,b). See also Property D of Model 1 (p.36) and Property E of Model 2 (p.66).

E. *If $\beta < 1$, for a certain x there exists an offer w which should be reserved or accepted at time $t+1$ but should be passed up at time t.*

Due to Corollary 6.2.3(a) we get $W_t(x) \not\supseteq W_{t+1}(x)$ for a certain x. This is different from Property E of Model 1 (p.36).

F. *If $\beta < 1$, there exists a certain time t^* such that if the search starts before the time, or time t with $t > t^*$, then continuation of the search is far from the optimal decision.*

Due to Theorem 6.2.4(a1) we have a certain time t^* such that $\theta_t < 0$ for any $t > t^*$. From this and the optimal decision rule (b1) on p.74 we get the property.

G. *If $r(w)$ is concave, the indifferent point between reserving an offer and passing up an offer is determined at one critical point.*

This is derived from Theorem 6.2.2. See Property F of Model 1 (p.36).

H. *Let $\ddot{\theta} \leq -\infty$ be the limit of θ_t. If the planning horizon is infinite, continue the search until an offer superior to $\ddot{\theta}$ is gotten.*

This is a restatement of the optimal decision rule on p.85. Similarly to Model 1 and Model 2, we have no need to reserve offers if an infinite planning horizon is given. See Property G of Model 1 (p.36) and Property H of Model 2 (p.66).

6.4.2 The Case where d_t is β-additive in t

I. *All of the properties of* Model 1 (p.35) *are inherited except for the point that $\alpha \leq a$ and θ are replaced with $\alpha + \beta d_0 \leq a + d_1$ and the solution of $K(x) = \beta\sigma$, respectively.*

The same property as Property B of Model 1 is already gotten as the optimal decision rule on p.75; Property C is from Theorem 6.2.5(a); Property D is from Theorem 6.2.1(a,b); Property E is from Corollary 6.2.4; Property F is from Theorem 6.2.2; Property G is the the optimal decision rule on p.85.

From the above, the optimal decision rule of can be summarized as follows:

◇ **Optimal Decision Rule:** Suppose that you are at time t with the leading offer x and have just drawn an offer w. Let x^0 be the initial leading offer, thus $x = x^0$ if time t is the start point of the search process. Then, the choices are:

(a) If $\alpha + \beta d_0 \leq a + d_1$ or $\theta_t \leq x^0$, then:

1. AS if the offer w found at the start is such that $x^0 \leq w$ (accept it and stop the search).
2. PS otherwise (accept the initial offer x^0 and stop the search).

(b) If $a + d_1 < \alpha + \beta d_0$ and $x^0 < \theta_t$, then:

1. If $t = 0$ (deadline), then:
   i AS if $x \leq w$ (accept the current offer w and stop the search).
   ii PS otherwise (accept the leading offer x and stop the search).
2. If $t \geq 1$, then:
   i AS if $w \in W_t(x)$ and $\lambda_t < w$ (accept the current offer w and stop the search).
   ii RC if $w \in W_t(x)$ and $w \leq \lambda_t$ (reserve the current offer w and continue the search).
   iii PC if $w \notin W_t(x)$ (pass up the current offer w and continue the search).
3. If $t = \infty$ (infinite planning horizon), then:

i `AS` if $\ddot{\theta} \leq w$ where $\ddot{\theta} \leq -\infty$ is the limit of θ_t (accept the current offer w and stop the search).

ii `PC` otherwise (pass up the current offer w and continue the search).

Chapter 7

Conclusions and Future Studies

We have discussed discrete-time optimal stopping problems with reservation from some viewpoints and revealed some properties of the optimal decision rules.

One of the most important results obtained from the three models is that no reserved offer should be recalled and accepted while it remains recallable at the next point in time, in other words, the time when recalling and accepting an reserved offer can become an optimal decision is restricted only to the maturity of its reservation. In the models with infinite-period reservation (Chapters 4 and 6), the deadline is regarded as the maturity of each reservation.

When the author obtained the result in the model with infinite-period reservation (Chapter 4), he thought that some mistakes might have been made somewhere in the proof. His reasoning was that since a search cost $s > 0$ is required and the expectation of finding better offers in the future attenuates every time the search is continued, he believed it was possible that recalling and accepting a reserved offer would become an optimal decision even before the deadline. After confirming that there were no mistakes, the author found that the reservation of an offer is insurance against any dire situation which may be awaiting at the deadline. This lead him to wonder whether the result would hold for other models.

In the model with finite-period reservation (Chapter 5), he verified the truth of the result and discovered that the reservation of an offer had the additional benefit of expediting the search process as well as avoiding the risk at the deadline.

In the model with remaining time value (Chapter 6), it seemed more attractive to stop the search early in the process than to do so in the previous two models. In the model, however, we should also wait to recall a reserved offer until the deadline comes.

In this thesis we have established the methodology for optimal stopping problems with the basic ideas of reservation and have obtained an intriguing result as stated above. Naturally there arise some questions. Is the result always effective in any optimal stopping problem where reservation is taken into account? If there are counterexamples, what conditions would be added to or deleted from our models? Answering the questions clarifies further the meanings and importance of reservation in optimal stopping problems. In order to achieve this, the future direction of this study should take in models with the following assumptions:

1. The length of the reserving periods depends on the offer value and/or the reserving cost.
2. The value of a reserved offer will deteriorate as time goes by.
3. Any reserved offer can be canceled with a certain probability but a cancellation fee will be received.
4. There exists a budget constraint for search costs and reserving costs to be invested.
5. The renewal of a reserved offer is allowed at its maturity.
6. The reward gained by accepting an offer can be invested in another economic activity after the acceptance.

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