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Towards a categorical boson-fermion correspondence. (English) Zbl 07184840

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The boson-fermion correspondence establishes an isomorphism between a bosonic Fock space and a fermionic Fock space, a Heisenberg algebra acting on the bosonic Fock space $V_B = \mathbb{Z}[x_1, x_2, \dots]$ (a ring of polynomials of infinite variables), and a Clifford algebra acts on the fermionic Fock space V_F (a free abelian group with a basis of semi-infinite monomials). The correspondence also provides maps between the Heisenberg and Clifford algebras via vertex operators [*I. B. Frenkel*, J. Funct. Anal. 44, 259–327 (1981; Zbl 0479.17003)].

The author has already constructed a DG categorification of a Clifford algebra [*Y. Tian*, Int. Math. Res. Not. 2015, No. 21, 10872–10928 (2015; Zbl 1344.18010)]. This paper aims to give an algebraic interpretation of the geometric structure underlying the Clifford categorification. *M. Khovanov* [Fundam. Math. 225, 169–210 (2014; Zbl 1304.18019)] constructed a $\mathbf{k}$-linear additive categorification of the Heisenberg algebra, where $\mathbf{k}$ is a field of characteristic zero. The Heisenberg category acts on the category of $\bigoplus_{n=0}^{\infty} \mathbf{k}[S(n)]$-modules, where $S(n)$ is the n-th symmetric group. This paper provides a modification of of Khovanov's Heisenberg category, showing that it is the Heisenberg counterpart of the Clifford category [*Y. Tian*, Int. Math. Res. Not. 2015, No. 21, 10872–10928 (2015; Zbl 1344.18010)] under a categorical boson-fermion correspondence.

On the Heisenberg facet, a $\mathbf{k}$-algebra B containing $\bigoplus_{n=0}^{\infty} \mathbf{k}[S(n)]$ as a subalgebra is constructed with generators of B not in $\bigoplus_{n=0}^{\infty} \mathbf{k}[S(n)]$ being closely related to contact structures on $(\mathbb{R} \times [0,1]) \times [0,1]$. The homotopy category $\mathcal{B} = \mathsf{Kom}(B)$ of finite-dimensional projective B-modules admits a monoidal structure given by the derived tensor product over B, there being two distinguished bimodules P and Q which correspond to the induction and restriction functors of B. The Heisenberg category $\mathcal{DH}$ is defined as a full triangulated monoidal subcategory of the derived category $D(B^e)$ which is generated by B, P and Q.

On the Clifford facet, the construction in the author's previous paper is generalized from $\mathbb{F}_2$ to $\mathbf{k}$, a DG $\mathbf{k}$-algebra $R = \bigoplus_{k \in \mathbb{Z}} R_k$ being defined where all R_k's are isomorphic to each other. A homotopy category of certain DG R-modules categorifies the fermionic Fock space, there being a family of distinguished DG R-bimodules $T(i)$ for $i \in \mathbb{Z}$ which correspond to certain contact geometric objects. The Clifford category $\mathcal{CL}$ is defined as a full triangulated monoidal subcategory of the derived category $D(R^e)$ which is generated by R and $T(i)$'s.

The main results of the paper goes as follows.

- The DG algebra R_0 is quasi-isomorphic to its cohomology algebra $H(R_0)$ with the trivial differential, which is derived Morita equivalent to a DG algebra $\widetilde{H}(R_0)$ with the trivial differential and concentrated in degree zero. It is shown that it is isomorphic to a quiver algebra F, and that the algebras B and F are Morita equivalent. Certain categories of B-modules and R_0-modules are equivalent, which categorifies the isomorphism of the Fock spaces (Theorem 5.1).
- There are some B-bimodule homomorphisms and extensions between B, P and Q which do not exist in Khovanov's Heisenberg category. These extra morphisms enable one to construct an infinite chain of adjoint pairs in $\mathcal{DH}$ containing the bimodules P and Q (Theorem 3.28).
- The bimodules $T(i)$ for $i \in \mathbb{Z}$ form a chain of adjoint pairs in the Clifford category, their classes $t_i = [T(i)]$ in the Grothendieck group generating a Clifford algebra Cl with the relation

  $$t_i t_j + t_j t_i = \delta_{|i-j|,1} 1$$

  Using a variation of vertex operator construction, one can express the Heisenberg generators p, q abiding by

  $$qp - pq = 1$$

in terms of the Clifford generators as

$$g(q) = \sum_{i\le 0} t_{2i}t_{2i-1} - \sum_{i>0} t_{2i-1}t_{2i}$$
$$g(p) = \sum_{i\le 0} t_{2i+1}t_{2i} - \sum_{i>0} t_{2i+1}t_{2i}$$

One constructs two objects $\overline{Q}, \overline{P}$ in $D(R_0^e)$ lifting the expressions $g(q), g(p)$. The chain

$$R_0 \longleftrightarrow H(R_0) \longleftrightarrow \widetilde{H}(R_0) \cong F \longleftrightarrow B$$

induces an equivalence

$$\mathcal{G} : D(B^e) \to D(R_0^e)$$

of categories. It is shown that $\mathcal{G}(Q)$ and $\mathcal{G}(P)$ are isomorphic to $\overline{Q}$ and $\overline{P}$, respectively (Theorem 5.3).

- One can consider two generating series

$$\bar{t}(z) = \sum_{i\in\mathbb{Z}} t_{2i+1}z^{i}$$
$$t(z) = \sum_{i\in\mathbb{Z}} t_{2i}z^{-i}$$

associated to Cl. The expresions $\bar{t}(z)\mid_{z=-1}$ and $t(z)\mid_{z=-1}$ define two linear operators of the Fock space, which are categorified to certain endofunctors of the Fock space categorification $\mathcal{B}$ (Theorem 6.5 and 6.9).

This review closes with comments on related works.

- Frenkel, Penkov and Serganova [Zbl 1355.17032] gave a categorification of the boson-fermion correspondece via the representation theory of $\mathfrak{sl}(\infty)$.
- Based upon the previous work of *S. Cautis* and *A. Licata* [Duke Math. J. 161, No. 13, 2469–2547 (2012; Zbl 1263.14020); "Vertex operators and 2-representations of quantum affine algebras", Preprint, arXiv:1112.6189], *S. Cautis* and *J. Sussan* [Commun. Math. Phys. 336, No. 2, 649–669 (2015; Zbl 1327.17009)] constructed another categorical version of the correspondence whose Heisenberg facet is Khovanov's categoricfication.
- The algebra B has already appeared in work on the stability of representation of symmetric groups, e.g., in [*T. Church* et al., Duke Math. J. 164, No. 9, 1833–1910 (2015; Zbl 1339.55004)].

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