Introduction

In this thesis we shall deal in two subjects. In Chapter 1 we study some geometric properties of orbits of Lie transformation groups in compact symmetric spaces. In the latter, Chapter 2 and Chapter 3, we apply the theory of transformation group to integral geometry, and describe about recent results for the Poincaré formula and the kinematic formula, which are important objects to research in integral geometry.

A mapping from a manifold M to itself is called a transformation. If a set G of transformations has a group structure when we define the product by the composition of mappings, then G is called a transformation group of M and we say that the group G acts on M. If M is furnished some geometric structures, then we will only consider the transformations which preserve the structures of M. In particular when M is a Riemannian manifold we say that G acts isometrically on M if it preserves the Riemannian metric. It is a well-known fact that in general the group of all isometries of a Riemannian manifold is a Lie group. Now let G be a Lie group acting on a manifold M. We say that G acts transitively on M if and only if for any points x and y in M there exists an element of G which translates x to y. Then M is called a homogeneous space. For $x \in M$ we consider the G-orbit through x, that is, the set of all points of M which translated from x by the action of G. Then each orbit is a homogeneous submanifold of M. The purpose of Chapter 1 is to investigate the geometric properties of some orbits as submanifolds.

Let G/K be a Riemannian symmetric space. We denote by S the unit hypersphere in the tangent space $T_o(G/K)$ at the origin o. Then K acts isometrically on $T_o(G/K)$ as the adjoint representation. So for each point H in S, the orbit $\mathrm{Ad}(K)H$ is a homogeneous submanifold of S, which is called an R-space. The R-spaces have been studied by many geometers and obtained several remarkable properties as submanifolds in S, from the view-point of differential geometry. For instance all R-spaces have parallel mean curvature vectors, which was proved by Kitagawa-Ohnita [22]. Ohnita [27] considered the parallel translations of the normal bundles of R-spaces and represent such parallel translations by the group actions. One can prove the result of Kitagawa-Ohnita mentioned above by this. Heintze-Olmos [8] also considered such parallel translations and described the normal holonomy groups of R-spaces. On the other hand K also acts isometrically on G/K. For compact symmetric spaces G/K, Hirohashi-Song-Takagi-Tasaki [12] and Hirohashi-Ikawa-Tasaki [11] considered some geometric properties of R-spaces and orbits of K-action on G/K.

We shall review some definitions and previous results concerning isometric group actions on compact symmetric spaces. Let (G, K_1) and (G, K_2) be compact symmetric pairs. Then K_2 acts isometrically on G/K_1, which is a compact symmetric space. This action is called a Hermann action. The Hermann actions are examples of hyperpolar actions, which is defined in the following. Let G be a Lie group acting isometrically on a Riemannian manifold M. A closed submanifold Σ of M is called a section, if all orbits of the action of G meet Σ perpendicularly. The action of G on M is said to be hyperpolar, if there exists a section which is flat with respect to the induced Riemannian metric. The codimension of the orbit of highest dimension is called the cohomogeneity. The isometric actions on compact symmetric spaces of cohomogeneity one are another examples of hyperpolar actions. Kollross [24] proved that the hyperpolar actions on compact symmetric spaces are Hermann actions or cohomogeneity one actions. In Section 1.3 we consider the parallel translations of the normal bundles of the orbits of Hermann actions on compact symmetric spaces and represent such parallel translations by the group actions. Using this we can show that their mean curvature vectors are parallel, moreover those of hyperpolar actions are parallel.

In the latter two chapters we deal in the topics of integral geometry. Let G be a Lie group acting isometrically on a Riemannian manifold M. Then a submanifold of M is moved by G-action. Consider a geometric character for a submanifold of M and evaluate its average, as an integration, for every position by G-action. Blaschke dealt this problem and called "*integral geometry*". These problems have originated with "*geometric probabilities*" such as the Buffon needle problem.

Let G be a Lie group and K a closed subgroup of G. We assume that G has a left invariant Riemannian metric that is also right invariant under K-action, then G/K is a homogeneous space with an invariant Riemannian metric. Consider now two submanifolds M and N of G/K satisfying $\dim M + \dim N \geq \dim(G/K)$, one fixed and the other moving under the action of $g \in G$. Then the intersection $M \cap gN$ becomes a submanifold in G/K of dimension $(\dim M + \dim N - \dim(G/K))$ for almost all $g \in G$. Now we define $I(M \cap gN)$, an "*integral invariant*" for the submanifold $M \cap gN$. Then it is called the kinematic formula, named by S. S. Chern at first, that represents the equality between the integral

$$\int_G I(M \cap gN) dg \tag{$*$}$$

and some geometric invariants of submanifolds M and N. This integral has been studied by many geometers from various viewpoints. When we put $I(M \cap gN) = \mathrm{vol}(M \cap gN)$, the evaluation of $(*)$ is especially called the Poincaré formula. In the case M and N are submanifolds of a real space form it was studied by Poincaré, Blaschke, and others (see [31] for references), and obtained the result that the integrl is equal to a constant times the product of the volumes of M and N. Under the complex submanifolds in a complex projective space, Santaló [30] showed the same result. For real submanifolds in a complex projective space, it was investigated by Howard, Kang and Tasaki ([13], [19], [20], [32], [33], [34]). Howard [13] formulated the generalized Poincaré formula for Riemannian homogeneous spaces G/K. And he asserted that if G is unimodular and acts transitively on the sets of tangent spaces to each of submanifolds M and N, then the integral is equal to a constant times the product of the volumes of submanifolds. For example, when G/K is a complex projective space and M and N are complex submanifolds of G/K, it holds this condition. This leads to Santaló's theorem. In Chapter 2, we attempt to describe the Poincaré formula for two almost complex submanifolds M and N in an almost Hermitian homogeneous space G/K. And we will show that it can be expressed as a constant times the product of the volumes of M and N if K acts irreducibly on an exterior algebra. Furthermore in the case where G/K is an irreducible Hermitian symmetric space, we can determine this constant. This is an extension of Santaló's conclusion.

Next we shall review some well-known results as the kinematic formula besides the Poincaré formula. In the case where $G/K = \mathbb{R}^n$ if we take $I(M \cap gN) = \mu(M \cap gN)$, an integral invariant from the Weyl tube formula, then the integral form $(*)$ leads to the kinematic formula of Chern [5] and Federer [7]. In particular, let M and N are closed surfaces in $\mathbb{R}^3$, and we take

$$I(M \cap gN) = \int_{M \cap gN} \kappa^2 \, ds,$$

where κ is the curvature of the curve $M \cap gN$. Then, $(*)$ leads to the kinematic formula of Chen [3]. Afterward Howard [13] defined integral invariants induced from invariant homogeneous polynomials of the second fundamental form of $M \cap gN$, and he achieved more general kinematic formula, where G is unimodular and acts transitively on the sets of tangent spaces to each of M and N. He put the kinematic formulas listed above into a uniform shape. However, the reader must have a nontrivial calculation.

In Chapter 3, we attempt to obtain the explicit expression of kinematic formulas stated by Howard. Let M and N be hypersurfaces of a real space form. We denote by h and H the second fundamental form and mean curvature of $M \cap gN$ respectively. When we put

$$I(M \cap gN) = \int_{M \cap gN} \|h\|^2 d\mu,$$

and

$$I(M \cap gN) = \int_{M \cap gN} H^2 d\mu,$$

as integral invariants of $M \cap gN$, we will express the kinematic formula by integral invariants of M and N in Section 3.2. In the case where $G/K = \mathbb{R}^3$, these formulas become the kinematic formula of Chen [3]. In Section 3.3, we will give two corollaries as an application of our theorems, and emphasize that in this case we have obtained the complete form of formulas asserted by Howard.