**Liu, Bo**; **Ma, Xiaonan**

**Differential $K$-theory and localization formula for $\eta$-invariants.** (English) Zbl 07269003

Invent. Math. 222, No. 2, 545-613 (2020).

The celebrated *Atiyah-Singer index theorem* [*M. F. Atiyah* and *I. M. Singer*, Bull. Am. Math. Soc. 69, 422–433 (1963; Zbl 0118.31203)] claims that, for an elliptic differential operator on a compact manifold, the analytical index is equal to the topological index, allowing us to view the index as a primitive spectral invariant of an elliptic operator. On the other hand, global spectral invariants such as the $\eta$-*invariant* of *M. F. Atiyah* et al. [Math. Proc. Camb. Philos. Soc. 77, 43–69 (1975; Zbl 0297.58008); Math. Proc. Camb. Philos. Soc. 78, 405–432 (1975; Zbl 0314.58016); Math. Proc. Camb. Philos. Soc. 79, 71–99 (1976; Zbl 0325.58015)] and the *analytical torsion* of *D. B. Ray* and *I. M. Singer* [Ann. Math. (2) 98, 154–177 (1973; Zbl 0267.32014); Proc. Sympos. Pure Math. 23, 167–181 (1973; Zbl 0273.58014)] are to be seen as the secondary spectral invariants of an elliptic operator. A *localization formul*a, computing the equivariant index via the contribution of the *fixed point set* of the group action, was established in [*M. F. Atiyah* and *G. B. Segal*, Ann. Math. (2) 87, 531–545 (1968; Zbl 0164.24201)] in use of *topological $K$-theory*. Therefore it occurs naturally in mind whether the localization property perseveres for these secondary spectral invariants.

This paper establishes a localization formula in *differential $K$-theory*, which is similar to Kõhler and Roessler's Theorem 4.4 in [*K. Köhler* and *D. Roessler*, Invent. Math. 145, No. 2, 333–396 (2001; Zbl 0999.14002)] from a genuinely formal viewpoint. For $S^1$-actions, one gets a pointwise identification between the equivariant $\eta$-invariant and the fixed point set contribution to the $\eta$-invariant modulo the values at this element of rational functions with integral coefficients, the definition of the fixed point set contribution being actually an important part of the localization formula. By combining this identification with the authors' recent extension [*B. Liu* and *X. Ma*, "Comparison of two equivariant $\eta$-forms", Preprint, `arXiv:1808.04044`] of Goette's comparison formula for two kinds of equivariant $\eta$-invariants [*S. Goette*, J. Reine Angew. Math. 526, 181–236 (2000; Zbl 0974.58021)], the authors have arrived at the main result of this paper claiming that the difference of the equivariant $\eta$-invariant and its fixed point set contribution, as a function on the complement of a finite subset of the circle $S^1$, is the restriction of a rational function on $S^1$ with integral coefficients, which seems the first geometric application of differential $K$-theory.

Kõhler and Roessler's Theorem 4.4 in [*K. Köhler* and *D. Roessler*, Invent. Math. 145, No. 2, 333–396 (2001; Zbl 0999.14002)], which is a kind of the Lefschetz type fixed point formula in the equivariant *arithmetic $K$-theory*, was established in their proof of the equivariant arithmetic *Riemann-Roch theorem*, relating the equivariant holomorphic torsion with the contribution of the fixed point set corresponding to the $n$-th roots of unity.

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**MSC:**

58J28 Eta-invariants, Chern-Simons invariants
58J20 Index theory and related fixed-point theorems on manifolds
53C20 Global Riemannian geometry, including pinching

**Keywords:**

eta-invariants; differential K-theory

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