Chapter 1. Weakly Hyperbolic Equations of Third Order with Holder Continuous Coefficients in Time

1.1 Introduction

For the hyperbolic equations of second order, F. Colombini, E. De Giorgi, E. Jannelli and S. Spagnolo got the results concerned with the relation between the Gevrey wellposedness and the regularity of the coefficients(see [CDS], [CJS] and see also [D], [N]). In this chapter we shall generalize their results to the weakly hyperbolic equations of third order.

We shall first consider the equation of third order in $[0,T] \times \mathbf{R}^n_x$

$$(1.1) \qquad \begin{cases} u_{ttt} + \sum_{i=1}^n a_i(t) u_{ttx_i} + \sum_{i,j=1}^n b_{ij}(t) u_{tx_ix_j} = 0 \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad u_{tt}(0,x) = u_2(x), \end{cases}$$

where $a_i(t)$ and $b_{ij}(t)$ are the real coefficients satisfying

$$(1.2) \qquad \begin{matrix} a_i(t) \in C^{\frac{k+\alpha}{2}}([0,T]) \\ |a_i(t)| \in C^{\frac{k+\alpha}{2}}([0,T]) \end{matrix} \quad \text{with} \quad \begin{matrix} k = 0 \text{ or } 1 \\ k \ \text{ integer } \geq 2 \end{matrix} \quad \text{and } 0 \leq \alpha \leq 1 \quad (i = 1, \cdots, n),$$

$$(1.3) \qquad b_{ij}(t) \in C^{k+\alpha}([0,T]) \quad \text{with } \ k \ \text{ integer } \geq 0 \ \text{ and } 0 \leq \alpha \leq 1 \quad (i,j = 1, \cdots, n).$$

Now we assume the restricted type of the weakly hyperbolic condition for the third order equation (1.1)

$$(1.4) \qquad \sum_{i,j=1}^n b_{ij}(t)\xi_i\xi_j \leq 0 \qquad \text{for } {}^\forall t \in [0,T], \quad {}^\forall \xi \in \mathbf{R}^n_\xi.$$

In general the most suitable weakly hyperbolic condition for (1.1) is

$$(1.5) \qquad \Big(\sum_{i=1}^n a_i(t)\xi_i\Big)^2 - 4\sum_{i,j=1}^n b_{ij}(t)\xi_i\xi_j \geq 0 \qquad \text{for } {}^\forall t \in [0,T], \quad {}^\forall \xi \in \mathbf{R}^n_\xi.$$

We can easily see that the condition (1.4) is stronger than the condition (1.5). The third order equations have 3 real characteristic roots $\lambda_1(t,\xi), \lambda_2(t,\xi), \lambda_3(t,\xi)$ such that $\lambda_1(t,\xi) \leq \lambda_2(t,\xi) \leq \lambda_3(t,\xi)$ for ${}^\forall t \in [0,T]$, ${}^\forall \xi \in \mathbf{R}^n_\xi$. Thanks to the condition (1.4), we find that the characteristic roots of the equation (1.1) satisfy $\lambda_1(t,\xi) \leq 0$, $\lambda_2(t,\xi) \equiv 0$, $\lambda_3(t,\xi) \geq 0$.

Then we can prove the following theorem.

Theorem 1.1. *Let $T > 0$, $\mu_0 > 0$. The coefficients satisfy (1.2), (1.3) and (1.4). Then for any u_0, u_1 and $u_2 \in G^s$, the Cauchy problem (1.1) has a unique (global) solution $u \in C^3([0,T], G^s)$, provided*

$$1 \le s < 1 + \frac{k+\alpha}{2}. \tag{1.6}$$

Moreover when u_0, u_1 and $u_2 \in G_0^s$ $(s > 1)$, there exist the constant $\nu > 0$ and the positive function $\mu(t)$ satisfying $\mu(0) = \mu_0$, such that for $^\forall \xi \in \mathbf{R}_\xi^n$

$$e^{\mu(t)\langle\xi\rangle_\nu^{\frac{1}{s}}}\left(\langle\xi\rangle_\nu^{\frac{4}{2+k+\alpha}}|\hat{u}| + \langle\xi\rangle_\nu^{\frac{2}{2+k+\alpha}}|\hat{u}_t| + |\hat{u}_{tt}|\right) \le^{\exists} C e^{\mu_0\langle\xi\rangle_\nu^{\frac{1}{s}}}\left(\langle\xi\rangle_\nu^2|\hat{u}_0| + \langle\xi\rangle_\nu|\hat{u}_1| + |\hat{u}_2|\right). \tag{1.7}$$

Remark 1. If $k = \alpha = 0$, (1.6) doesn't make sense. However whenever the coefficients $a_i(t)$, $b_{ij}(t)$ belong to $C^0([0,T])$, or even to $L^1([0,T])$, the Cauthy problem (1.1) is wellposed in G^1(see [J2] and see also [CDS]).

Remark 2. If one replaced the weakly hyperbolic condition (1.4) by the condition (1.5), the same regularity as the coefficients $b_{ij}(t)$ would be needed for the coefficients $a_i(t)$, i.e., $a_i(t)$ or $|a_i(t)| \in C^{k+\alpha}([0,T])$.

Remark 3. Precisely the positive function $\mu(t)$ is a strictly decreasing function. Therefore $\mu(T)$ is less than $\mu_0(= \mu(0))$. However if we take large enough $\nu > 0$, $\mu(T)$ can be chosen arbitrarily close to μ_0.

With a different method, Y. Ohya and S. Tarama got more geneal results for the weakly hyperbolic equations of higher order(see [OT] and see also [Tar]). They assume that all the coefficients of the principal part belong to the same Hölder class with respect to time. For the third order equation (1.1) whose coefficients satisfy (1.2), (1.3) when $k = 0$, we relax the regularity of the coefficients $a_i(t)$ from C^α to $C^{\frac{\alpha}{2}}$. Moreover according to their result, in order that the Cauthy problem for the weakly hyperbolic equations of third order is wellposed in G^s, it is necessary that the Gevrey exponent s satisfies

$$1 \le s < 1 + \frac{\alpha}{r} \quad (\text{ the multiplicity } r = 3). \tag{1.8}$$

The multiplicity of the characteristic roots for the equation (1.1), is also 3, but the range (1.6) when $k = 0$ is wider than the range (1.8). We know that the range (1.6) for the third order equation (1.1) coincides the range for the second order equations(see [CJS]). This improvement is due to the fact that one of the characteristic roots is identically equal to 0 and the regularities of the other two characteristic roots become more smooth.

1.2. Preliminaries

When $s=1$, the problem (1.1) is well-posed in G^1 which is the topological vector space of analytic functions on $\mathbf{R}^n$(see [CJS] for the weakly hyperbolic equations of second order and see [J2] for weakly hyperbolic systems including the third order equations). Therefore we can suppose $s>1$ for the proof.

In virtue of Holmgren's theorem we get the uniqueness of solutions to (1.1) and can suppose that $u_0(x)$, $u_1(x)$ and $u_2(x)$ belong to G_0^s. Hence by Paley-Wiener theorem we shall assume that

$$\sup_{\xi\in\mathbf{R}^n_\xi} e^{\mu_0\langle\xi\rangle_\nu^{\frac{1}{s}}}\left(\langle\xi\rangle_\nu^2|\hat{u}_0|+\langle\xi\rangle_\nu|\hat{u}_1|+|\hat{u}_2|\right)<+\infty. \tag{1.9}$$

Moreover Ovciannikov theorem gives the existence of solutions(see [CJS], [J3], [Ov]). Our task is to investigate the regularity for x of the solution.

In order to derive the energy inequality, the following lemma is useful.

Lemma 1.2. *(Colombini, Jannelli, Spagnolo) Let $f(t)$ be a real function of class $C^{k+\alpha}$ on some compact interval $I\subset\mathbf{R}$, with k integer ≥ 1 and $0\leq\alpha\leq 1$, and assume that $f(t)\geq 0$ on I. Then the function $f^{\frac{1}{k+\alpha}}$ is absolutely continuous on I. Moreover*

$$\|\,(f^{\frac{1}{k+\alpha}})'\,\|_{L^1(I)}^{k+\alpha}\leq C(k,\alpha,I)\,\|\,f\,\|_{C^{k+\alpha}(I)}\,.$$

For the proof, refer to [CJS].

By Fourier transform the Cauchy problem (1.1) is changed to

$$\begin{cases} v_{ttt}+ia(t,\xi)v_{tt}-b(t,\xi)v_t=0\\ v(0,\xi)=v_0(\xi),\quad v_t(0,\xi)=v_1(\xi),\quad v_{tt}(0,\xi)=v_2(\xi),\end{cases} \tag{1.10}$$

where $v=\hat{u}$, and $v_l=\hat{u}_l$ $(l=0,1,2)$, and $a(t,\xi)=\sum_{i=1}^n a_i(t)\xi_i$, $b(t,\xi)=\sum_{i,j=1}^n b_{ij}(t)\xi_i\xi_j$.

1.3 Energy inequality in case of $k=0$

We first treat the csae of $k=0$ which implies that both coefficients $a(t,\xi)$ and $b(t,\xi)$ belong to Hölder classes in t. Since these coefficients are not differentiable, they can not enter into the definition of the energy directly. Therefore we shall regularize them as follows.

$$a_\varepsilon(t,\xi)=\frac{1}{\varepsilon}\int_{-\infty}^{\infty}a(t+\tau,\xi)\varphi\Big(\frac{\tau}{\varepsilon}\Big)d\tau, \tag{1.11}$$

$$b_\varepsilon(t,\xi)=\frac{1}{\varepsilon}\int_{-\infty}^{\infty}b(t+\tau,\xi)\varphi\Big(\frac{\tau}{\varepsilon}\Big)d\tau, \tag{1.12}$$

$(0<\varepsilon<1)$, where $\varphi(t)\in C_0^\infty(\mathbf{R}_t^1)$ satisfies $0\le\varphi(t)<\infty$ and $\int_{-\infty}^{\infty}\varphi(t)dt=1$.

Then there exists $C_0>0$ such that for $^\forall\xi\in\mathbf{R}_n^\xi$

$$|a_\varepsilon'(t,\xi)|\le C_0\varepsilon^{\frac{\alpha}{2}-1}\langle\xi\rangle_\nu,\quad |a_\varepsilon(t,\xi)-a(t,\xi)|\le C_0\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu, \tag{1.13}$$

$$|b_\varepsilon'(t,\xi)|\le C_0\varepsilon^{\alpha-1}\langle\xi\rangle_\nu^2,\quad |b_\varepsilon(t,\xi)-b(t,\xi)|\le C_0\varepsilon^{\alpha}\langle\xi\rangle_\nu^2. \tag{1.14}$$

With the coefficients $a_\varepsilon(t,\xi)$, $b_\varepsilon(t,\xi)$ we shall define the following energy.

$$\begin{aligned}E_{\varepsilon,\nu}(t,\xi)^2 =&e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{|v_{tt}+ia_\varepsilon(t,\xi)v_t-b_\varepsilon(t,\xi)v+\varepsilon^\alpha\langle\xi\rangle_\nu^2v|^2\\ &+\Big|v_{tt}+\frac{i}{2}a_\varepsilon(t,\xi)v_t\Big|^2+\Big(\frac{a_\varepsilon(t,\xi)^2}{4}-b_\varepsilon(t,\xi)+\varepsilon^\alpha\langle\xi\rangle_\nu^2\Big)|v_t|^2\Big\}.\end{aligned} \tag{1.15}$$

Here $\rho(t)$ is positive and determined later on. Thanks to the conditions (1.4) and the term $\varepsilon^\alpha\langle\xi\rangle_\nu^2$, this energy can be bounded from below by the absolute values of v, v_t and v_{tt}. While we can also easily see that this energy is bounded (from above) by the absolute values of them. Therefore the energy inequality based on (1.15) can be changed into the one based on the absolute values.

Defferentiating (1.15) in t, by (1.10) we get

$$\begin{aligned}\frac{d}{dt}(E_{\varepsilon,\nu}^2) =& \rho'(t)\langle\xi\rangle_\nu^\kappa E_{\varepsilon,\nu}^2\\ &+2e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(i(a_\varepsilon-a)v_{tt}-(b_\varepsilon-b)v_t+\varepsilon^\alpha\langle\xi\rangle_\nu^2v_t+ia_\varepsilon'v_t-b_\varepsilon'v,\\ &\qquad\qquad v_{tt}+ia_\varepsilon v_t-b_\varepsilon v+\varepsilon^\alpha\langle\xi\rangle_\nu^2v\Big)\\ &+2e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(-iav_{tt}+bv_t+\frac{i}{2}a_\varepsilon v_{tt}+\frac{i}{2}a_\varepsilon'v_t,\ v_{tt}+\frac{i}{2}a_\varepsilon v_t\Big)\\ &+2\Big(\frac{a_\varepsilon^2}{4}-b_\varepsilon+\varepsilon^\alpha\langle\xi\rangle_\nu^2\Big)e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re(v_t,\ v_{tt})+e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big(\frac{1}{2}a_\varepsilon a_\varepsilon'-b_\varepsilon'\Big)|v_t|^2\\ &(\equiv\rho'(t)\langle\xi\rangle_\nu^\kappa E_{\varepsilon,\nu}^2+I+II+III+IV).\end{aligned} \tag{1.16}$$

In order to further estimate the derivative of the energy, we shall pick up the term I. We first rewrite I as follows.

$$
\begin{aligned}
(1.17)\ I =& 2(a_\varepsilon - a)e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(i\big(v_{tt}+\frac{i}{2}a_\varepsilon v_t\big),\ v_{tt}+ia_\varepsilon v_t - b_\varepsilon v + \varepsilon^\alpha\langle\xi\rangle_\nu^2 v\Big)\\
&+2(a_\varepsilon - a)e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(\frac{1}{2}a_\varepsilon v_t,\ v_{tt}+ia_\varepsilon v_t - b_\varepsilon v + \varepsilon^\alpha\langle\xi\rangle_\nu^2 v\Big)\\
&-2(b_\varepsilon - b)e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(\varepsilon^{\frac{\alpha}{4}}\langle\xi\rangle_\nu^{\frac{1}{2}} v_t,\ \varepsilon^{-\frac{\alpha}{4}}\langle\xi\rangle_\nu^{-\frac{1}{2}}\big(v_{tt}+ia_\varepsilon v_t - b_\varepsilon v + \varepsilon^\alpha\langle\xi\rangle_\nu^2 v\big)\Big)\\
&+2\varepsilon^\alpha\langle\xi\rangle_\nu^2 e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(\varepsilon^{\frac{\alpha}{4}}\langle\xi\rangle_\nu^{\frac{1}{2}} v_t,\ \varepsilon^{-\frac{\alpha}{4}}\langle\xi\rangle_\nu^{-\frac{1}{2}}\big(v_{tt}+ia_\varepsilon v_t - b_\varepsilon v + \varepsilon^\alpha\langle\xi\rangle_\nu^2 v\big)\Big)\\
&+2a'_\varepsilon e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(i\varepsilon^{\frac{\alpha}{4}}\langle\xi\rangle_\nu^{\frac{1}{2}} v_t,\ \varepsilon^{-\frac{\alpha}{4}}\langle\xi\rangle_\nu^{-\frac{1}{2}}\big(v_{tt}+ia_\varepsilon v_t - b_\varepsilon v + \varepsilon^\alpha\langle\xi\rangle_\nu^2 v\big)\Big)\\
&+2\frac{-b'_\varepsilon}{-b_\varepsilon+\varepsilon^\alpha\langle\xi\rangle_\nu^2}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(-v_{tt}-\frac{i}{2}a_\varepsilon v_t,\ v_{tt}+ia_\varepsilon v_t - b_\varepsilon v + \varepsilon^\alpha\langle\xi\rangle_\nu^2 v\Big)\\
&+2\frac{-b'_\varepsilon}{-b_\varepsilon+\varepsilon^\alpha\langle\xi\rangle_\nu^2}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(-\frac{i}{2}a_\varepsilon v_t,\ v_{tt}+ia_\varepsilon v_t - b_\varepsilon v + \varepsilon^\alpha\langle\xi\rangle_\nu^2 v\Big)\\
&+2\frac{-b'_\varepsilon}{-b_\varepsilon+\varepsilon^\alpha\langle\xi\rangle_\nu^2}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\big|v_{tt}+ia_\varepsilon v_t - b_\varepsilon v + \varepsilon^\alpha\langle\xi\rangle_\nu^2 v\big|^2.
\end{aligned}
$$

For this expression, we remark thet the denominator of the fraction in the last three terms is not zero, since $-b_\varepsilon$ is non-negative by the condition (1.4).

Noting the definition of the energy (1.15), by(1.13), (1.14), (1.17) we obtain

$$
\begin{aligned}
(1.18)\ I \le& C_0\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\big|v_{tt}+\frac{i}{2}a_\varepsilon v_t\big|^2 + |v_{tt}+ia_\varepsilon v_t - b_\varepsilon v + \varepsilon^\alpha\langle\xi\rangle_\nu^2 v|^2\Big\}\\
&+C_0\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\frac{1}{4}a_\varepsilon^2|v_t|^2 + |v_{tt}+ia_\varepsilon v_t - b_\varepsilon v + \varepsilon^\alpha\langle\xi\rangle_\nu^2 v|^2\Big\}\\
&+C_0\varepsilon^\alpha\langle\xi\rangle_\nu^2 e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu|v_t|^2 + \varepsilon^{-\frac{\alpha}{2}}\langle\xi\rangle_\nu^{-1}\big|v_{tt}+ia_\varepsilon v_t - b_\varepsilon v + \varepsilon^\alpha\langle\xi\rangle_\nu^2 v\big|^2\Big\}\\
&+\varepsilon^\alpha\langle\xi\rangle_\nu^2 e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu|v_t|^2 + \varepsilon^{-\frac{\alpha}{2}}\langle\xi\rangle_\nu^{-1}\big|v_{tt}+ia_\varepsilon v_t - b_\varepsilon v + \varepsilon^\alpha\langle\xi\rangle_\nu^2 v\big|^2\Big\}\\
&+C_0\varepsilon^{\frac{\alpha}{2}-1}\langle\xi\rangle_\nu e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu|v_t|^2 + \varepsilon^{-\frac{\alpha}{2}}\langle\xi\rangle_\nu^{-1}\big|v_{tt}+ia_\varepsilon v_t - b_\varepsilon v + \varepsilon^\alpha\langle\xi\rangle_\nu^2 v\big|^2\Big\}\\
&+C_0\varepsilon^{-1}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\big|v_{tt}+\frac{i}{2}a_\varepsilon v_t\big|^2 + |v_{tt}+ia_\varepsilon v_t - b_\varepsilon v + \varepsilon^\alpha\langle\xi\rangle_\nu^2 v|^2\Big\}\\
&+C_0\varepsilon^{-1}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\frac{1}{4}a_\varepsilon^2|v_t|^2 + |v_{tt}+ia_\varepsilon v_t - b_\varepsilon v + \varepsilon^\alpha\langle\xi\rangle_\nu^2 v|^2\Big\}\\
&+2C_0\varepsilon^{-1}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}|v_{tt}+ia_\varepsilon v_t - b_\varepsilon v + \varepsilon^\alpha\langle\xi\rangle_\nu^2 v|^2\\
\le& C_0\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2 + C_0\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2 + C_0\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2 + \varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2\\
&+C_0\varepsilon^{-1}E_{\varepsilon,\nu}^2 + C_0\varepsilon^{-1}E_{\varepsilon,\nu}^2 + C_0\varepsilon^{-1}E_{\varepsilon,\nu}^2 + 2C_0\varepsilon^{-1}E_{\varepsilon,\nu}^2\\
=& (3C_0+1)\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2 + 5C_0\varepsilon^{-1}E_{\varepsilon,\nu}^2,
\end{aligned}
$$

here we used $2\Re(z_1,\ z_2) \le |z_1|^2 + |z_2|^2$ for ${}^\forall z_1, {}^\forall z_2 \in \mathbf{C}^1$.

Secondly we shall estimate the other terms. From the definition of the energy (1.15) we also find that

$$E_{\varepsilon,\nu}^2 \geq e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\frac{1}{4}a_\varepsilon^2 + \varepsilon^\alpha\langle\xi\rangle_\nu^2\Big\}|v_t|^2 \geq e^{\rho(t)\langle\xi\rangle_\nu^\kappa}|a_\varepsilon|\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu|v_t|^2. \tag{1.19}$$

Dealing with the terms II, III and IV together, by (1.13), (1.14) and (1.19) we obtain

$$\begin{aligned}
II+III+IV &= 2e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(\big(-\frac{1}{2}aa_\varepsilon + b + \frac{1}{4}a_\varepsilon^2\big)v_t,\ v_{tt}\Big) + e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\big(ia_\varepsilon' v_t,\ v_{tt}\big)\\
&\quad + \frac{1}{2}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}a_\varepsilon' a_\varepsilon|v_t|^2 + III + IV\\
&= a_\varepsilon(a_\varepsilon - a)e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\big(v_t,\ v_{tt}\big) + 2(b - b_\varepsilon + \varepsilon^\alpha\langle\xi\rangle_\nu^2)e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\big(v_t,\ v_{tt}\big)\\
&\quad + a_\varepsilon a_\varepsilon' e^{\rho(t)\langle\xi\rangle_\nu^\kappa}|v_t|^2 + (-b_\varepsilon')e^{\rho(t)\langle\xi\rangle_\nu^\kappa}|v_t|^2 + a_\varepsilon' e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\big(iv_t,\ v_{tt}\big)\\
&= 2(a_\varepsilon - a)e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(\frac{1}{2}a_\varepsilon v_t,\ v_{tt} + \frac{i}{2}a_\varepsilon v_t\Big)\\
&\quad + 2\big(b - b_\varepsilon + \varepsilon^\alpha\langle\xi\rangle_\nu^2\big)e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(\varepsilon^{\frac{\alpha}{4}}\langle\xi\rangle_\nu^{\frac{1}{2}}v_t,\ \varepsilon^{-\frac{\alpha}{4}}\langle\xi\rangle_\nu^{-\frac{1}{2}}\big(v_{tt} + \frac{i}{2}a_\varepsilon v_t\big)\Big)\\
&\quad + a_\varepsilon'\varepsilon^{-\frac{\alpha}{2}}\langle\xi\rangle_\nu^{-1}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}a_\varepsilon\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu|v_t|^2 + (-b_\varepsilon')\varepsilon^{-\alpha}\langle\xi\rangle_\nu^{-2}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\varepsilon^\alpha\langle\xi\rangle_\nu^2|v_t|^2\\
&\quad + 2a_\varepsilon' e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(\frac{i}{2}\varepsilon^{\frac{\alpha}{4}}\langle\xi\rangle_\nu^{\frac{1}{2}}v_t,\ \varepsilon^{-\frac{\alpha}{4}}\langle\xi\rangle_\nu^{-\frac{1}{2}}\big(v_{tt} + \frac{i}{2}a_\varepsilon v_t\big)\Big)\\
&\quad + 2a_\varepsilon' e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(\frac{i}{2}v_t,\ -\frac{i}{2}a_\varepsilon v_t\Big)\Big(= -\frac{1}{2}\times\ \text{“third term”}\Big)\\
&\leq C_0\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\frac{1}{4}a_\varepsilon^2|v_t|^2 + \big|v_{tt} + \frac{i}{2}a_\varepsilon v_t\big|^2\Big\}\\
&\quad + (C_0+1)\varepsilon^\alpha\langle\xi\rangle_\nu^2 e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu|v_t|^2 + \varepsilon^{-\frac{\alpha}{2}}\langle\xi\rangle_\nu^{-1}\big|v_{tt} + \frac{i}{2}a_\varepsilon v_t\big|^2\Big\}\\
&\quad + \frac{C_0}{2}\varepsilon^{\frac{\alpha}{2}-1}\langle\xi\rangle_\nu\varepsilon^{-\frac{\alpha}{2}}\langle\xi\rangle_\nu^{-1}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}|a_\varepsilon|\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu|v_t|^2\\
&\quad + C_0\varepsilon^{\alpha-1}\langle\xi\rangle_\nu^2\varepsilon^{-\alpha}\langle\xi\rangle_\nu^{-2}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\varepsilon^\alpha\langle\xi\rangle_\nu^2|v_t|^2\\
&\quad + C_0\varepsilon^{\frac{\alpha}{2}-1}\langle\xi\rangle_\nu e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu|v_t|^2 + \varepsilon^{-\frac{\alpha}{2}}\langle\xi\rangle_\nu^{-1}\big|v_{tt} + \frac{i}{2}a_\varepsilon v_t\big|^2\Big\}\\
&\leq C_0\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2 + (C_0+1)\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2\\
&\qquad\qquad + \frac{C_0}{2}\varepsilon^{-1}E_{\varepsilon,\nu}^2 + C_0\varepsilon^{-1}E_{\varepsilon,\nu}^2 + C_0\varepsilon^{-1}E_{\varepsilon,\nu}^2\\
&= (2C_0+1)\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2 + \frac{5}{2}C_0\varepsilon^{-1}E_{\varepsilon,\nu}^2,
\end{aligned} \tag{1.20}$$

here we used $\Re\big(iz,\ z\big) = \Re\{i|z|^2\} = 0$ for ${}^\forall z \in \mathbf{C}^1$.

Therefore by (1.16), (1.18) and (1.20) we have the estimate

$$\frac{d}{dt}\big(E_{\varepsilon,\nu}^2\big) \leq \rho'(t)\langle\xi\rangle_\nu^\kappa E_{\varepsilon,\nu}^2 + C_1\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2 + C_2\varepsilon^{-1}E_{\varepsilon,\nu}^2, \tag{1.21}$$

where $C_1 = 5C_0 + 2$, $C_2 = \frac{15}{2}C_0$.

Thus Gronwall's inequality yields

$$E_{\varepsilon,\nu}^2(t,\xi) \le E_{\varepsilon,\nu}^2(0,\xi)\exp\Big[\int_0^t \{\rho'(\tau)\langle\xi\rangle_\nu^\kappa + C_1\varepsilon^{\frac{\alpha}{2}}\langle\xi\rangle_\nu + C_2\varepsilon^{-1}\}d\tau\Big] \qquad \text{for } {}^\forall t\in[0,T].$$

Since $s(=\kappa^{-1}) < 1+\frac{\alpha}{2}$, we find $\frac{2}{2+\alpha}-\kappa<0$ and $\langle\xi\rangle_\nu^{\frac{2}{2+\alpha}-\kappa} \le \nu^{\frac{2}{2+\alpha}-\kappa}$. Hence we take $\varepsilon = \langle\xi\rangle_\nu^{-\frac{2}{2+\alpha}}$ and obtain

$$\begin{aligned}E_{\varepsilon,\nu}^2(t,\xi) &\le E_{\varepsilon,\nu}^2(0,\xi)\exp\Big[\langle\xi\rangle_\nu^\kappa\int_0^t\Big\{\rho'(\tau)+(C_1+C_2)\nu^{\frac{2}{2+\alpha}-\kappa}\Big\}d\tau\Big]\\ &= E_{\varepsilon,\nu}^2(0,\xi)\exp\Big[\langle\xi\rangle_\nu^\kappa\Big\{\rho(t)-\rho_0+C_3t\nu^{\frac{2}{2+\alpha}-\kappa}\Big\}\Big] \qquad \text{for } {}^\forall t\in[0,T],\end{aligned}$$

where $\rho_0=\rho(0)$, $C_3=C_1+C_2$.

Moreover we determine $\rho(t)=\rho_0-C_3t\nu^{\frac{2}{2+\alpha}-\kappa}$, and choose $\nu>0$ such that $\rho(T)\big(=\rho_0-C_3T\nu^{\frac{2}{2+\alpha}-\kappa}\big)>0$ for any given $T>0$.

Finally we have the energy inequality

$$(1.22)\qquad E_{\varepsilon,\nu}^2(t,\xi)\le E_{\varepsilon,\nu}^2(0,\xi) \qquad \text{for } {}^\forall t\in[0,T] \text{ and } {}^\forall\xi\in\mathbf{R}_\xi^n.$$

1.4 Energy inequality in case of $k=1$

We next treat the case of $k=1$ which implies that the coefficient $a(t,\xi)$ belongs to Hölder class while $b(t,\xi)$ belongs to C^1 class in t at least. Therefore we shall only regularize the coefficient $a(t,\xi)$ as (1.11), and get

$$(1.13)'\qquad |a_\varepsilon'(t,\xi)|\le C_0\varepsilon^{\frac{\alpha+1}{2}-1}\langle\xi\rangle_\nu,\quad |a_\varepsilon(t,\xi)-a(t,\xi)|\le C_0\varepsilon^{\frac{\alpha+1}{2}}\langle\xi\rangle_\nu$$

With the coefficients $a_\varepsilon(t,\xi)$, $b(t,\xi)$ we shall define the following energy.

$$\begin{aligned}(1.23)\ E_{\varepsilon,\nu}(t,\xi)^2 =& e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{|v_{tt}+ia_\varepsilon(t,\xi)v_t-b(t,\xi)v+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2v|^2\\ &+\Big|v_{tt}+\frac{i}{2}a_\varepsilon(t,\xi)v_t\Big|^2+\Big(\frac{a_\varepsilon(t,\xi)^2}{4}-b(t,\xi)+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2\Big)|v_t|^2\Big\}.\end{aligned}$$

Defferentiating (1.23) in t, by (1.10) we get

$$\begin{aligned}(1.24)\ \frac{d}{dt}(E_{\varepsilon,\nu}^2) =& \rho'(t)\langle\xi\rangle_\nu^\kappa E_{\varepsilon,\nu}^2 + 2e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(i(a_\varepsilon-a)v_{tt}+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2v_t+ia_\varepsilon'v_t-b'v,\\ &\qquad v_{tt}+ia_\varepsilon v_t-bv+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2v\Big)\\ &+2e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(-iav_{tt}+bv_t+\frac{i}{2}a_\varepsilon v_{tt}+\frac{i}{2}a_\varepsilon'v_t,\ v_{tt}+\frac{i}{2}a_\varepsilon v_t\Big)\\ &+2\Big(\frac{a_\varepsilon^2}{4}-b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2\Big)e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re(v_t,\ v_{tt})+e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big(\frac{1}{2}a_\varepsilon a_\varepsilon'-b'\Big)|v_t|^2\\ &(\equiv\rho'(t)\langle\xi\rangle_\nu^\kappa E_{\varepsilon,\nu}^2+I'+II'+III'+IV').\end{aligned}$$

Noting that

$$\frac{-b'}{-b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2} \le \frac{\left|\left(-b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2\right)'\right|}{\left(-b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2\right)^{1-\frac{1}{1+\alpha}}\varepsilon\langle\xi\rangle_\nu^{\frac{2}{1+\alpha}}} = (1+\alpha)\left|\left\{\left(-b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2\right)^{\frac{1}{1+\alpha}}\right\}'\right|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{1+\alpha}},$$

similarly as §1.3, we can estimate I' and $II'+III'+IV'$ as follows.

$$\begin{aligned}
(1.25)\ I' =&2(a_\varepsilon-a)e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(i\big(v_{tt}+\frac{i}{2}a_\varepsilon v_t\big),\ v_{tt}+ia_\varepsilon v_t-bv+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2 v\Big)\\
&+2(a_\varepsilon-a)e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(\frac{1}{2}a_\varepsilon v_t,\ v_{tt}+ia_\varepsilon v_t-bv+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2 v\Big)\\
&+2\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2 e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(\varepsilon^{\frac{1+\alpha}{4}}\langle\xi\rangle_\nu^{\frac{1}{2}}v_t,\ \varepsilon^{-\frac{1+\alpha}{4}}\langle\xi\rangle_\nu^{-\frac{1}{2}}\big(v_{tt}+ia_\varepsilon v_t-bv+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2 v\big)\Big)\\
&+2a_\varepsilon' e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(i\varepsilon^{\frac{1+\alpha}{4}}\langle\xi\rangle_\nu^{\frac{1}{2}}v_t,\ \varepsilon^{-\frac{1+\alpha}{4}}\langle\xi\rangle_\nu^{-\frac{1}{2}}\big(v_{tt}+ia_\varepsilon v_t-bv+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2 v\big)\Big)\\
&+2\frac{-b'}{-b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(-v_{tt}-\frac{i}{2}a_\varepsilon v_t,\ v_{tt}+ia_\varepsilon v_t-bv+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2 v\Big)\\
&+2\frac{-b'}{-b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(-\frac{i}{2}a_\varepsilon v_t,\ v_{tt}+ia_\varepsilon v_t-bv+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2 v\Big)\\
&+2\frac{-b'}{-b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\left|v_{tt}+ia_\varepsilon v_t-bv+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2 v\right|^2\\
\le&C_0\varepsilon^{\frac{1+\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2+C_0\varepsilon^{\frac{1+\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2+\varepsilon^{\frac{1+\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2+C_0\varepsilon^{-1}E_{\varepsilon,\nu}^2\\
&+(1+\alpha)\left|\left\{\left(-b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2\right)^{\frac{1}{1+\alpha}}\right\}'\right|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{1+\alpha}}E_{\varepsilon,\nu}^2\\
&+(1+\alpha)\left|\left\{\left(-b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2\right)^{\frac{1}{1+\alpha}}\right\}'\right|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{1+\alpha}}E_{\varepsilon,\nu}^2\\
&+2(1+\alpha)\left|\left\{\left(-b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2\right)^{\frac{1}{1+\alpha}}\right\}'\right|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{1+\alpha}}E_{\varepsilon,\nu}^2\\
=&(2C_0+1)\varepsilon^{\frac{1+\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2+C_0\varepsilon^{-1}E_{\varepsilon,\nu}^2+4(1+\alpha)\left|\left\{\left(-b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2\right)^{\frac{1}{1+\alpha}}\right\}'\right|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{1+\alpha}}E_{\varepsilon,\nu}^2.
\end{aligned}$$

$$\begin{aligned}
(1.26)\ &II'+III'+IV'\\
=&2(a_\varepsilon-a)e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(\frac{1}{2}a_\varepsilon v_t,\ v_{tt}+\frac{i}{2}a_\varepsilon v_t\Big)\\
&+2\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2 e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(\varepsilon^{\frac{1+\alpha}{4}}\langle\xi\rangle_\nu^{\frac{1}{2}}v_t,\ \varepsilon^{-\frac{1+\alpha}{4}}\langle\xi\rangle_\nu^{-\frac{1}{2}}\big(v_{tt}+\frac{i}{2}a_\varepsilon v_t\big)\Big)\\
&+\frac{1}{2}a_\varepsilon'\varepsilon^{-\frac{1+\alpha}{2}}\langle\xi\rangle_\nu^{-1}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}a_\varepsilon\varepsilon^{\frac{1+\alpha}{2}}\langle\xi\rangle_\nu|v_t|^2+\frac{-b'}{-b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\left(-b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2\right)|v_t|^2\\
&+2a_\varepsilon' e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(\frac{i}{2}\varepsilon^{\frac{1+\alpha}{4}}\langle\xi\rangle_\nu^{\frac{1}{2}}v_t,\ \varepsilon^{-\frac{1+\alpha}{4}}\langle\xi\rangle_\nu^{-\frac{1}{2}}\big(v_{tt}+\frac{i}{2}a_\varepsilon v_t\big)\Big)\\
\le&C_0\varepsilon^{\frac{1+\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2+\varepsilon^{\frac{1+\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2+\frac{C_0}{2}\varepsilon^{-1}E_{\varepsilon,\nu}^2\\
&+(1+\alpha)\left|\left\{\left(-b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2\right)^{\frac{1}{1+\alpha}}\right\}'\right|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{1+\alpha}}E_{\varepsilon,\nu}^2+C_0\varepsilon^{-1}E_{\varepsilon,\nu}^2\\
=&(C_0+1)\varepsilon^{\frac{1+\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2+\frac{3}{2}C_0\varepsilon^{-1}E_{\varepsilon,\nu}^2+(1+\alpha)\left|\left\{\left(-b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2\right)^{\frac{1}{1+\alpha}}\right\}'\right|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{1+\alpha}}E_{\varepsilon,\nu}^2.
\end{aligned}$$

Therefore by (1.24)-(1.26) we have the estimate coresponding to (1.21)

$$\frac{d}{dt}(E_{\varepsilon,\nu}^2) \le \rho'(t)\langle\xi\rangle_\nu^\kappa E_{\varepsilon,\nu}^2 + C_4\varepsilon^{\frac{1+\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2 + C_5\varepsilon^{-1}E_{\varepsilon,\nu}^2$$
$$+ C_6\left|\left\{\left(-b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2\right)^{\frac{1}{1+\alpha}}\right\}'\right|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{1+\alpha}}E_{\varepsilon,\nu}^2,$$

where $C_4 = 3C_0 + 1$, $C_5 = \frac{5}{2}C_0$, $C_6 = 5(1+\alpha)$.

Thus Gronwall's inequality yields

$$(1.27)\ E_{\varepsilon,\nu}^2(t,\xi) \le E_{\varepsilon,\nu}^2(0,\xi)\exp\Big[\int_0^t\Big\{\rho'(\tau)\langle\xi\rangle_\nu^\kappa + C_4\varepsilon^{\frac{1+\alpha}{2}}\langle\xi\rangle_\nu + C_5\varepsilon^{-1}$$
$$+ C_6\left|\left\{\left(-b(\tau)+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2\right)^{\frac{1}{1+\alpha}}\right\}'\right|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{1+\alpha}}\Big\}d\tau\Big] \quad \text{for } {}^\forall t \in [0,T],$$

Now from the condition (1.4) we can apply Lemma 1.2 to (1.27). Since $s(=\kappa^{-1}) < 1 + \frac{1+\alpha}{2}$, we find $\frac{2}{3+\alpha} - \kappa < 0$ and $\langle\xi\rangle_\nu^{\frac{2}{3+\alpha}-\kappa} \le \nu^{\frac{2}{3+\alpha}-\kappa}$. Hence we take $\varepsilon = \langle\xi\rangle_\nu^{-\frac{2}{3+\alpha}}$ and obtain

$$E_{\varepsilon,\nu}^2(t,\xi) \le E_{\varepsilon,\nu}^2(0,\xi)\exp\Big[\langle\xi\rangle_\nu^\kappa\int_0^t\Big\{\rho'(\tau) + (C_4+C_5)\nu^{\frac{2}{3+\alpha}-\kappa}$$
$$+ C_6\left|\left\{\left(-b(\tau)+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2\right)^{\frac{1}{1+\alpha}}\right\}'\right|\langle\xi\rangle_\nu^{-\frac{2}{1+\alpha}}\nu^{\frac{2}{3+\alpha}-\kappa}\Big\}d\tau\Big]$$
$$\le E_{\varepsilon,\nu}^2(0,\xi)\exp\Big[\langle\xi\rangle_\nu^\kappa\Big\{\rho(t)-\rho_0+\big(C_7t+\phi_1(t)\big)\nu^{\frac{2}{3+\alpha}-\kappa}\Big\}\Big] \qquad \text{for } {}^\forall t \in [0,T],$$

where $C_7 = C_4 + C_5$, $\phi_1(t)$ is a bounded function independent of ξ and satisfy $\phi_1(0) = 0$ and $\phi_1(t) \ge C_6C(t)\ \| -b+\varepsilon^{1+\alpha}\langle\xi\rangle_\nu^2 \|_{C^{1+\alpha}}^{\frac{1}{1+\alpha}}\ \langle\xi\rangle_\nu^{-\frac{2}{1+\alpha}}$.

Moreover we determine $\rho(t) = \rho_0 - \big(C_7t+\phi_1(t)\big)\nu^{\frac{2}{3+\alpha}-\kappa}$, and choose $\nu > 0$ such that $\rho(T)\big(=\rho_0 - \big(C_7T+\phi_1(T)\big)\nu^{\frac{2}{3+\alpha}-\kappa}\big) > 0$ for any given $T > 0$.

Finally we have the energy inequality

$$(1.28) \qquad E_{\varepsilon,\nu}^2(t,\xi) \le E_{\varepsilon,\nu}^2(0,\xi) \qquad \text{for } {}^\forall t \in [0,T] \text{ and } {}^\forall \xi \in \mathbf{R}_\xi^n.$$

1.5 Energy inequality in case of $k \ge 2$

We finally treat the case of $k \ge 2$ which implies that both coefficients $a(t,\xi)$, $b(t,\xi)$ belong to C^1 class in t at least. Therefore we need not regularize the coefficients $a(t,\xi)$, $b(t,\xi)$.

With the coefficients $a(t,\xi)$, $b(t,\xi)$ we shall define the following energy.

$$(1.29)\ E_{\varepsilon,\nu}(t,\xi)^2 = e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{|v_{tt}+ia(t,\xi)v_t-b(t,\xi)v+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v|^2 + \Big|v_{tt}+\frac{i}{2}a(t,\xi)v_t\Big|^2 + \Big(\frac{a(t,\xi)^2}{4}-b(t,\xi)+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\Big)|v_t|^2\Big\}.$$

Defferentiating (1.29) in t, by (1.10) we get

$$\begin{aligned}(1.30)\ \frac{d}{dt}(E_{\varepsilon,\nu}^2) &= \rho'(t)\langle\xi\rangle_\nu^\kappa E_{\varepsilon,\nu}^2 \\ &+ 2e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v_t + ia'v_t - b'v,\ v_{tt}+iav_t-bv+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v\Big) \\ &+ 2e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(-\frac{i}{2}av_{tt}+bv_t+\frac{i}{2}a'v_t,\ v_{tt}+\frac{i}{2}av_t\Big) \\ &+ 2\Big(\frac{a^2}{4}-b+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\Big)e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re(v_t,\ v_{tt}) + e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big(\frac{1}{2}aa'-b'\Big)|v_t|^2 \\ &(\equiv \rho'(t)\langle\xi\rangle_\nu^\kappa E_{\varepsilon,\nu}^2 + I'' + II'' + III'' + IV'').\end{aligned}$$

Before we estimate the terms I'' and $II''+III''+IV''$, we shall culculate the parts concerned with the coefficient $a(t)$ in advance. From the condition $|a(t)| \in C^{\frac{k+\alpha}{2}}([0,T])(\subset C^1([0,T]))$, we can see $|a'(t)| = \big|\,|a(t)|'\big|$. Hence it holds that for $w(t,\xi) = v_{tt}+ia(t,\xi)v_t-b(t,\xi)v+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v$ or $v_{tt}+\frac{i}{2}a(t,\xi)v_t$

$$\begin{aligned}(1.31)\quad & 2a'e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(i\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^{\frac{1}{2}}v_t,\ \Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^{-\frac{1}{2}}w\Big) \\ &\le |a'|e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)|v_t|^2 + \Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^{-1}|w|^2\Big\} \\ &= \frac{\Big|\Big\{\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^2\Big\}'\Big|}{\frac{a^2}{4}+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2}e^{\rho\langle\xi\rangle_\nu^\kappa}\Big(\frac{a^2}{4}+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\Big)|v_t|^2 + 2\frac{\Big|\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)'\Big|}{\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu}e^{\rho\langle\xi\rangle_\nu^\kappa}|w|^2 \\ &\le 2\frac{\Big|\Big\{\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^2\Big\}'\Big|}{\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^2}E_{\varepsilon,\nu}^2 + 2\frac{\Big|\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)'\Big|}{\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu}E_{\varepsilon,\nu}^2 \\ &\le 3(k+\alpha)\Big|\Big\{\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^{\frac{2}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}E_{\varepsilon,\nu}^2.\end{aligned}$$

$$\begin{aligned}(1.32)\ aa'e^{\rho(t)\langle\xi\rangle_\nu^\kappa}|v_t|^2 &\le \frac{2\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)\cdot 2\Big|\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)'\Big|}{\frac{a^2}{4}+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2}e^{\rho\langle\xi\rangle_\nu^\kappa}\Big(\frac{a^2}{4}+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\Big)|v_t|^2 \\ &\le 4\frac{\Big|\Big\{\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^2\Big\}'\Big|}{\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^2}E_{\varepsilon,\nu}^2 \le 4(k+\alpha)\Big|\Big\{\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^{\frac{2}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}E_{\varepsilon,\nu}^2.\end{aligned}$$

In (1.31), (1.32) we used $\frac{a^2}{4}+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 \ge \frac{1}{2}\big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\big)^2$.

Noting that $\frac{-b'}{-b+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2} \le (k+\alpha)\Big|\Big\{\big(-b+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\big)^{\frac{1}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}$, and using (1.31) with $w(t,\xi) = v_{tt} + ia(t,\xi)v_t - b(t,\xi)v + \varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v$, similarly as §1.3 and §1.4, we can estimate I'' as follows.

$$
\begin{aligned}
(1.33)\quad I'' =& 2\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(\varepsilon^{\frac{k+\alpha}{4}}\langle\xi\rangle_\nu^{\frac{1}{2}}v_t,\ \varepsilon^{-\frac{k+\alpha}{4}}\langle\xi\rangle_\nu^{-\frac{1}{2}}\big(v_{tt}+iav_t-bv+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v\big)\Big)\\
&+2a'e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(i\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^{\frac{1}{2}}v_t,\\
&\qquad\qquad\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^{-\frac{1}{2}}\big(v_{tt}+iav_t-bv+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v\big)\Big)\\
&+2\frac{-b'}{-b+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(-v_{tt}-\frac{i}{2}av_t,\ v_{tt}+iav_t-bv+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v\Big)\\
&+2\frac{-b'}{-b+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(-\frac{i}{2}av_t,\ v_{tt}+iav_t-bv+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v\Big)\\
&+2\frac{-b'}{-b+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\big|v_{tt}+iav_t-bv+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v\big|^2\\
\le&\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2+3(k+\alpha)\Big|\Big\{\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^{\frac{2}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}E_{\varepsilon,\nu}^2\\
&+(k+\alpha)\Big|\Big\{\big(-b+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\big)^{\frac{1}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}E_{\varepsilon,\nu}^2\\
&+(k+\alpha)\Big|\Big\{\big(-b+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\big)^{\frac{1}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}E_{\varepsilon,\nu}^2\\
&+2(k+\alpha)\Big|\Big\{\big(-b+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\big)^{\frac{1}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}E_{\varepsilon,\nu}^2\\
=&\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2+3(k+\alpha)\Big|\Big\{\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^{\frac{2}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}E_{\varepsilon,\nu}^2\\
&+4(k+\alpha)\Big|\Big\{\big(-b+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\big)^{\frac{1}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}E_{\varepsilon,\nu}^2.
\end{aligned}
$$

As for $II''+III''+IV''$, using (1.31) $w(t,\xi)$ with $v_{tt}+\frac{i}{2}a(t,\xi)v_t$ and (1.32), similarly as §1.3 and §1.4, we can get

$$
\begin{aligned}
(1.34)\,II''+III''+IV'' =& 2\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Re\Big(\varepsilon^{\frac{k+\alpha}{4}}\langle\xi\rangle_\nu^{\frac{1}{2}}v_t,\ \varepsilon^{-\frac{k+\alpha}{4}}\langle\xi\rangle_\nu^{-\frac{1}{2}}\big(v_{tt}+\frac{i}{2}av_t\big)\Big)\\
&+\frac{1}{2}aa'e^{\rho(t)\langle\xi\rangle_\nu^\kappa}|v_t|^2+\frac{-b'}{-b+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\big(-b+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\big)|v_t|^2\\
&+a'e^{\rho\langle\xi\rangle_\nu^\kappa}\Re\Big(i\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^{\frac{1}{2}}v_t,\ \Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^{-\frac{1}{2}}\big(v_{tt}+\frac{i}{2}av_t\big)\Big)\\
\le&\ \varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu E_{\varepsilon,\nu}^2+2(k+\alpha)\Big|\Big\{\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^{\frac{2}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}E_{\varepsilon,\nu}^2\\
&+(k+\alpha)\Big|\Big\{\big(-b+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\big)^{\frac{1}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}E_{\varepsilon,\nu}^2\\
&+\frac{3}{2}(k+\alpha)\Big|\Big\{\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^{\frac{2}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}E_{\varepsilon,\nu}^2
\end{aligned}
$$

$$= \varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu E^2_{\varepsilon,\nu} + (k+\alpha)\Big|\Big\{\big(-b+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\big)^{\frac{1}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}E^2_{\varepsilon,\nu}$$
$$+\frac{7}{2}(k+\alpha)\Big|\Big\{\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^{\frac{2}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}E^2_{\varepsilon,\nu}.$$

Therefore by (1.30), (1.33) and (1.34) we have the estimate

$$\frac{d}{dt}(E^2_{\varepsilon,\nu}) \le \rho'(t)\langle\xi\rangle_\nu^\kappa E^2_{\varepsilon,\nu} + C_8\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu E^2_{\varepsilon,\nu} + C_9\Big|\Big\{\big(-b+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\big)^{\frac{1}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}E^2_{\varepsilon,\nu}$$
$$+C_{10}\Big|\Big\{\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^{\frac{2}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}E^2_{\varepsilon,\nu},$$

where $C_8 = 2$, $C_9 = 5(k+\alpha)$, $C_{10} = \frac{13}{3}(k+\alpha)$.

Thus Gronwall's inequality yields

$$E^2_{\varepsilon,\nu}(t) \le E^2_{\varepsilon,\nu}(0)\exp\Big[\int_0^t \Big\{\rho'(\tau)\langle\xi\rangle_\nu^\kappa + C_8\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu + C_9\Big|\Big\{\big(-b+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\big)^{\frac{1}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}$$
$$+C_{10}\Big|\Big\{\Big(\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^{\frac{2}{k+\alpha}}\Big\}'\Big|\varepsilon^{-1}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}\Big\}d\tau\Big] \qquad \text{for } {}^\forall t\in[0,T].$$

Hence we shall also apply Lemma 1.2 to this inequality. Noting that $s(=\kappa^{-1}) < 1+\frac{k+\alpha}{2}$ $\left(\frac{2}{2+k+\alpha}-\kappa<0\right)$, and taking $\varepsilon = \langle\xi\rangle_\nu^{-\frac{2}{2+k+\alpha}}$, we obtain

$$E^2_{\varepsilon,\nu}(t,\xi) \le E^2_{\varepsilon,\nu}(0,\xi)\exp\Big[\langle\xi\rangle_\nu^\kappa\int_0^t\Big\{\rho'(\tau)+C_8\nu^{\frac{2}{2+k+\alpha}-\kappa}$$
$$+C_9\Big|\Big\{\big(-b(\tau)+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\big)^{\frac{1}{k+\alpha}}\Big\}'\Big|\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}\nu^{\frac{2}{2+k+\alpha}-\kappa}$$
$$+C_{10}\Big|\Big\{\Big(\frac{|a(\tau)|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big)^{\frac{2}{k+\alpha}}\Big\}'\Big|\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}\nu^{\frac{2}{2+k+\alpha}-\kappa}\Big\}d\tau\Big]$$
$$= E^2_{\varepsilon,\nu}(0,\xi)\exp\Big[\langle\xi\rangle_\nu^\kappa\Big\{\rho(t)-\rho_0+\big(C_8t+\phi_2(t)\big)\nu^{\frac{2}{2+k+\alpha}-\kappa}\Big\}\Big] \qquad \text{for } {}^\forall t\in[0,T],$$

where $\phi_2(t)$ is a bounded function independent of ξ and satisfies $\phi_2(0)=0$ and

$$\phi_2(t) \ge C_9C(t)\,\|-b+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\|_{C^{k+\alpha}}^{\frac{1}{k+\alpha}}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}} + C_{10}\tilde{C}(t)\,\Big\|\frac{|a|}{2}+\varepsilon^{\frac{k+\alpha}{2}}\langle\xi\rangle_\nu\Big\|_{C^{k+\alpha}}^{\frac{2}{k+\alpha}}\langle\xi\rangle_\nu^{-\frac{2}{k+\alpha}}.$$

Similarly as §1.3 and §1.4 we determine $\rho(t) = \rho_0-\big(C_8t+\phi_2(t)\big)\nu^{\frac{2}{2+k+\alpha}-\kappa}$, and choose $\nu>0$ such that $\rho(T)\big(=\rho_0-\big(C_8T+\phi_2(T)\big)\nu^{\frac{2}{2+k+\alpha}-\kappa}\big)>0$ for any given $T>0$.

Finally we have the energy inequality

$$(1.35) \qquad E^2_{\varepsilon,\nu}(t,\xi) \le E^2_{\varepsilon,\nu}(0,\xi) \qquad \text{for } {}^\forall t\in[0,T] \text{ and } {}^\forall\xi\in\mathbf{R}^n_\xi.$$

1.6 Proof of Theorem 1.1

Putting

$$(1.36)\quad a_{(\varepsilon)}(t,\xi)=\begin{cases} a_\varepsilon(t,\xi) & \text{for } k=0,1\\ a(t,\xi) & \text{for } k=2,3,\cdots,\end{cases}\qquad b_{(\varepsilon)}(t,\xi)=\begin{cases} b_\varepsilon(t,\xi) & \text{for } k=0\\ b(t,\xi) & \text{for } k=1,2,\cdots,\end{cases}$$

we can alter the definitions of energies (1.15), (1.23) and (1.29) into the following.

$$\begin{aligned}(1.37)\ E_{\varepsilon,\nu}(t,\xi)^2 =& e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{|v_{tt}+ia_{(\varepsilon)}(t,\xi)v_t-b_{(\varepsilon)}(t,\xi)v+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v|^2\\ &+\Big|v_{tt}+\frac{i}{2}a_{(\varepsilon)}(t,\xi)v_t\Big|^2+\Big(\frac{a_{(\varepsilon)}(t,\xi)^2}{4}-b_{(\varepsilon)}(t,\xi)+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\Big)|v_t|^2\Big\}.\end{aligned}$$

Since the energy inequalities (1.22), (1.28) and (1.35) are same, it holds that

$$(1.38)\qquad E_{\varepsilon,\nu}^2(t,\xi)\le E_{\varepsilon,\nu}^2(0,\xi)\qquad \text{for } {}^\forall t\in[0,T] \text{ and } {}^\forall\xi\in\mathbf{R}_\xi^n.$$

We shall change the energy inequality (1.38) into the inequality based on the absolute values of v, v_t and v_{tt}. For this aim we must investigate the relations between the energy (1.37) and the absolute values of v, v_t and v_{tt}.

From the definition of energy (1.37) we can easily see

$$(1.39)\quad E_{\varepsilon,\nu}^2\ge e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big(\frac{a_{(\varepsilon)}(t,\xi)^2}{4}-b_{(\varepsilon)}(t,\xi)+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\Big)|v_t|^2\ge e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\langle\xi\rangle_\nu^{\frac{4}{2+k+\alpha}}|v_t|^2.$$

We can also find

$$\begin{aligned}(1.40)\qquad E_{\varepsilon,\nu}^2 &\ge e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\Big|v_{tt}+\frac{i}{2}a_{(\varepsilon)}v_t\Big|^2+\frac{a_{(\varepsilon)}^2}{4}|v_t|^2\Big\}\\ &= e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\frac{1}{2}|v_{tt}|^2+\frac{1}{2}|v_{tt}|^2+\Re\Big(v_{tt},\ ia_{(\varepsilon)}v_t\Big)+\frac{a_{(\varepsilon)}^2}{2}|v_t|^2\Big\}\\ &= e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\frac{1}{2}|v_{tt}|^2+\frac{1}{2}|v_{tt}+ia_{(\varepsilon)}v_t|^2\Big\}\ge\frac{1}{2}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}|v_{tt}|^2.\end{aligned}$$

Moreover we can see

$$\begin{aligned}(1.41)\qquad E_{\varepsilon,\nu}^2 &\ge e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\Big|\Big(v_{tt}+\frac{i}{2}a_{(\varepsilon)}v_t\Big)+\Big(\frac{i}{2}a_{(\varepsilon)}v_t-b_{(\varepsilon)}v+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v\Big)\Big|^2\\ &\quad+\Big|v_{tt}+\frac{i}{2}a_{(\varepsilon)}v_t\Big|^2+\frac{a_{(\varepsilon)}^2}{4}|v_t|^2\Big\}\\ &= e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\Big|\sqrt{2}\Big(v_{tt}+\frac{i}{2}a_{(\varepsilon)}v_t\Big)+\frac{1}{\sqrt{2}}\Big(\frac{i}{2}a_{(\varepsilon)}v_t-b_{(\varepsilon)}v+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v\Big)\Big|^2\\ &\quad+\frac{1}{2}\Big|\frac{i}{2}a_{(\varepsilon)}v_t-b_{(\varepsilon)}v+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v\Big|^2+\frac{a_{(\varepsilon)}^2}{4}|v_t|^2\Big\}\\ &\ge e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\Big\{\frac{1}{2}\Big|\frac{i}{2}a_{(\varepsilon)}v_t-b_{(\varepsilon)}v+\varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v\Big|^2+\frac{a_{(\varepsilon)}^2}{4}|v_t|^2\Big\}\end{aligned}$$

$$= e^{\rho\langle\xi\rangle_\nu^\kappa}\left\{\frac{1}{2}\left|\frac{\sqrt{3}i}{2}a_{(\varepsilon)}v_t + \frac{1}{\sqrt{3}}\left(-b_{(\varepsilon)}v + \varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v\right)\right|^2 + \frac{1}{3}\left|-b_{(\varepsilon)}v + \varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v\right|^2\right\}$$
$$\geq e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\frac{1}{3}\left(-b_{(\varepsilon)} + \varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\right)^2|v|^2$$
$$\geq \frac{1}{3}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\varepsilon^{2(k+\alpha)}\langle\xi\rangle_\nu^4|v|^2 = \frac{1}{3}e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\langle\xi\rangle_\nu^{\frac{8}{2+k+\alpha}}|v|^2.$$

While the energy $E_{\varepsilon,\nu}(0,\xi)$ can be also dominated by the absolute values of the initial data. From the definitions of the energy (1.37) we get

$$(1.42)\quad E_{\varepsilon,\nu}(0)^2 = e^{\rho_0\langle\xi\rangle_\nu^\kappa}\Big\{|v_2 + ia_{(\varepsilon)}(0)v_1 - b_{(\varepsilon)}(0)v_0 + \varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2 v_0|^2$$
$$+ \left|v_2 + \frac{i}{2}a_{(\varepsilon)}(0)v_1\right|^2 + \Big(\frac{a_{(\varepsilon)}(0)^2}{4} - b_{(\varepsilon)}(0) + \varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\Big)|v_1|^2\Big\}$$
$$= e^{\rho_0\langle\xi\rangle_\nu^\kappa}\Big\{2|v_2|^2 + \Big(\frac{3}{2}a_{(\varepsilon)}(0)^2 - b_{(\varepsilon)}(0) + \varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\Big)|v_1|^2 + 2\Re\Big(v_2,\ \frac{3i}{2}a_{(\varepsilon)}(0)v_1\Big)$$
$$+ \big(-b_{(\varepsilon)}(0) + \varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\big)^2|v_0|^2 + 2\Re\Big(v_2,\ \big(-b_{(\varepsilon)}(0) + \varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\big)v_0\Big)$$
$$+ 2\Re\big(ia_{(\varepsilon)}(0)v_1,\ (-b_{(\varepsilon)}(0) + \varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2)v_0\big)\Big\}$$
$$\leq e^{\rho_0\langle\xi\rangle_\nu^\kappa}\Big\{4|v_2|^2 + \Big(\frac{19}{4}a_{(\varepsilon)}(0)^2 - b_{(\varepsilon)}(0) + \varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\Big)|v_1|^2$$
$$+ 3\big(-b_{(\varepsilon)}(0) + \varepsilon^{k+\alpha}\langle\xi\rangle_\nu^2\big)^2|v_0|^2\Big\}$$
$$\leq Ce^{\rho_0\langle\xi\rangle_\nu^\kappa}\big(\langle\xi\rangle_\nu^4|v_0|^2 + \langle\xi\rangle_\nu^2|v_1|^2 + |v_2|^2\big).$$

Using the energy inequalities (1.38), by (1.9) and (1.39)-(1.42) we have the energy inequality based on the absolute values of v, v_t and v_{tt}

$$e^{\rho(t)\langle\xi\rangle_\nu^\kappa}\big(\langle\xi\rangle_\nu^{\frac{8}{2+k+\alpha}}|v|^2 + \langle\xi\rangle_\nu^{\frac{4}{2+k+\alpha}}|v_t|^2 + |v_{tt}|^2\big)$$
$$\leq^{\exists}\ C_{ab}e^{\rho_0\langle\xi\rangle_\nu^\kappa}\big(\langle\xi\rangle_\nu^4|v_0|^2 + \langle\xi\rangle_\nu^2|v_1|^2 + |v_2|^2\big)(<\infty),$$

and by square root of the both sides we also have

$$(1.43)\qquad e^{\frac{\rho(t)}{2}\langle\xi\rangle_\nu^\kappa}\big(\langle\xi\rangle_\nu^{\frac{4}{2+k+\alpha}}|v| + \langle\xi\rangle_\nu^{\frac{2}{2+k+\alpha}}|v_t| + |v_{tt}|\big)$$
$$\leq^{\exists}\ C_{ab}e^{\frac{\rho_0}{2}\langle\xi\rangle_\nu^\kappa}\big(\langle\xi\rangle_\nu^2|v_0| + \langle\xi\rangle_\nu|v_1| + |v_2|\big)(<\infty).$$

Putting $\mu(t) = \frac{1}{2}\rho(t)$, $\mu_0 = \frac{1}{2}\rho_0$, we can see that (1.43) implies (1.7). In vitue of Paley-Wiener theorem, $\{u(\cdot,t); t \in [0,T]\}$ is bounded in G_0^s. Thus taking into account that u is a solution of (1.1), we find $u \in C^3([0,T], G_0^s)$. This concludes the proof of Theorem 1.1.