$O(4)$ Symmetry in Feynman Amplitudes

YOICHI IWASAKI
Department of Physics, University of Tokyo, Tokyo, Japan
(Received 29 April 1968)

It is shown that any Feynman amplitude of pole diagrams at zero invariant mass does indeed possess the $O(4)$ symmetry and that there are poles which do not correspond to real particles.

I. INTRODUCTION

RECENTLY, it has been recognized that Regge trajectories occur in families with definite requirements on the spacing of members of a family and on the behavior of residue functions at zero values of the invariant mass.[1,2] These parent-daughter phenomena have been elucidated by Toller,[3] and by Freedman and Wang,[4] exploiting the $O(3,1)$ or $O(4)$ invariance possessed by the equal-mass forward-scattering amplitude. These conclusions have been derived only from the Lorentz-invariance and analytic properties of scattering amplitudes, without specific assumptions of dynamical behaviors. It seems, however, that there are some confusions about the physical interpretation of these conclusions. For example, it is not clear whether or not daughter trajectories really correspond to "real" particle trajectories.[5] Furthermore, it has been said[6] that we do not see any immediate possibility of deriving the $O(4)$ symmetry from a Lagrangian approach. The aim of this paper is to show that any Feynman amplitude of pole diagrams at zero invariant mass does indeed possess the $O(4)$ symmetry and that there are poles which do not correspond to real particles.

II. SCALAR-SCALAR SCATTERING

We begin by discussing a simple example. Massless particles with spin 2, "gravitons," are described by the following set of equations:

$$\Box\phi_{\mu\nu}=0\,, \tag{1}$$

$$\phi_{\mu\nu}=\phi_{\nu\mu}\,, \tag{2}$$

$$\phi_{\mu\mu}=0\,, \tag{3}$$

$$\partial_\mu\phi_{\mu\nu}=0\,. \tag{4}$$

In quantization of massless particles there are some complications as in the case of the photon. We must regard Eq. (4) as a supplementary condition and use the indefinite metric. The result obtained by the quantization which is necessary in this paper is the expression of the propagator of the "graviton."[7] The expression in the momentum space is given by

$$d_{\mu_1\mu_2,\nu_1\nu_2}(p)=\frac{1}{p^2-i\epsilon}\left[\tfrac{1}{2}(\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}+\delta_{\mu_1\nu_2}\delta_{\mu_2\nu_1})-\tfrac{1}{4}\delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2}\right], \tag{5}$$

which is compatible with the conditions (2) and (3) ($d_{\mu_1\mu_2\nu_1\nu_2}=d_{\mu_2\mu_1\nu_1\nu_2}$, $d_{\mu\mu\nu_1\nu_2}=0$).

Then, the Feynman amplitude of the pole diagram (Fig. 1) where all particles 1–4 are spinless is

$$g^2\frac{(2k)^4}{s}(\cos^2\theta-\tfrac{1}{4})=\frac{2g^2(2k)^4}{3}\frac{1}{s}\left[P_2(\cos\theta)+\tfrac{1}{8}P_0(\cos\theta)\right], \tag{6}$$

where k and θ are the momentum and the scattering angle in the c.m. system, respectively.[8] The essential point of (6) is that it contains the pole term proportional to $P_0(\cos\theta)$ with the ratio 1/8 to the term proportional to $P_2(\cos\theta)$, which is exactly the ratio derived from the $O(4)$ symmetry.

Next, we go to the case of massive spin-2 particles, in which we have the following equation instead of Eq. (1):

$$(\Box-\mu^2)\phi_{\mu\nu}=0\,, \tag{1'}$$

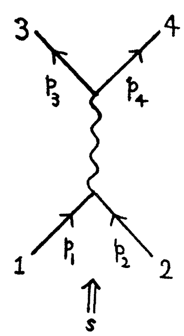

FIG. 1. Diagram of a pole amplitude.

[1] D. V. Volkov and V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **44**, 1068 (1963) [English transl.: Soviet Phys.—JETP **17**, 720 (1963)].
[2] D. Z. Freedman and J. M. Wang, Phys. Rev. **153**, 1596 (1967); D. Z. Freedman, C. E. Jones, and J. M. Wang, *ibid.* **155**, 1645 (1967).
[3] M. Toller, Nuovo Cimento **37**, 631 (1965); University of Rome Reports No. 76, 1965, and No. 84, 1966 (unpublished); M. Toller and A. Sciarrino, University of Rome Report No. 108 1966 (unpublished).
[4] D. Z. Freedman and J. M. Wang, Phys. Rev. **160**, 1560 (1967).
[5] R. Oakes, Phys. Letters **24B**, 154 (1967); L. Durand, III, Phys. Rev. **154**, 1537 (1967); Phys. Rev. Letters **18**, 58 (1967).
[6] G. Domokos, Phys. Rev. **159**, 1387 (1967).
[7] Details will be published elsewhere. Our results are different from those of S. N. Gupta, Proc. Phys. Soc. (London) **63**, 681 (1950). His starting point is different from our Eqs. (1)–(4). See also S. Weinberg, in *Brandeis Summer Institute in Theoretical Physics* (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1965), Vol. 2.
[8] We take the interaction as $g\phi_{\mu_1\mu_2}\phi\overleftrightarrow{\partial}_{\mu_1}\overleftrightarrow{\partial}_{\mu_2}\phi$. Even if we had taken another interaction, the difference would be the term which becomes zero as s becomes zero. So the following arguments remain unchanged.

and the others are the same as (2)–(4). The correct propagator is as follows[9]:

$$d_{\mu_1\mu_2,\nu_1\nu_2}(p)=\frac{1}{s-\mu^2}\Bigg\{\tfrac{1}{2}(\tilde{\delta}_{\mu_1\nu_1}\tilde{\delta}_{\mu_2\nu_2}+\tilde{\delta}_{\mu_1\nu_2}\tilde{\delta}_{\mu_2\nu_1})-\tfrac{1}{3}\tilde{\delta}_{\mu_1\mu_2}\tilde{\delta}_{\nu_1\nu_2}+\frac{(s-\mu^2)}{6\mu^4}\big[\delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2}(\tfrac{1}{4}s-\tfrac{1}{2}\mu^2)+(\delta_{\mu_1\mu_2}p_{\nu_1}p_{\nu_2}+p_{\mu_1}p_{\mu_2}\delta_{\nu_1\nu_2})\big]\Bigg\}, \quad (7)$$

where $s=-p^2$ and $\tilde{\delta}_{\mu\nu}=\delta_{\mu\nu}+p_\mu p_\nu/\mu^2$. The term proportional to $(s-\mu^2)$ which is zero on the mass shell is important to the discussion of the detailed properties of amplitudes[10] and Eq. (7) is compatible with Eqs. (2) and (3) ($d_{\mu_1\mu_2\nu_1\nu_2}=d_{\mu_2\mu_1\nu_1\nu_2}$, $d_{\mu\mu\nu_1\nu_2}=0$). Usually this term is dropped and the propagator is incompatible with Eq. (3) at $s\neq\mu^2$. The Feynman amplitude of the pole diagram (Fig. 1) is

$$\frac{g^2(2k)^4}{s-\mu^2}\left[\tfrac{2}{3}P_2(\cos\theta)+\frac{s-\mu^2}{6\mu^4}(\tfrac{1}{4}s-\tfrac{1}{2}\mu^2)P_0(\cos\theta)\right]. \quad (8)$$

The ratio of the term proportional to P_2 and P_0 at $s=0$ is again equal to $\frac{1}{8}$. The essential difference between (6) and (8) is the fact that in Eq. (6) two terms are both poles, while on the other hand, in Eq. (8) only the term proportional to P_2 is the pole term.

These facts show that at $s=0$, a Feynman amplitude corresponding to a pole diagram does have the $O(4)$ symmetry irrespective of the zero or nonzero value of the particle mass. For the case of a zero-mass spin-2 particle, there appears a pole at $s=0$ for the $l=0$ partial-wave amplitude. However, this does not correspond to a real scalar particle of zero mass: When the particle line is external in Feynman amplitudes, the wave function for the particle satisfies Eqs. (1)–(4), and behaves only as a massless particle of spin 2. The spin-0 component appears only in intermediate states as a pole proportional to $P_0(\cos\theta)$. So we call this pole a "shadow pole."

The above argument can be easily extended to the case of arbitrary spin. The propagator of massless particles with arbitrary integer spin can be calculated as follows:

$$d_{\{\mu\},\{\nu\}}=\frac{1}{s}\sum_{r=0}^{[n/2]}(-)^r\frac{(n-r)!r!}{(n!)^2}\sum_C[2r]\{n-2r\}, \quad (9)$$

where n is the spin of the particle, $[2r]$ is r pairs of $\delta_{\mu_i\mu_j}$ and r pairs of $\delta_{\nu_i\nu_j}$, $\{n-2r\}$ is $(n-2r)$ pairs of $\delta_{\mu_i\nu_j}$, and C means all possible combinations. Then the Feynman amplitude of Fig. 1 is

$$f(k,\cos\theta)=\frac{g^2(2k)^{2n}}{2^n}\frac{1}{s}C_n^1(\cos\theta), \quad (10)$$

where C_n^1 is the Gegenbauer polynomial. Equation (10) can be expanded in terms of $P_l(\cos\theta)$ as follows:

$$f(k,\cos\theta)=\frac{2^n(n!)^2(2k)^{2n}g^2}{(2n)!}\times\frac{1}{s}\left(P_n+\frac{2n-3}{4n}P_{n-2}+\cdots\right), \quad (11)$$

which is exactly the result derived from the $O(4)$ symmetry. Note that $C_n^1(z)$ is the spherical function of a four-dimensional space, that $P_n(z)$ $[=C_n^{1/2}(z)]$ is the spherical function of a three-dimensional space, and that the single-pole contribution can be expressed simply in terms of the C_n^1's.

We can also discuss the case of the massive intermediate particle. The part of the propagator composed of only the Kronecker δ contributes even at $s=0$, while the other part composed of p_μ becomes zero at $s=0$. The former part of the propagator of the massive particle is the same as Eq. (9), because this part is determined by the symmetric and traceless properties of the propagator. The propagator adopted usually does not satisfy these properties. Terms such as the second term of Eq. (7) are essential for these properties. Using the correct propagator, we again obtain the ratio of the partial waves at $s=0$ that is derived from the $O(4)$ symmetry.

III. NUCLEON-ANTINUCLEON SCATTERING

We proceed to discuss the case of N-$\bar{N}$ scattering which is extensively examined in Refs. 1, 3, and 4. The vertex factors for the coupling of the nucleon-antinucleon system to the intermediate particle are linear combinations of $P_{\mu_1}\cdots P_{\mu_n}$ and $\gamma_{\mu_1}P_{\mu_2}\cdots P_{\mu_n}$ for the states $C=P=(-)^n$, $\gamma_5\gamma_{\mu_1}P_{\mu_2}\cdots P_{\mu_n}$ for the states $P=C=-(-)^n$, and $\gamma_5 P_{\mu_1}\cdots P_{\mu_n}$ for the states $P=-C=-(-)^n$. In these expressions, $P_\mu=(p_2-p_1)_\mu$ at the 1,2 vertex, and $P_\mu=(p_3-p_4)_\mu$ at the 3,4 vertex. We can easily construct the pole Feynman amplitudes as in the case of the scalar-scalar scattering.

Concerning the coupling type of $P_{\mu_1}\cdots P_{\mu_n}$, the calculation is exactly the same as above, and gives

$$f_{11}(k,z)=c(1/s)C_n^1(z), \quad (12)$$

[9] L. M. Nath, Nucl. Phys. **68**, 660 (1965); S. C. Bhargava and H. Watanabe, *ibid.* **87**, 273 (1966). Our results correspond to the limit ($\alpha\to\infty$, $\delta\to\infty$) of their results. This limiting process is necessary for the consistency of the theory. Details will be published elsewhere. For the same reason, the propagator of particles with spin $\frac{3}{2}$ becomes the following:

$$d_{\mu\nu}=\frac{1}{\Box-m^2}\Bigg\{-(\gamma\partial-m)\left[\delta_{\mu\nu}-\tfrac{1}{3}\gamma_\mu\gamma_\nu+\frac{1}{3m}(\gamma_\mu\partial_\nu-\gamma_\nu\partial_\mu)-\frac{2}{3m^2}\partial_\mu\partial_\nu\right]-\frac{\Box-m^2}{3m^2}\big[\tfrac{1}{2}\partial_\mu\gamma_\nu+\tfrac{1}{2}\gamma_\mu\partial_\nu+\tfrac{1}{4}m\gamma_\mu\gamma_\nu-\tfrac{1}{8}\gamma_\mu(\gamma\partial)\gamma_\nu\big]\Bigg\},$$

which is compatible with the condition $\gamma_\mu\psi_\mu=0$ ($\gamma_\mu d_{\mu\nu}=0$).

[10] Y. Takahashi and H. Umezawa, Progr. Theoret. Phys. (Kyoto) **9**, 14 (1953); **9**, 50 (1953).

where n is the magnitude of the spin of the intermediate particle. This is exactly of type I as it is called by Toller, Freedman, and Wang. For the coupling type of $\gamma_{\mu_1}P_{\mu_2}\cdots P_{\mu_n}$ and the crossed term, we obtain also Eq. (12), i.e., the type I, ignoring the term proportional to s, which becomes automatically zero at $s=0$. After a similar calculation, we obtain the following results for the coupling $\gamma_5\gamma_{\mu_1}P_{\mu_2}\cdots P_{\mu_n}$:

$$f_1{}^{j=n}/f_0{}^{j=n-1}\xrightarrow[s\to 0]{} n(n+1)/(2n+1)\,, \tag{13}$$

which is the ratio required for type II. The situation is quite different for the coupling $\gamma_5 P_{\mu_1}\cdots P_{\mu_n}$. The amplitudes become zero as s becomes zero. This phenomenon is called "evasion."

Although until now we have been describing the mesons by symmetric tensors [$(\frac{1}{2}j,\frac{1}{2}j)$ representation of the homogeneous Lorentz group], we can also describe them by other quantities. Spin-1 mesons, for example, can also be expressed by an antisymmetric tensor of the second rank [(1,0)+(0,1)],

$$(\Box-\mu^2)\epsilon_{[\mu,\nu]}=0\,, \tag{14}$$

$$\epsilon_{[\mu,\nu]}=-\epsilon_{[\nu,\mu]}\,, \tag{15}$$

$$\partial_\mu\epsilon_{[\mu,\nu]}=0\,. \tag{16}$$

We obtain the following amplitudes as the contribution from the pole diagram[11]:

$$f_0{}^{j=1}=\tfrac{1}{3}g^2\frac{2s-8m^2}{s}\,, \tag{17}$$

$$f_{22}{}^{j=1}=\tfrac{2}{3}g^2\frac{-8m^2}{s}\,, \tag{18}$$

when the mass of the intermediate particle is zero. The ratio f_{22}/f_0 at $s=0$ is 2, which was obtained in the case "type III" of the $O(4)$ symmetry. The parities of f_{22} and f_0 are opposite to each other and this fact is called parity doubling. We want to stress the fact that both pole terms of f_0 and f_{22} are the contributions of the pole diagram of only one particle. We can understand this situation from the equaltity $\bar\psi\gamma_5\sigma_{\mu\nu}\psi\bar\psi\gamma_5\sigma_{\mu\nu}\psi=\bar\psi\sigma_{\mu\nu}\psi\bar\psi\sigma_{\mu\nu}\psi$. Investigating the cases of all kinds of interactions, we can conclude that the concept of parity of this massless particle makes no sense even in a theory with parity conservation. We can also obtain the ratio $f_{22}/f_0=2$ at $s=0$ when the intermediate particle is massive.

Higher-spin ($j\geq 1$) particles can be described by this type [$(\frac{1}{2}(j+1),\frac{1}{2}(j-1))+(\frac{1}{2}(j-1),\frac{1}{2}(j+1))$] and the ratio of each partial wave at $s=0$ is exactly the value predicted by the $O(4)$ symmetry. Spin-0 particles cannot be represented by this type, and spin 0–spin 0 cannot couple to the particle described by this type. In the Regge-pole theory this fact still remains. So it seems difficult to classify the π meson as type III.

We summarize the above statements as follows:

$$P_{\mu_1}P_{\mu_2}\cdots P_{\mu_n}\,,\quad \gamma_{\mu_1}P_{\mu_2}\cdots P_{\mu_n}\quad \text{type I}\,, \tag{19a}$$

$$\gamma_5\gamma_{\mu_1}P_{\mu_2}\cdots P_{\mu_n}\quad \text{type II}\,, \tag{19b}$$

$$\gamma_5 P_{\mu_1}\cdots P_{\mu_n}\quad \text{evasion}\,, \tag{19c}$$

$$\sigma_{\mu_1\mu_2}P_{\mu_3}\cdots P_{\mu_{n+1}}\quad \text{type III}\,, \tag{19d}$$

where the left-hand side represents the interaction vertex.

IV. GENERAL CASES

We can examine more general cases with particles of arbitrary spin as internal and external particles. In general, a free particle is described by a unitary representation (s,m) of the inhomogeneous Lorentz group,[12] and for a definite s, m there is one unique representation (apart from the equivalences). To express a particle of spin s in a covariant way, however, we use a field which transforms according to a representation (j,k) of the homogeneous Lorentz group, where j and k can be any pair of integers or half-integers, which satisfy $j-k\leq s\leq j+k$ [if s is an integer (or a half-integer), $j+k$ must be an integer (or a half-integer)]. For a definite spin s, different expressions are equivalent to each other for a "free" particle. However, there remains the question of whether or not they are equivalent in the presence of interaction. We shall show that (j,k) corresponds to (n,M) of Toller, Freedman, and Wang, where $n=j+k$ and $M=j-k$. This means that fields of different (j,k) are not equivalent to each other when they appear as virtual intermediate particles. To prove this correspondence, it is more transparent to use the spinor representation[13] than the tensor representation. A field (j,k) is expressed by $\chi_{\dot\alpha_1\cdots\dot\alpha_j\beta_1\cdots\beta_k}$, and an interaction by $\chi_{\dot\alpha\cdots\beta\cdots}\psi_{\dot\alpha\cdots\beta\cdots}\phi_{\dot\alpha\cdots\beta\cdots}p_{\dot\alpha}{}^{b}\cdots p_{\dot\alpha}{}^{b}$, which is Lorentz-invariant. This means that $\psi\cdots\phi\cdots p\cdots$ transforms as (j,k). Then the problem is how to construct a state (j,k) from two particles at $(p_1+p_2)_\mu=0$. This is a purely group-theoretical problem and was treated by Freedman and Wang in Ref. 4, Eq. (25). Thus we conclude that the amplitude is the same as that of Ref. 4, unless it becomes zero.

---

[11] We have taken $g\bar\psi\sigma_{\mu\nu}\psi\epsilon_{[\mu,\nu]}$ as the coupling. The propagator is calculated to be

$$d_{[\mu_1\mu_2],[\nu_1\nu_2]}(p)$$
$$=\frac{1}{s-\mu^2}\Big[\tfrac{1}{2}(\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}-\delta_{\mu_1\nu_2}\delta_{\mu_2\nu_1})+\frac{1}{2\mu^2}(\delta_{\mu_1\nu_1}p_{\mu_2}p_{\nu_2}+p_{\mu_1}p_{\nu_1}\delta_{\mu_2\nu_2}-\delta_{\mu_1\nu_2}p_{\mu_2}p_{\nu_1}-p_{\mu_1}p_{\nu_2}\delta_{\mu_2\nu_1})\Big],\ \mu^2\neq 0$$

and

$$d_{[\mu_1\mu_2],[\nu_1\nu_2]}(p)=(1/s)[\tfrac{1}{2}(\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2}-\delta_{\mu_1\nu_2}\delta_{\mu_2\nu_1})],\ \mu^2=0.$$

[12] E. P. Wigner, Ann. Math. 40, 149 (1939).

[13] See, for example, H. Umezawa, *Quantum Field Theory* (North-Holland Publishing Co., Amsterdam, 1956).

V. CONCLUSIONS

We have shown that the Feynman amplitude of a pole diagram mediated by a particle (j,k) at $s=0$ is the same as the result of the $O(4)$ symmetry, and that, in addition, (j,k) corresponds in a one-to-one manner to (n,M) introduced by Toller, Freedman, and Wang. With these considerations we have concluded that it is difficult to assign the π meson to the class III.

Another point we want to stress is the fact that the pole terms appear in more than one partial wave, even if a single particle is exchanged. This fact indicates that there are poles in the S matrix which do not correspond to a "real" particle. We call this pole a "shadow pole." The fact that there is a pole which does not correspond to a "real" particle forces one to modify the usual assumption that a pole in the S matrix corresponds to a real particle.

The relations between shadow poles and abnormal solutions of the Bethe-Salpeter equation, and the phenomenological effects of shadow poles will be discussed in subsequent papers.

ACKNOWLEDGMENTS

The author wishes to thank Professor H. Miyazawa for valuable discussions and reading through the manuscript, and Professor K. Nishizima and Professor S. Kamefuchi for discussions. He also wishes to thank Professor Y. Takahashi for his kind correspondence.

PHYSICAL REVIEW VOLUME 173, NUMBER 5 25 SEPTEMBER 1968

Rescattering Model Applied to Y^* Production

C. P. SINGH AND B. K. AGARWAL
Physics Department, Allahabad University, Allahabad, India
(Received 29 March, 1968; revised manuscript received 14 May 1968)

The production angular distributions for $Y_1^*(1385)$, $Y_0^*(1520)$, and $Y_1^*(1660)$ in K^--p scattering are characterized by forward and backward peakings. As the single-particle-exchange model is unable to account for these features, we have attempted to explain them by considering rescattering square diagrams. We find that the use of a coincident-pole method leads to a simple prescription for evaluating the production angular distributions. Our results show agreement with the observed data when spin-parity assignments are $\frac{3}{2}^+$ for $Y_1^*(1385)$ and $\frac{3}{2}^-$ for $Y^*(1520, 1660)$.

1. INTRODUCTION

IN K^--p scattering, the following quasi-two-body final states have been observed[1-3]:

$$K^-+p \to Y_1^{*+}(1385)+\pi^-, \tag{1}$$

$$K^-+p \to Y_1^{*-}(1385)+\pi^+, \tag{2}$$

$$K^-+p \to Y_0^{*0}(1520)+\pi^0, \tag{3}$$

$$K^-+p \to Y_1^{*0}(1660)+\pi^0. \tag{4}$$

In all these processes, a characteristic feature of the center-of-mass production angular distributions for the various Y^* is that there is an approximate symmetry at about 90° due to the presence of both forward and backward peakings. Such a characteristic defies explanation in terms of either the one-meson-exchange model or the one-baryon-exchange model. For reaction (1) only $\bar{K}^{*0}$ can be exchanged, for (2) only the nucleon can be exchanged, and for (3) and (4) both can be exchanged. Therefore, for (1) and (2) we cannot hope to get all the observed features from a one-particle-exchange model.[4] For reactions (3) and (4), one may combine the two single-particle-exchange diagrams and use *ad hoc*, drastic, form factors to obtain the observed structure. We have shown[5] that in such cases the rescattering square diagrams can offer a natural explanation. The purpose of this paper is to consider such diagrams for reactions (1)–(4), to explain the structure of the production angular distributions and thereby to fix the spin-parity assignments.

2. METHOD OF CALCULATION

The rescattering diagram for the general process $A+B \to C+D$ is shown in Fig. 1. The various momenta have been labeled in the diagram. In Table I, we summarize the intermediate states possible for the reactions (1)–(4). The invariant amplitude for the diagram shown

[1] Birmingham-Glasgow-London (I.C.)-Oxford-Rutherford Collaboration, Phys. Rev. **152**, 1148 (1966).
[2] W. A. Cooper, H. Filthuth, A. Fridman, E. Malamud, H. Schneider, E. S. Gelsema, J. C. Kluyver, and A. G. Tenner, in *Proceedings oj the Sienna International Conference on Elementary Particles and High-Energy Physics, 1963*, edited by G. Bernardini and G. P. Puppi (Società Italiana di Fisica, Bologna, 1963), p. 160.
[3] R. P. Ely, S. Y. Fung, G. Gidal, Y. L. Pan, W. M. Powell, and M. S. White, Phys. Rev. Letters **7**, 461 (1961).
[4] Y. M. Gupta and B. K. Agarwal, Nuovo Cimento **40**, 434 (1966).
[5] C. P. Singh and B. K. Agarwal, Nuovo Cimento **54A**, 497 (1968).