Chapter 3

Kinematic formulas

3.1 Second fundamental form of an intersection

In this section we provide some lemmas in order to prove our theorems in the following section.

Let M be a p dimensional submanifold of a Riemannian manifold X. By second fundamental form of M at $x \in M$, we shall use a symmetric bilinear mapping

$$h_x : T_xM \times T_xM \to T_x^{\perp}M.$$

If we choose an orthonormal basis $e_1, \ldots, e_n$ of T_xX such that $e_1, \ldots, e_p$ is a basis of T_xM and $e_{p+1}, \ldots, e_n$ is a basis of $T_x^{\perp}M$, then the components of h_x for this basis are defined by

$$(h_x)_{ij}^k = \langle h_x(e_i, e_j), e_k \rangle \qquad 1 \leq i, j \leq p,\ p+1 \leq k \leq n,$$

where $\langle \cdot, \cdot \rangle$ is the Riemannian metric on X.

We now work on the hypersurface in Riemannian manifold X. Let M and N be C^2-class hypersurfaces in X. We assume that M and N are intersecting transversely at x, then $M \cap N$ is a submanifold of dimension $(n-2)$. We shall denote the second fundamental forms of M and N by h^M and h^N respectively. Unless stated otherwise, the second fundamental forms of $M \cap N$ will be denoted by h. Take an orthonormal frame $\{e_1, \ldots, e_{n-2}, e_{n-1}, e_n\}$ at x in $M \cap N$ with

$$e_1, \ldots, e_{n-2} \in T_x(M \cap N),\ e_{n-1} \in T_xM,\ e_n \in T_x^{\perp}M.$$

We also take another orthonormal frame $\{e_1, \ldots, e_{n-2}, e'_{n-1}, e'_n\}$ at x in $M \cap N$ such that

$$e_1, \ldots, e_{n-2} \in T_x(M \cap N), \; e'_{n-1} \in T_x(N), \; e'_n \in T_x^{\perp}(N).$$

Let ϕ be the angle between M and N at x, that is, $\cos\phi = \langle e_n, e'_n \rangle$. Then we may give the following equalities:

$$e_{n-1} = \cos\phi e'_{n-1} - \sin\phi e'_n, \quad e_n = \sin\phi e'_{n-1} + \cos\phi e'_n, \tag{3.1.1}$$
$$e'_{n-1} = \cos\phi e_{n-1} + \sin\phi e_n, \quad e'_n = -\sin\phi e_{n-1} + \cos\phi e_n. \tag{3.1.2}$$

Using these frames, we define the components of h by

$$h^k_{ij} = \langle h(e_i, e_j), e_k \rangle \qquad \text{or} \qquad h'^k_{ij} = \langle h(e_i, e_j), e'_k \rangle,$$

where $1 \le i, j \le n-2$ and $n-1 \le k \le n$. Furthermore we put

$$h^n_{i,n-1} = \langle h^M(e_i, e_{n-1}), e_n \rangle, \qquad h'^n_{i,n-1} = \langle h^N(e_i, e'_{n-1}), e'_n \rangle,$$

when $1 \le i \le n-1$. By the choice of orthonormal frames, h^M and h^N are represented as the following matrixes,

$$h^M = \begin{bmatrix} h^n_{11} & \cdots & h^n_{1,n-1} \\ \vdots & & \vdots \\ h^n_{n-1,1} & \cdots & h^n_{n-1,n-1} \end{bmatrix}, \quad h^N = \begin{bmatrix} h'^n_{11} & \cdots & h'^n_{1,n-1} \\ \vdots & & \vdots \\ h'^n_{n-1,1} & \cdots & h'^n_{n-1,n-1} \end{bmatrix}.$$

It is a well-known fact that h, h^M and h^N are all symmetric tensors.

Having set up these notations we can now give two lemmas.

Lemma 3.1.1. *For $1 \le i, j \le n-2$, we have*

$$\sin^2\phi h(e_i, e_j) = (h^n_{ij} - h'^n_{ij}\cos\phi)e_n + (h'^n_{ij} - h^n_{ij}\cos\phi)e'_n.$$

Proof. It is obvious that e_n and e'_n are linearly independent with each other since M and N are intersecting transversely. This implies that the normal space $T_x^{\perp}(M \cap N)$ is spanned by e_n and e'_n. So, there exist α_{ij} and β_{ij} such that

$$h(e_i, e_j) = \alpha_{ij} e_n + \beta_{ij} e'_n \qquad (1 \le i, j \le n-2).$$

From (3.1.1), we have

$$h(e_i, e_j) = \alpha_{ij} \sin \phi e'_{n-1} + (\alpha_{ij} \cos \phi + \beta_{ij}) e'_n.$$

Hence we get

$$h'^n_{ij} = \langle h(e_i, e_j), e'_n \rangle = \alpha_{ij} \cos \phi + \beta_{ij}. \tag{3.1.3}$$

Similarly, from (3.1.2) we obtain

$$h^n_{ij} = \langle h(e_i, e_j), e_n \rangle = \alpha_{ij} + \beta_{ij} \cos \phi. \tag{3.1.4}$$

From (3.1.3) and (3.1.4), we have the following:

$$\begin{aligned} \sin^2 \phi \alpha_{ij} &= h^n_{ij} - h'^n_{ij} \cos \phi, \\ \sin^2 \phi \beta_{ij} &= h'^n_{ij} - h^n_{ij} \cos \phi. \end{aligned}$$

These equalities bring the proof to a conclusion.

We can see the following lemma by a direct computation from Lemma 3.1.1.

Lemma 3.1.2. *For $1 \leq i, j, k, l \leq n-2$, we have*

$$\sin^2 \phi \langle h(e_i, e_j), h(e_k, e_l) \rangle = h^n_{ij} h^n_{kl} + h'^n_{ij} h'^n_{kl} - \cos \phi (h^n_{ij} h'^n_{kl} + h'^n_{ij} h^n_{kl}).$$

The principal directions of a hypersurface are the directions which diagonalize its second fundamental form. It is a well-known fact that at each point of a hypersurface in X there exist $(n-1)$ principal directions and $(n-1)$ principal curvatures.

Let $\{\xi_1, \ldots, \xi_{n-1}, e_n\}$ be an orthonormal frame at x in M such that $\xi_1, \ldots, \xi_{n-1}$ are principal directions of M and e_n is normal to M. We denote by κ_i $(1 \leq i \leq n-1)$ the principal curvature for the principal direction ξ_i. Then, we have

$$\langle h^M(\xi_i, \xi_j), e_n \rangle = \delta_{ij} \kappa_i. \tag{3.1.5}$$

Using this frame, we have

$$e_i = \sum_{j=1}^{n-1} a_{ji} \xi_j \quad (i = 1, 2, \cdots, n-1).$$

Since the $(n-1)\times(n-1)$ matrix (a_{ji}) is an orthogonal matrix, one can see the following equalities by putting $a_{k,n-1} = v_k$:

$$v_j^2 + \sum_{i=1}^{n-2} a_{ji}^2 = 1, \quad v_j v_k + \sum_{i=1}^{n-2} a_{ji} a_{ki} = 0 \quad (j \neq k). \tag{3.1.6}$$

We thus have from (3.1.5)

$$h^n_{n-1,n-1} = \langle h^M(e_{n-1}, e_{n-1}), e_n\rangle = \sum_{i,j=1}^{n-1} v_i v_j \langle h(\xi_i, \xi_j), e_n\rangle = \sum_{j=1}^{n-1} v_j^2 \kappa_j. \tag{3.1.7}$$

Similarly for $i = 1, 2, \cdots, n-2$, we also have

$$\begin{aligned} h^n_{ii} &= \sum_{j,k=1}^{n-1} a_{ji} a_{ki} \langle h^M(\xi_j, \xi_k), e_n\rangle = \sum_{j=1}^{n-1} a_{ji}^2 \kappa_j, && (3.1.8) \\ h^n_{i,n-1} &= \sum_{j,k=1}^{n-1} a_{ji} v_k \langle h^M(\xi_j, \xi_k), e_n\rangle = \sum_{j=1}^{n-1} a_{ji} v_j \kappa_j. && (3.1.9) \end{aligned}$$

3.2 Kinematic formulas for hypersurfaces in real space forms

In this section, we will prove the following kinematic formulas.

Let M and N be C^2-class hypersurfaces of an n dimensional real space form G/K and G the group of all orientation preserving isometries of G/K. We denote by h^X and H^X the second fundamental form and the mean curvature of a submanifold X respectively.

Theorem 3.2.1. ([17]) *Let h be the second fundamental form of submanifold $M \cap gN$. Then we have*

$$\begin{aligned} &\int_G \left(\int_{M\cap gN} \|h\|^2 d\mu \right) dg && (3.2.10) \\ = \quad & C(n)\mathrm{vol}(N) \int_M \left((n^2-2n-1)\|h^M\|^2 + (n-1)^2 (H^M)^2 \right) d\mu_M \\ & + C(n)\mathrm{vol}(M) \int_N \left((n^2-2n-1)\|h^N\|^2 + (n-1)^2 (H^N)^2 \right) d\mu_N. \end{aligned}$$

Theorem 3.2.2. ([17]) *Let H be the mean curvature of $M \cap gN$. Then we have*

$$\begin{aligned} & \int_G \left(\int_{M \cap gN} H^2 d\mu \right) dg \qquad (3.2.11) \\ = & \frac{C(n)}{(n-2)^2} \mathrm{vol}(N) \int_M \left(2\|h^M\|^2 + (n^2-2n-2)(n-1)^2 (H^M)^2 \right) d\mu_M \\ & + \frac{C(n)}{(n-2)^2} \mathrm{vol}(M) \int_N \left(2\|h^N\|^2 + (n^2-2n-2)(n-1)^2 (H^N)^2 \right) d\mu_N . \end{aligned}$$

Here the constant $C(n)$ is

$$C(n) = \frac{\mathrm{vol}(SO(n-1))\mathrm{vol}(S^{n-2})^2}{(n-1)(n+1)\mathrm{vol}(S^{n-3})}.$$

Remark 3.2.3. Theorem 3.2.2 has been studied by Zhou, but the numerical coefficients here do not agree with those in Zhou's paper [36] and [37]. Roughly speaking, the expression of the normal curvature in those papers are erroneous. Here, we will amend Zhou's coefficients.

It is sufficient to prove our theorems only when the homogeneous space G/K coincides with $\mathbb{R}^n$. The reason why we do so is that we have the theory of "*transfer principle*" in integral geometry. In other words, the transfer principle allows us to move kinematic formulas proven for a homogeneous space G/K to any other homogeneous spaces with the same isotropy subgroup K. (See [13] for a detailed discussion.) Thus, the transfer principle states that our formulas hold in all real space forms if we can show the case of $G/K = \mathbb{R}^n$. Throughout this section, M and N will be hypersurfaces in $\mathbb{R}^n$ unless otherwise stated.

From now on, we work on the proof of Theorem 3.2.1. Namely our goal is to give a detailed calculation of the following integral form:

$$\int_G \left(\int_{M \cap gN} \|h\|^2 d\mu \right) dg.$$

To do this, we start with the following two lemmas.

Lemma 3.2.4. *With the notation of Section 3.1, we have*

$$\begin{aligned}\sin^2\phi\|h\|^2 &= \|h^M\|^2 - 2\sum_{i=1}^{n-2}(h^n_{i,n-1})^2 - (h^n_{n-1,n-1})^2 \\ &\quad + \|h^N\|^2 - 2\sum_{i=1}^{n-2}({h'}^n_{i,n-1})^2 - ({h'}^n_{n-1,n-1})^2 - 2\cos\phi\sum_{i,j=1}^{n-2} h^n_{ij}{h'}^n_{ij}\end{aligned}$$

Proof. From Lemma 3.1.2 we have

$$\begin{aligned}\sin^2\phi\|h\|^2 &= \sin^2\phi\sum_{i,j=1}^{n-2}\left((h^{n-1}_{ij})^2 + (h^n_{ij})^2\right) \\ &= \sin^2\phi\sum_{i,j=1}^{n-2}\langle h(e_i,e_j), h(e_i,e_j)\rangle \\ &= \sum_{i,j=1}^{n-2}\left((h^n_{ij})^2 + ({h'}^n_{ij})^2 - 2\cos\phi h^n_{ij}{h'}^n_{ij}\right) \\ &= \|h^M\|^2 - 2\sum_{i=1}^{n-2}(h^n_{i,n-1})^2 - (h^n_{n-1,n-1})^2 \\ &\quad + \|h^N\|^2 - 2\sum_{i=1}^{n-2}({h'}^n_{i,n-1})^2 - ({h'}^n_{n-1,n-1})^2 - 2\cos\phi\sum_{i,j=1}^{n-2} h^n_{ij}{h'}^n_{ij}.\end{aligned}$$

Lemma 3.2.5. (Santaló [31] p.262 (15.35)) *We denote the kinematic density of M, N and $M\cap gN$ by dT_M, dT_N and dT respectively. Then we have*

$$dT\wedge dg = \sin^{n-1}\phi\, d\phi\wedge dT_M\wedge dT_N,$$

where ϕ is the angle between M and gN.

From Lemmas 3.2.4 and 3.2.5, we have

$$\begin{aligned}&\|h\|^2 dT\wedge dg \qquad (3.2.12)\\ &= \left(\begin{array}{l} \|h^M\|^2 - 2\sum_{i=1}^{n-2}(h^n_{i,n-1})^2 - (h^n_{n-1,n-1})^2 \\ + \|h^N\|^2 - 2\sum_{i=1}^{n-2}({h'}^n_{i,n-1})^2 - ({h'}^n_{n-1,n-1})^2 \\ - 2\cos\phi\sum_{i,j=1}^{n-2} h^n_{ij}{h'}^n_{ij}\end{array}\right)\sin^{n-3}\phi\, d\phi\wedge dT_M\wedge dT_N\end{aligned}$$

Let us integrate both terms of (3.2.12). The integral of left hand side gives

$$\mathrm{vol}(SO(n-2)) \int_G \left(\int_{M \cap gN} \|h\|^2 d\mu \right) dg,$$

where $d\mu$ is the volume element of the intersection submanifold $M \cap gN$ and dg is the kinematic density for $\mathbb{R}^n$.

On the right-hand side, we have

$$\int_0^{\pi} \sin^{n-3}\phi \cos\phi \, d\phi = 0,$$

thus it is enough to calculate the first and second line of (3.2.12). Put

$$c_n = \int_0^{\pi} \sin^{n-3}\phi \, d\phi = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} = \frac{\mathrm{vol}(S^{n-2})}{\mathrm{vol}(S^{n-3})} \tag{3.2.13}$$

then we may consider the first part of the right-hand side of (3.2.12), that is,

$$c_n \int \left(\|h^M\|^2 - 2\sum_{i=1}^{n-2} (h^n_{i,n-1})^2 - (h^n_{n-1,n-1})^2 \right) dT_M \int dT_N.$$

The kinematic densities dT_M and dT_N will be written as

$$dT_M = dk \wedge d\mu_M, \qquad dT_N = dk \wedge d\mu_N,$$

where $d\mu_M$ and $d\mu_N$ are the volume elements of M and N respectively, and dk is the invariant measure of $SO(n-1)$. Hence, we have

$$\begin{aligned}
&\int \left(\|h^M\|^2 - 2\sum_{i=1}^{n-2} (h^n_{i,n-1})^2 - (h^n_{n-1,n-1})^2 \right) dT_M \\
= \;& \mathrm{vol}(SO(n-1)) \int_M \|h^M\|^2 d\mu_M \\
& - \int_M \int_{SO(n-1)} \left(2\sum_{i=1}^{n-2} (h^n_{i,n-1})^2 + (h^n_{n-1,n-1})^2 \right) dk \wedge d\mu_M.
\end{aligned}$$

From equations (3.1.6) and (3.1.9), we have

$$\sum_{i=1}^{n-2}(h^n_{i,n-1})^2 = \sum_{j=1}^{n-1}(1-v_j^2)v_j^2\kappa_j^2 - 2\sum_{j<k}v_j^2v_k^2\kappa_j\kappa_k. \tag{3.2.14}$$

Furthermore from equation (3.1.7), we obtain

$$(h^n_{n-1,n-1})^2 = \sum_{j=1}^{n-1}v_j^4\kappa_j^2 + 2\sum_{j<k}v_j^2v_k^2\kappa_j\kappa_k. \tag{3.2.15}$$

We first integrate on $SO(n-1)$. Then we have

$$\begin{aligned}
&\int_{SO(n-1)}\left(2\sum_{i=1}^{n-2}(h^n_{i,n-1})^2 + (h^n_{n-1,n-1})^2\right)dk \\
= \;& \mathrm{vol}(SO(n-2))\int_{S^{n-2}}\left(\sum_{j=1}^{n-1}(2-v_j^2)v_j^2\kappa_j^2 - 2\sum_{j<k}v_j^2v_k^2\kappa_j\kappa_k\right)dv \\
= \;& \frac{\mathrm{vol}(SO(n-2))\mathrm{vol}(S^{n-2})}{(n-1)(n+1)}\left((2n-1)\sum_{j=1}^{n-1}\kappa_j^2 - 2\sum_{j<k}\kappa_j\kappa_k\right),
\end{aligned}$$

where the step going from the first to second line uses equations (3.2.14) and (3.2.15), the fibering of $SO(n-1)$ over S^{n-2} with the fiber $SO(n-2)$, and the last step used the known values of the integrals of Weyl [35],

$$\int_{S^{n-2}} v_1^{2i_1}\cdots v_{n-1}^{2i_{n-1}}dv = \mathrm{vol}(SO(n-2))\frac{(2i_1-1)!!\cdots(2i_{n-1}-1)!!}{(n-1)(n+1)\cdots(n+2p-3)},$$

where $p=\sum_{k=1}^{n-1}i_k$. Since

$$\begin{aligned}
\|h^M\|^2 &= \sum_{j=1}^{n-1}\kappa_j^2, \\
(n-1)^2(H^M)^2 &= \left(\sum_{j=1}^{n-1}\kappa_j\right)^2 = \sum_{j=1}^{n-1}\kappa_j^2 + 2\sum_{j<k}\kappa_j\kappa_k,
\end{aligned}$$

we have

$$(2n-1)\sum_{j=1}^{n-1}\kappa_j^2 - 2\sum_{j<k}\kappa_j\kappa_k = 2n\|h^M\|^2 - (n-1)^2(H^M)^2.$$

Therefore, we obtain

$$\begin{aligned}
&\int \left(\|h^M\|^2 - 2\sum_{i=1}^{n-2} (h^n_{i,n-1})^2 - (h^n_{n-1,n-1})^2 \right) dT_M \\
= \; & \frac{\mathrm{vol}(SO(n-1))}{(n-1)(n+1)} \int_M \left((n^2-2n-1)\|h^M\|^2 + (n-1)^2 (H^M)^2 \right) d\mu_M.
\end{aligned}$$

By the same calculation, we also have

$$\begin{aligned}
&\int \left(\|h^N\|^2 - 2\sum_{i=1}^{n-2} (h'^n_{i,n-1})^2 - (h'^n_{n-1,n-1})^2 \right) dT_N \\
= \; & \frac{\mathrm{vol}(SO(n-1))}{(n-1)(n+1)} \int_N \left((n^2-2n-1)\|h^N\|^2 + (n-1)^2 (H^N)^2 \right) d\mu_N.
\end{aligned}$$

Thus the integral of the right-hand side of (3.2.12) becomes

$$\frac{c_n \mathrm{vol}(SO(n-1))^2}{(n-1)(n+1)} \times \begin{pmatrix} \mathrm{vol}(N) \displaystyle\int_M \left((n^2-2n-1)\|h^M\|^2 + (n-1)^2 (H^M)^2 \right) d\mu_M \\ +\mathrm{vol}(M) \displaystyle\int_N \left((n^2-2n-1)\|h^N\|^2 + (n-1)^2 (H^N)^2 \right) d\mu_N \end{pmatrix}.$$

Put

$$C(n) = \frac{\mathrm{vol}(SO(n-1))\mathrm{vol}(S^{n-2})^2}{(n-1)(n+1)\mathrm{vol}(S^{n-3})}.$$

then this completes the proof of the formula (3.2.10).

We now turn to the proof of Theorem 3.2.2. The main idea of this proof is the same as that of Theorem 3.2.1. In this case, we have to consider the integral

$$\int_G \left(\int_{M \cap gN} H^2 d\mu \right) dg,$$

which requires the following lemma.

Lemma 3.2.6. *With the notation of Section 3.1, we have*

$$\begin{aligned}\sin^2\phi(n-2)^2H^2 &= (n-1)^2(H^M)^2 - 2\sum_{i=1}^{n-2} h^n_{ii}h^n_{n-1,n-1} - (h^n_{n-1,n-1})^2\\ &\quad + (n-1)^2(H^N)^2 - 2\sum_{i=1}^{n-2} h'^n_{ii}h'^n_{n-1,n-1} - (h'^n_{n-1,n-1})^2\\ &\quad - \cos\phi\sum_{i,j=1}^{n-2}(h^n_{ii}h'^n_{jj} + h'^n_{ii}h^n_{jj}).\end{aligned}$$

Proof. From Lemma 3.1.2 we have

$$\begin{aligned}\sin^2\phi(n-2)^2H^2 &= \sin^2\phi\sum_{i,j=1}^{n-2}\langle h(e_i,e_i), h(e_j,e_j)\rangle\\ &= \sum_{i,j=1}^{n-2}\left(h^n_{ii}h^n_{jj} + h'^n_{ii}h'^n_{jj} - \cos\phi(h^n_{ii}h'^n_{jj} + h'^n_{ii}h^n_{jj})\right)\\ &= (n-1)^2(H^M)^2 - 2\sum_{i=1}^{n-2} h^n_{ii}h^n_{n-1,n-1} - (h^n_{n-1,n-1})^2\\ &\quad + (n-1)^2(H^N)^2 - 2\sum_{i=1}^{n-2} h'^n_{ii}h'^n_{n-1,n-1} - (h'^n_{n-1,n-1})^2\\ &\quad - \cos\phi\sum_{i,j=1}^{n-2}(h^n_{ii}h'^n_{jj} + h'^n_{ii}h^n_{jj}).\end{aligned}$$

By Lemmas 3.2.6 and 3.2.5

$$\begin{aligned}&(n-2)^2H^2 dT\wedge dg \qquad (3.2.16)\\ &= \left(\begin{aligned}&(n-1)^2(H^M)^2 - 2\sum_{i=1}^{n-2} h^n_{ii}h^n_{n-1,n-1} - (h^n_{n-1,n-1})^2\\ &+(n-1)^2(H^N)^2 - 2\sum_{i=1}^{n-2} h'^n_{ii}h'^n_{n-1,n-1} - (h'^n_{n-1,n-1})^2\\ &-\cos\phi\sum_{i,j=1}^{n-2}(h^n_{ii}h'^n_{jj} + h'^n_{ii}h^n_{jj}).\end{aligned}\right)\\ &\sin^{n-3}\phi\, d\phi\wedge dT_M\wedge dT_N.\end{aligned}$$

The integral of the left-hand side of (3.2.16) is

$$(n-2)^2\mathrm{vol}(SO(n-2))\int_G\left(\int_{M\cap gN}H^2d\mu\right)dg.$$

On the other hand, the integral of the right-hand side of (3.2.16) becomes

$$\begin{aligned}
& c_n\int\left((n-1)^2(H^M)^2-2\sum_{i=1}^{n-2}h^n_{ii}h^n_{n-1,n-1}-(h^n_{n-1,n-1})^2\right)dT_M\int dT_N \\
+ \;& c_n\int dT_M\int\left((n-1)^2(H^N)^2-2\sum_{i=1}^{n-2}h'^n_{ii}h'^n_{n-1,n-1}-(h'^n_{n-1,n-1})^2\right)dT_N.
\end{aligned}$$

Here, we have

$$\begin{aligned}
& \int\left((n-1)^2(H^M)^2-2\sum_{i=1}^{n-2}h^n_{ii}h^n_{n-1,n-1}-(h^n_{n-1,n-1})^2\right)dT_M \\
= \;& \mathrm{vol}(SO(n-1))\int_M\left((n-2)^2(H^M)^2\right)d\mu_M \\
& -\int_M\int_{SO(n-1)}\left(2\sum_{i=1}^{n-2}h^n_{ii}h^n_{n-1,n-1}+(h^n_{n-1,n-1})^2\right)dk\wedge d\mu_M.
\end{aligned}$$

From equations (3.1.7), (3.1.8) and (3.1.6), we obtain

$$\sum_{i=1}^{n-2}h^n_{ii}h^n_{n-1,n-1}=\sum_{j=1}^{n-1}(1-v_j^2)v_j^2\kappa_j^2+2\sum_{j<k}(1-v_j^2)v_k^2\kappa_j\kappa_k. \tag{3.2.17}$$

Hence, from equations (3.2.17) and (3.2.15) we have

$$\begin{aligned}
& \int\left((n-1)^2(H^M)^2-2\sum_{i=1}^{n-2}h^n_{ii}h^n_{n-1,n-1}-(h^n_{n-1,n-1})^2\right)dT_M \\
= \;& \frac{\mathrm{vol}(SO(n-1))}{(n-1)(n+1)}\int_M\left(2\|h^M\|^2+(n^2-2n-2)(n-1)^2(H^M)^2\right)d\mu_M.
\end{aligned}$$

By the same calculation, we also have

$$\begin{aligned}
& \int\left((n-1)^2(H^N)^2-2\sum_{i=1}^{n-2}h'^n_{ii}h'^n_{n-1,n-1}-(h'^n_{n-1,n-1})^2\right)dT_N \\
= \;& \frac{\mathrm{vol}(SO(n-1))}{(n-1)(n+1)}\int_N\left(2\|h^N\|^2+(n^2-2n-2)(n-1)^2(H^N)^2\right)d\mu_N.
\end{aligned}$$

Thus, the integral of the right-hand side of (3.2.16) becomes

$$\frac{c_n \mathrm{vol}(SO(n-1))^2}{(n-1)(n+1)}$$
$$\times \left(\begin{array}{l} \mathrm{vol}(N) \int_M \left(2\|h^M\|^2 + (n^2-2n-2)(n-1)^2(H^M)^2\right) d\mu_M \\ +\mathrm{vol}(M) \int_N \left(2\|h^N\|^2 + (n^2-2n-2)(n-1)^2(H^N)^2\right) d\mu_N \end{array} \right).$$

Hence we obtain the kinematic formula (3.2.11).

Corollary 3.2.7. *Under the hypothesis of Theorems 3.2.1 and 3.2.2, if $n = 3$ then we have*

$$\begin{aligned} \int_G \left(\int_{M \cap gN} \kappa^2 ds \right) dg &= \pi^3 \mathrm{vol}(N) \int_M \left(\|h^M\|^2 + 2(H^M)^2\right) d\mu_M \\ &\quad + \pi^3 \mathrm{vol}(M) \int_N \left(\|h^N\|^2 + 2(H^N)^2\right) d\mu_N, \end{aligned}$$

where κ is the curvature of the curve $M \cap gN$.

Actually, this is the well-known kinematic formula of Chen [3].

3.3 Further remarks

In this section, we shall review some definitions and fundamental properties with respect to the kinematic formula on real space forms and give the Corollaries 3.3.5 and 3.3.7 as an application of Theorems 3.2.1 and 3.2.2. We will use the notation in Howard [14].

Let G be a Lie group and K a closed subgroup of G. We denote by o the origin of a homogeneous space G/K. Let V_o be a linear subspace of $T_o(G/K)$. We define a vector space $\mathrm{II}(V_o)$ to be

$$\mathrm{II}(V_o) = \left\{ h \mid h : V_o \times V_o \to V_o^{\perp}; \text{symmetric bilinear} \right\}.$$

The element $h \in \mathrm{II}(V_o)$ can be thought of as the second fundamental form of submanifolds of G/K which pass through o and have V_o as the tangent space at o. Let $K(V_o)$ be the stabilizer of V_o in K, that is, $K(V_o) = \{a \in K \mid a_* V_o = V_o\}$. This group $K(V_o)$ acts on $\mathrm{II}(V_o)$ in the following manner:

$$(ah)(u,v) = a_* h(a_*^{-1}u, a_*^{-1}v) \qquad (u, v \in V_o)$$

for $a \in K(V_o)$ and $h \in \mathrm{II}(V_o)$. Here we can consider a polynomial $\mathcal{P}$ on $\mathrm{II}(V_o)$, since $\mathrm{II}(V_o)$ is a vector space. Then a polynomial $\mathcal{P}$ is *invariant* under $K(V_o)$ if and only if $\mathcal{P}(ah) = \mathcal{P}(h)$ for all $a \in K(V_o)$. In addition, let M be a submanifold of G/K of type V_o. Since we can take $g \in G$ with $g_* V_o = T_x M$ for each $x \in M$, $g^{-1}M$ becomes a submanifold of G/K through o whose tangent space at o coincides with V_o. Thus $h_o^{g^{-1}M} \in \mathrm{II}(V_o)$. If g_1 is another element of G such that $g_{1*} V_o = T_x M$ then there exists $a \in K(V_o)$ with $g_1 = ga$. Since a is an isometry, we obtain $h_o^{g_1^{-1}M} = a^{-1} h_o^{g^{-1}M}$. An this time, if $\mathcal{P}$ is polynomial on $\mathrm{II}(V_o)$ invariant under $K(V_o)$ then we have $\mathcal{P}(h_o^{g^{-1}M}) = \mathcal{P}(h_o^{g_1^{-1}M})$. Therefore, we can define $\mathcal{P}(h_x^M)$ at $x \in M$ by

$$\mathcal{P}(h_x^M) = \mathcal{P}(h_o^{g^{-1}M}),$$

where g is any element of G such that $g_* V_o = T_x M$.

Definition 3.3.1. Let V_o be a subspace of $T_o(G/K)$ and $\mathcal{P}$ a polynomial on $\mathrm{II}(V_o)$ which is invariant under $K(V_o)$. Then, for each submanifold M of G/K of type V_o, we define

$$I^{\mathcal{P}}(M) = \int_M \mathcal{P}(h_x^M) d\mu_M.$$

We remark that $I^{\mathcal{P}}$ is invariant under G. If G/K is a real space form, then many of the integral invariants that are usually encountered are of the form $I^{\mathcal{P}}$. For example, if $\mathcal{P} \equiv 1$ then $I^{\mathcal{P}}(M) = \mathrm{vol}(M)$.

With these preliminaries, the kinematic formula can now be stated as below. Hereafter, unless otherwise stated, G/K will be an n dimensional real space form.

Theorem 3.3.2. ([13]) *Let p and q be integers such that $1 \leq p, q \leq n$ and $p + q \geq n$. Let $\mathcal{P}$ be an invariant homogeneous polynomial of degree l defined on the second fundamental forms of $(p + q - n)$ dimensional submanifolds with $l \leq p + q - n + 1$. Then there exists a finite set of pairs $\{\mathcal{Q}_\alpha, \mathcal{R}_\alpha\}_{\alpha \in A}$ such that*

(1) *each $\mathcal{Q}_\alpha$ is an invariant homogeneous polynomial on the second fundamental forms of p dimensional submanifolds,*

(2) *each $\mathcal{R}_\alpha$ is an invariant homogeneous polynomial on the second fundamental forms of q dimensional submanifolds,*

(3) $\deg \mathcal{Q}_\alpha + \deg \mathcal{R}_\alpha = l$ *for each* α,

(4) *for all submanifolds M^p and N^q of G/K*

$$\int_G I^{\mathcal{P}}(M \cap gN)dg = \sum_\alpha I^{\mathcal{Q}_\alpha}(M) I^{\mathcal{R}_\alpha}(N).$$

Let G be the full isometry group of G/K and $O(T_o(G/K))$ the orthogonal group of the inner product space $T_o(G/K)$. Then we will identify K with $O(T_o(G/K))$ via the isomorphism of K to $O(T_o(G/K))$ with $a \mapsto a_*$. It is easy to check that if V_o is any p dimensional subspace of $T_o(G/K)$ then $K(V_o) = O(V_o) \times O(V_o^\perp)$. Here, we note that there are no homogeneous polynomials of odd degree on $\mathrm{II}(V_o)$ invariant under $K(V_o)$. The homogeneous polynomials of degree 2 invariant under $K(V_o)$ are spanned by the two polynomials

$$\mathcal{Q}_1(h) = \sum_{i,j,k} (h_{ij}^k)^2, \qquad \mathcal{Q}_2(h) = \sum_k \left(\sum_i h_{ii}^k \right)^2,$$

where $1 \le i,j \le p$, $p+1 \le k \le n$, and if $2 < p < n-1$ these polynomials are independent. Geometrically, $\mathcal{Q}_1(h)$ is the square of the norm of the second fundamental form, and $\mathcal{Q}_2(h)$ is p^2 times the square of the mean curvature.

We now turn to the group of orientation preserving isometries of G/K. The case where G is the full isometry group of G/K can be dealt with in the same manner, while the value of constants will be twice.

Now we want to rewrite our kinematic formulas in the sense of above polynomials $\mathcal{Q}_1(h)$ and $\mathcal{Q}_2(h)$. Then Theorems 3.2.1 and 3.2.2 state:

Theorem 3.3.3. *We have*

$$\begin{aligned} \int_G I^{\mathcal{Q}_1}(M \cap gN)dg &= C(n)\left((n^2-2n-1)I^{\mathcal{Q}_1}(M) + I^{\mathcal{Q}_2}(M)\right)\mathrm{vol}(N) \\ &\quad + C(n)\mathrm{vol}(M)\left((n^2-2n-1)I^{\mathcal{Q}_1}(N) + I^{\mathcal{Q}_2}(N)\right), \end{aligned}$$

and

$$\begin{aligned}\int_G I^{\mathcal{Q}_2}(M\cap gN)dg &= C(n)\left(2I^{\mathcal{Q}_1}(M)+(n^2-2n-2)I^{\mathcal{Q}_2}(M)\right)\mathrm{vol}(N)\\ &\quad +C(n)\mathrm{vol}(M)\left(2I^{\mathcal{Q}_1}(N)+(n^2-2n-2)I^{\mathcal{Q}_2}(N)\right).\end{aligned}$$

In Theorem 3.3.3, we present the final form in the case of degree 2 for hypersurfaces of real space forms.

For the rest of this section, we define the homogeneous polynomials of degree $2l$ on $\mathrm{II}(V_o)$ by

$$\mathcal{W}_{2l}(h)=2^l\sum\det\begin{bmatrix} h^{k_1}_{i_1i_1} & h^{k_1}_{i_1i_2} & \cdots & h^{k_1}_{i_1i_{2l}}\\ h^{k_1}_{i_2i_1} & h^{k_1}_{i_2i_2} & \cdots & h^{k_1}_{i_2i_{2l}}\\ \vdots & \vdots & \ddots & \vdots\\ h^{k_l}_{i_{2l-1}i_1} & h^{k_l}_{i_{2l-1}i_2} & \cdots & h^{k_l}_{i_{2l-1}i_{2l}}\\ h^{k_l}_{i_{2l}i_1} & h^{k_l}_{i_{2l}i_2} & \cdots & h^{k_l}_{i_{2l}i_{2l}} \end{bmatrix},$$

where the summation is over $1\le i_1,i_2,\ldots,i_{2l}\le p$ and $p+1\le k_1,k_2,\ldots,k_l\le n$. We note that the (a,b)-component of above matrix is $h^{k_{[(a+1)/2]}}_{i_ai_b}$, where $[x]$ means the greatest integer $[x]$ not greater than x. It is remark that these polynomials are characterized as the invariant polynomials which vanish on the second fundamental forms of generalized cylinders. For a detailed discussion, the reader is referred to the paper [13] of Howard.

Theorem 3.3.4. ([5], [7]) *Assume that* $0\le 2l\le p+q-n$.

$$\int_G I^{\mathcal{W}_{2l}}(M^p\cap gN^q)dg=\sum_{0\le k\le l}C(n,p,q,k,l)I^{\mathcal{W}_{2k}}(M)I^{\mathcal{W}_{2(l-k)}}(N),$$

where each constant $C(n,p,q,k,l)$ *depends only on the indicated parameters.*

For each submanifold M of G/K, we introduce the integral invariants $\mu_{2l}(M)$ which are defined by

$$\mu_{2l}(M)=I^{\mathcal{W}_{2l}}(M).$$

Indeed, these integral invariants $\mu_{2l}(M)$ appear in formula of Weyl for the volume of a tube [35].

We now turn to our case. If $l=1$, we have

$$\mathcal{W}_2(h)=2(\mathcal{Q}_2(h)-\mathcal{Q}_1(h)).$$

From Theorem 3.3.3, we have

Corollary 3.3.5. *For $n \geq 4$,*

$$\int_G I^{\mathcal{W}_2}(M \cap gN)dg = 2(n+1)(n-3)C(n)I^{\mathcal{W}_2}(M)\mathrm{vol}(N) + 2(n+1)(n-3)C(n)\mathrm{vol}(M)I^{\mathcal{W}_2}(N).$$

For each k with $2 \leq k \leq n-1$, let $\mathcal{U}_k$ be the invariant polynomial defined on the second fundamental forms of k dimensional submanifolds of G/K by

$$\mathcal{U}_k(h) = k\mathcal{Q}_1(h) - \mathcal{Q}_2(h).$$

This polynomial is characterized as the invariant polynomial which vanishes on the second fundamental forms of the k dimensional spheres in $\mathbb{R}^n$. The invariant polynomials on the second fundamental forms of k dimensional submanifolds of G/K has as a basis the polynomials $\mathcal{U}_k$ and $\mathcal{W}_2$. Using these invariant polynomials $\mathcal{U}_k(h)$, Howard formally gave the following proposition:

Proposition 3.3.6. ([13]) *If $p + q - n \geq 2$ then*

$$\int_G I^{\mathcal{U}_{p+q-n}}(M^p \cap gN^q)dg = c(p,q,n)I^{\mathcal{U}_p}(M)\mathrm{vol}(N) + c(q,p,n)\mathrm{vol}(M)I^{\mathcal{U}_q}(N),$$

where $c(p,q,n)$ are numerical constants depending only on p, q and n.

Using this invariant polynomial $\mathcal{U}_{n-2}(h) = (n-2)\mathcal{Q}_1(h) - \mathcal{Q}_2(h)$, we can give a conclusive expression for $c(n-1, n-1, n)$ from Theorem 3.3.3. Namely we have

Corollary 3.3.7. *For $n \geq 4$,*

$$\int_G I^{\mathcal{U}_{n-2}}(M \cap gN)dg = n(n-3)C(n)I^{\mathcal{U}_{n-1}}(M)\mathrm{vol}(N) + n(n-3)C(n)\mathrm{vol}(M)I^{\mathcal{U}_{n-1}}(N).$$