CHAPTER 4

Balanced n-ary Designs

We show in this chapter the construction of balanced n-ary designs based on algebraic curves over a finite field.

4.1. Introduction

Let V be a set of v elements and $\mathcal{B}$ be a collection of b multi-subsets of V. The elements of V and $\mathcal{B}$ are called *treatments* and *blocks*, respectively. As defined in Section 1.4, a *block design* $(V, \mathcal{B})$ is an arrangement of v treatments of V into b blocks of $\mathcal{B}$. When any treatment of $\mathcal{V}$ occurs at most once in any block of $\mathcal{B}$, the block design is said to be *binary*. A *balanced n-ary design* BnD is a pair $(V, \mathcal{B})$ satisfying the following conditions:

(B1): each block is of a constant size k,

(B2): each treatment occurs at most $n-1$ times in any block $B \in \mathcal{B}$, and

(B3): each unordered pair of distinct treatments occurs exactly λ times in the blocks of $\mathcal{B}$.

Let $\boldsymbol{N} = (n_{ij})$ be a $v \times b$ matrix such that n_{ij} is the number of occurrences of the i-th treatment in the j-th block, where b is the number of blocks. We consider $\boldsymbol{N}$ as the incidence matrix of a balanced n-ary design. Using the incidence matrix, the conditions in the definition of a balanced n-ary design can be reformulated in the following way:

(B1′): $\sum_i n_{ij} = k$ for any j,

(B2′): $0 \le n_{ij} \le n-1$ for any i, j, and

(B3′): $\sum_j n_{ij} n_{i'j} = \lambda$ for any unordered pair $\{i, i'\}$, $i \ne i'$.

Example 4.1.1. Let $V = \{\alpha, \beta, \gamma, \delta, \epsilon\}$ and $\mathcal{B}$ be the collection of the following blocks:

$$
\begin{aligned}
&\{\alpha,\beta,\beta,\delta,\delta,\epsilon,\epsilon,\epsilon\}, \quad \{\alpha,\alpha,\gamma,\gamma,\gamma,\delta,\epsilon,\epsilon\},\\
&\{\beta,\beta,\beta,\gamma,\gamma,\delta,\delta,\epsilon\}, \ \{\alpha,\alpha,\beta,\gamma,\gamma,\delta,\delta,\delta\},\\
&\{\alpha,\alpha,\alpha,\beta,\beta,\gamma,\epsilon,\epsilon\}, \ \{\alpha,\gamma,\gamma,\delta,\delta,\delta,\epsilon,\epsilon\},\\
&\{\alpha,\alpha,\beta,\beta,\beta,\gamma,\delta,\delta\}, \ \{\beta,\beta,\gamma,\gamma,\delta,\epsilon,\epsilon,\epsilon\},\\
&\{\alpha,\alpha,\beta,\beta,\gamma,\gamma,\gamma,\epsilon\}, \{\alpha,\alpha,\alpha,\beta,\delta,\delta,\epsilon,\epsilon\}.
\end{aligned}
$$

Then $(V, \mathcal{B})$ is a balanced 4-ary (quaternary) design with 5 treatments and 10 blocks, and the block size is 8. The incidence matrix of the above balanced 4-ary design is

$$
\begin{pmatrix}
1 & 2 & 0 & 2 & 3 & 1 & 2 & 0 & 2 & 3\\
2 & 0 & 3 & 1 & 2 & 0 & 3 & 2 & 2 & 1\\
0 & 3 & 2 & 2 & 1 & 2 & 1 & 2 & 3 & 0\\
2 & 1 & 2 & 3 & 0 & 3 & 2 & 1 & 0 & 2\\
3 & 2 & 1 & 0 & 2 & 2 & 0 & 3 & 1 & 2
\end{pmatrix},
$$

and for any pair $\{i, i'\}$, $i \ne i'$, we have

$$\sum_j n_{ij} n_{i'j} = 23 = \lambda.$$

From a combinatorial view of point, the word 'pairwise balanced', or simply 'balanced', means that the condition (B3′) is satisfied, that is, the inner product of any two distinct rows of the incidence matrix has a constant value. A *proper* block design is one which satisfies the condition (B1′), i.e., the block design has a constant block size. The number of occurrences of a treatment in a block design is called the *replication number* of the treatment. If every treatment of V has the same replication number, then the block design is said to be *equireplicate*.

Let ρ_a^x be the number of blocks in which the treatment x occurs exactly a times. If ρ_a^x is independent of the choice of x, then the balanced n-ary design is said to be *regular* and ρ_a^x is denoted by ρ_a. A regular balanced n-ary design is equireplicate since the replication number $r = \sum_{a=1}^{n-1} a\rho_a$ is a constant.

Balanced n-ary designs were first introduced by Tocher in a statistical paper [**Toc52**] in 1952, where he suggested the relaxation of two conditions of balanced incomplete block designs. Let V be a set of v treatments and $\mathcal{B}$ a collection of subsets of V. A *balanced incomplete block design (BIBD)* is a pair $(V, \mathcal{B})$ satisfying

(C1): every treatment occurs in precisely r blocks of $\mathcal{B}$,

(C2): every pair of distinct treatments occurs in precisely λ blocks, and

(C3): each block contains k treatments.

In one relaxation in Tocher's paper [**Toc52**], the third condition (C3) is omitted. Such designs satisfying only the conditions (C1) and (C2) are called (r, λ)*-designs*. Note that the incidence matrices of these designs are always binary. The other relaxation is that the blocks may be multisets, that is, the incidence matrices need not to be binary.

In his paper [**Toc52**] Tocher mainly considered the case where $n = 3$, namely, balanced ternary designs, and presented an existence analysis of these designs. Since then, many people have been involved in the study of balanced n-ary designs, especially balanced ternary designs. Billington made excellent surveys on balanced n-ary designs [**Bil84**] in 1984 and also on designs with repeated treatments in blocks [**Bil89**] in 1989. Most part of the second survey paper [**Bil89**] was devoted to the results on balanced n-ary designs since 1984, the year when the first one was published.

There are two aspects of studies on balanced n-ary designs; one is from the combinatorial point of view and the other is from the statistical one. Most of papers were concerned with proper and equireplicate balanced n-ary designs, although the original definition by Tocher [Toc52] did not require the design to be equireplicate. Several statistical papers, for example, Kageyama [Kag79], Kageyama and Tsuji [KT79, KT80], and Kulshreshtha, Dey and Saha [KDS72], dealt with balanced n-ary designs which are not necessarily proper nor equireplicate. They considered designs which are *variance balanced* for statistical uses. In fact, 'variance balanced' is not the same as 'pairwise balanced' in a non-proper design (see [CK96] for the details). This is why proper balanced n-ary designs are considered in most of the papers on balanced n-ary designs.

We assume here that balanced n-ary designs are proper and equireplicate. The parameters of a balanced n-ary design $(V, \mathcal{B})$ are v, b, k, r and λ, where v is the number of treatments of V, b the number of blocks of $\mathcal{B}$, k the block size, r the replication number and λ the number of occurrences of any pair for distinct treatments from V. It is well known that the equalities $vr = bk$ and $\lambda(v-1) = r(k-1)$ hold for any balanced incomplete block design (hence, in the binary case of a balanced n-ary design). The corresponding results for proper and equireplicate balanced n-ary designs are given below.

Lemma 4.1.2 ([Bil84]). *For any proper and equireplicate balanced n-ary design with parameters (v, b, k, r, λ), we have*

(i): $vr = bk$,

(ii): $\sum_{j=1}^{b} n_{ij}^2$ *is independent of i chosen, and*

(iii): $\lambda(v-1) = rk - \sum_{j=1}^{b} n_{ij}^2$.

Note that (ii) and (iii) were originally due to Saha [Sah75].

As another necessary condition for the existence of a balanced incomplete block design with parameters (v, b, k, r, λ), we have Fisher's inequality $b \geq v$. Moreover if the design is symmetric (that is, if $v = b$)

then Bruck-Ryser-Chowla Theorem also gives a necessary condition for its existence. The corresponding conditions for n-ary designs are given by Dey [**Dey75**], Kageyama [**Kag79, Kag80**], Kageyama and Tsuji [**KT79**], Morgan [**Mor77, Mor79**], Shafiq and Federer [**SF79**], and Pandian [**Pan81**].

Lemma 4.1.3 (Fisher's inequality for n-ary designs [**Bil84**]). *Let v, b, r, k and λ be the parameters of a balanced n-ary design. Then*

(i): *$b \geq v$, and*

(ii): *if $b = v$ then $\det N = k(k^2 - \lambda v)^{\frac{(v-1)}{2}}$.*

Lemma 4.1.4 (Bruck-Ryser-Chowla Theorem for n-ary designs [**Bil84**]). *In a symmetric balanced n-ary design with parameters v, k and λ,*

(i): *if v is even then $k^2 - \lambda v$ is a perfect square, and*

(ii): *if v is odd then*

$$z^2 = (k^2 - \lambda v)x^2 + (-1)^{\frac{(v-1)}{2}} \lambda y^2$$

has a solution in integers x, y, z not all zero.

Balanced n-ary designs are also closely related to other types of combinatorial structures. In a joint paper [**FHKMS**] with the present author, a relationship was established between balanced n-ary designs and certain (r, λ)-designs in which the blocks are divided into some subblocks. This is a modification of the connection between balanced arrays and (r, λ)-designs with mutually balanced nested designs studied in [**FHK91**] and [**KFH94**].

4.2. Algebraic curves and balanced n-ary designs

Let C be an irreducible curve defined over $\mathbb{F}_q$ and $\mathcal{B} = \{D_1, \cdots, D_b\}$ a collection of b divisors in $\mathrm{Div}(C)$. Let $V = \bigcup_{E \in \mathcal{B}} \mathrm{Supp}\ (E) = \{P_1, \cdots, P_v\}$. Denote by n_{ij} the coefficient of a point P_i with respect to a divisor D_j, that is, D_j has the form $\sum_i n_{ij} P_i$. Then $N = (n_{ij})$ is an

n-ary matrix whose rows and columns are indexed by the point set V and by the divisor set $\mathcal{B}$, respectively. We can regard the $v \times b$ matrix N as the incidence matrix of an n-ary design, the point set V as the set of treatments, and the divisor set $\mathcal{B}$ as the collection of blocks. For N to be the incidence matrix of a balanced n-ary design, we have to choose suitable C, $\mathcal{B}$ and V so that N satisfies the conditions (B1′)–(B3′).

Let C be a nonsingular curve defined over $\mathbb{F}_q$. The first condition (B1′) is required for the block size to be constant, i.e., $\sum_i n_{ij} = k$ for any curve $C_j \in \mathcal{B}$. For a divisor D on C, a divisor E is said to be *linearly equivalent to* D, denoted by $E \sim D$, if there exists a rational function $f \in \mathrm{Rat}(C)$ such that $E = D + \mathrm{div}(f)$. Let $\mathcal{B}$ be a collection of divisors each of which is linearly equivalent to a divisor D, that is, $\mathcal{B} = (\mathrm{div}(f) + D : f \in L(D) \setminus \{0\}) = (D_1, \cdots, D_b)$. Note that the number of divisors in $\mathcal{B}$ is equal to the number of nonzero rational functions in $L(D)$. It is well known (e.g., Corollary 3.2 in **[Uen97]**) that if a divisor E is linearly equivalent to a divisor D then $\deg E = \deg D$. Hence, any divisor in $\mathcal{B}$ has the same degree as D, which forces the block size to be constant.

Lemma 4.2.1. *Let $\mathcal{B}$ be a collection of divisors each of which is linearly equivalent to a divisor D, and let $V = \bigcup_{E\in\mathcal{B}} \mathrm{Supp}\ (E)$. Then the pair $(V, \mathcal{B})$ is a proper design.*

We assume here that $\mathcal{B}$ and V satisfy the conditions in Lemma 4.2.1. The second condition in the definition of a balanced n-ary design requires that each treatment of the design occurs at most $n-1$ times in any block of $\mathcal{B}$. Since all divisors in $\mathcal{B}$ are effective, all n_{ij}'s are positive. The block size $\sum_i n_{ij}$ is a finite number k. Hence the condition (B2′) is satisfied.

Lemma 4.2.2. *Let D be a $\mathbb{F}_q$-rational divisor with positive degree on a nonsingular curve C defined over $\mathbb{F}_q$. Let $\mathcal{B} = (\mathrm{div}(f) + D : f \in L(D) \setminus \{0\})$, $V = \bigcup_{E\in\mathcal{B}} \mathrm{Supp}\ (E)$, and n_{ij} the coefficient of the j-th divisor in $\mathcal{B}$ with respect to the i-th point in V. If the genus of C is* 0 *and if D has the form $D = \sum_{P \in V} mP + D'$, m being an integer, then $\boldsymbol{N} = (n_{ij})$ is the incidence matrix of an equireplicate balanced n-ary design $(V, \mathcal{B})$ with parameters*

$$\begin{aligned}
v &= |V|, \\
b &= q^{\deg D+1} - 1, \\
r &= \sum_{l=1}^{\deg D} q^l - \deg D, \\
k &= \deg D.
\end{aligned}$$

Proof. We only have to check whether the third condition of a balanced n-ary design is satisfied, that is, whether the design $(V, \mathcal{B})$ is pairwise balanced. Let E_j be the j-th element of $\mathcal{D}$. Assume that D and E_j's have the forms $D = \sum_{P_i \in V} mP_i + D'$ and $E_j = \mathrm{div}(f_j) + D$, respectively. Let $\mathrm{div}(f_j) = \sum_i e_{ij} P_i$. Then each entry n_{ij} of the incidence matrix $\boldsymbol{N}$ is $m + e_{ij}$, since $\mathrm{div}(f_j) + D = \sum_i e_{ij} P_i + D = \sum_i (m + e_{ij}) P_i$. For any pair $\{i, i'\}$, we have

$$\begin{aligned}
&\sum_j n_{ij} n_{i'j} \\
=&\sum_j (m + e_{ij})(m + e_{i'j}) \\
=&\sum_j (m^2 + me_{i'j} + me_{ij} + e_{ij} e_{i'j}) \\
=&bm^2 + m(\sum_j e_{i'j} + \sum_j e_{ij}) + \sum_j e_{ij} e_{i'j}.
\end{aligned}$$

For an arbitrarily fixed i, let $\mu(\alpha)$ be the number of e_{ij} such that $e_{ij} = \alpha$. It can be easily seen that

$$\sum_j e_{i'j} = \sum_j e_{ij} = \sum_\alpha \alpha \mu(\alpha).$$

Let $D_x = D - (m+x)P_i$. Since the order of any $f \in L(D_x)$ is greater than or equal to x, we have $\mu(\alpha) = |L(D_\alpha) - L(D_{\alpha+1})|$. From Lemma 2.5.3 and Theorem 2.6.4, if $\deg D_x \geq 0$ then $\dim L(D_x) = \deg D_x + 1 = \deg D - (m+x) + 1$, and if $\deg D_x < 0$ then $L(D_x) = \{0\}$. Hence, we have

$$\mu(\alpha) = \begin{cases} (q-1)q^{\deg D - m - \alpha} & \text{if } -m \leq \alpha \leq \deg D - m, \\ 0 & \text{if } \alpha > \deg D - m - 1, \end{cases} \tag{4.2.1}$$

which is independent of i chosen. Let $\mu(\alpha, \beta)$ be the number of pairs $(e_{ij}, e_{i'j})$ such that $(e_{ij}, e_{i'j}) = (\alpha, \beta)$, that is, $\mu(\alpha, \beta)$ is the number of rational functions in $L(D)$ whose orders at P_i and $P_{i'}$ are exactly α and β, respectively. Then we have

$$\sum_j e_{ij} e_{i'j} = \sum_{(\alpha,\beta)} \alpha\beta\mu(\alpha, \beta).$$

Let $D_{xy} = D - (m+x)P_i - (m+y)P_{i'}$. It is easy to see that $|L(D_{xy})| = q^{\deg D_{xy}+1} = q^{\deg D - 2m - x - y}$ if $x + y \leq \deg D - 2m$, and that $|L(D_{xy})| = |\{0\}| = 1$ if $x + y > \deg D - 2m$. Since $\mu(\alpha, \beta)$ can be obtained from the fact that $L(D_{\alpha\beta}) \setminus (L(D_{\alpha+1,\beta}) \cup L(D_{\alpha,\beta+1}))$ is the set of rational functions each of which has orders α and β at P_i and $P_{i'}$, respectively, it is readily checked that $\mu(\alpha, \beta)$ is independent of the pairs (i, i') chosen. Since both of $\mu(\alpha, \beta)$ and $\mu(\alpha)$ are independent of the choice of i and i', we have

$$\sum_j n_{ij} n_{i'j} = \lambda.$$

So we can conclude that the third condition in the definition of a balanced n-ary design is also satisfied.

Each parameter of the design can be obtained as follows. Since the number b of blocks is equal to the number of rational functions in $L(D) \setminus \{0\}$, we have $b = |L(D)| - 1 = q^{\deg D+1} - 1$. The replication number

r_i of the i-th treatment is the j-th column sum $\sum_j n_{ij} = \sum m + e_{ij} = bm + \sum_a a\mu(a)$. From (4.2.1) we have

$$\begin{aligned}\sum_a a\mu(a) &= \sum_{a=-m}^{\deg D-m} a(q-1)q^{\deg D-m-a} \\ &= \sum_{l=1}^{\deg D} q^l - \deg D - m(q-1)\sum_{l=0}^{\deg D} q^l,\end{aligned}$$

and we can say consequently that the design is equireplicate. The block size k can be obtained from Lemma 4.2.1. □

Let $\mathcal{B}' = \{\mathrm{div}(f)+D : f \in L(D)\setminus\{0\}\}$, and $V' = \bigcup_{E\in\mathcal{B}'} \mathrm{Supp}\ (E)$. Then we have the following.

Corollary 4.2.3. *The pair $(V', \mathcal{B}')$ is a balanced n-ary design with parameters*

$$\begin{aligned}v' &= |V'|,\\ b' &= \sum_{i=0}^{\deg D} q^i,\\ r' &= \sum_{i=0}^{\deg D-1} (\deg D - i)q^i,\\ k' &= \deg D.\end{aligned}$$

Proof. Let f and g be rational functions in $\mathbb{F}_q(C)$ on the curve C such that $f = \alpha g$, $\alpha \in \mathbb{F}_q \setminus \{0\}$. Then $\mathrm{div}(f) = \mathrm{div}(\alpha g) = \mathrm{div}(g)$. This means that for any rational function $f \in L(D)\setminus\{0\}$, there are $q-1$ rational functions which have the same divisor as $\mathrm{div}(f)$. On the other hand, for any two distinct rational functions f and g satisfying $f \neq \alpha g$, we have $\mathrm{div}(f) \neq \mathrm{div}(g)$. The pair $(V, \mathcal{B})$ in Lemma 4.2.2 is a balanced n-ary design with each block repeated exactly $q-1$ times. Hence the number b' of blocks and the replication number r' of each treatment can be reduced to $b/(q-1)$ and $r/(q-1)$, respectively. The new design $(V', \mathcal{B}')$ obtained in this way is obviously pairwise balanced. $\square$