Prest, Mike; Slávik, Alexander
Purity in categories of sheaves. (English) Zbl 07303578
Math. Z. 297, No. 1-2, 429-451 (2021).

Purity plays a key role in the model theory of modules, finding many uses in the representation theory of finite-dimensional algebras. The theory of purity is based on that of a pure-exact sequence, extending to many additive categories, including those of sheaves and those of quasicoherent sheaves. It should be stressed that this *categorical* purity differes, outside the affine case, from the *geometric* purity. The general relationship has been investigated in [*E. Enochs* et al., Proc. Edinb. Math. Soc., II. Ser. 59, No. 3, 623–640 (2016; Zbl 1370.18011)]. Looking more closely, this paper considers categorical and geometric purity for sheaves over a scheme abiding by some mild conditions, both for the category of all sheaves and for the category of quasicoherent sheaves.

A synopsis of the paper goes as follows. §1 explores the relations between the purities in $\mathcal{O}_X-\mathrm{Mod}$ and $\mathrm{QCoh}\,(X)$. §2 looks deeper into the purity-related notions in the category, investigating which of them are preserved or reflected by the three functors associated to an open subset, namely, the restriction, the extension by zero and the direct image. The main result therein is that the geometric pure-injective in $\mathcal{O}_X-\mathrm{Mod}$ are the skyscraper sheaves with an indecomposable module of sections. §3 presents an example of the Zuegler spectrum of the category $\mathcal{O}_X-\mathrm{Mod}$ over a local affine 1-dimensional scheme X.

§4 turns to quasicoherent sheaves, restricting to the case of quasicompact quasiseparated schemes. It is shown that such schemes are affine iff the two purities coincide in the category $\mathrm{QCoh}\,(X)$. The authors proceed by describing the geometric part of the Ziegler spectrum of $\mathrm{QCoh}\,(X)$, showing that this is always glued from afine pieces and forms a quasicompact closed subset of the spectrum. A definable subcategory $\mathcal{D}_X \subseteq \mathrm{QCoh}\,(X)$ is assigned to this closed set, such that its objects enjoy the property that every geometrically pure-exact sequence starting in them is categorically pure.

§5 is devoted to the computation of the Ziegler spectrum of the category of quasicoherent sheaves over a projective line. Both the points and the topology are described, noting that, unlike the affine case, the Ziegler spectrum is not quasicompact. The subcategory $\mathcal{D}_X$ allows of a more explicit description in this case.

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MSC:

14A15 Schemes and morphisms
18F20 Presheaves and sheaves, stacks, descent conditions (category-theoretic aspects)
03C60 Model-theoretic algebra
16G20 Representations of quivers and partially ordered sets
15A75 Exterior algebra, Grassmann algebras

Keywords:

scheme; sheaf; pure-exact sequence; Ziegler spectrum

Full Text: DOI

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