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String-net models for nonspherical pivotal fusion categories. (English) Zbl 07240986

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String-net models were introduced by *M. A. Levin* and *X.-G. Wen* ["String-net condensation: a physical mechanism for topological phases", Phys. Rev. B 71, No. 4, Article ID 045110, 21 p. (2005; doi:10.1103/PhysRevB.71.045110)] so as to describe states in what is called doubled topological phases. String-net spaces were shown to be equivalent to state spaces of *Turaev-Viro topological quantum field theories* (TQFTs) in [*Z. Kádár* et al., Adv. Math. Phys. 2010, Article ID 671039, 18 p. (2010; Zbl 1201.81099); *R. Koenig* et al., Ann. Phys. 325, No. 12, 2707–2749 (2010; Zbl 1206.81033); *A. Kirillov jun.*, "String-net model of Turaev-Viro invariants", Preprint, arXiv:1106.6033], where their construction began with a *pivotal fusion category* $\mathcal{C}$ which is *spherical*. This paper, consisting of seven sections together with an appendix, explores some of the effects of dropping the sphericality condtion.

Given a pivotal fusion category $\mathcal{C}$ over the complex numbers $\mathbb{C}$, whether it is spherical or not, the string-net construction assigns a $\mathbb{C}$-vector space

$$\mathrm{SN}\left(\Sigma,\mathcal{C}\right)$$

to an oriented surface Σ, possibly with boundary, which can contain marked points labeled by objects of $\mathcal{C}$. For spherical $\mathcal{C}$ the string-net for a given marked surface is canonically isomorphic to the state space of the Turaev-Viro TQFT for $\mathcal{C}$, which is in turn isomorphic to the state space of the *Reshetikhin-Turaev theory* for the Drinfeld dual $\mathcal{Z}\left(\mathcal{C}\right)$ of $\mathcal{C}$ [*A. Kirillov jun.* and *B. Balsam*, "Turaev-Viro invariants as an extended TQFT", Preprint, arXiv:1004.1533; *V. Turaev* and *A. Virelizier*, "On two approaches to 3-dimensional TQFTs", Preprint, arXiv:1006.3501; *B. Balsam*, "Turaev-Viro invariants as an extended TQFT III", Preprint, arXiv:1012.0560]. Then we have (Proposition 4.4)

For a pivotal fusion category $\mathcal{C}$, we have

$$\dim\,\mathrm{SN}\left(S^2,\mathcal{C}\right)=\left\{\begin{array}{c}1;\ \mathcal{C}\text{ spherical}\\ 0;\ \mathcal{C}\text{ not spherical}\end{array}\right\}$$

On the other hand, the string-net space on the torus is the same whether $\mathcal{C}$ is spherical or not (Proposition 4.8).

For a pivotal fusion category $\mathcal{C}$, the dimension of $\mathrm{SN}\left(T^2,\mathcal{C}\right)$ is given by the number of simple objects in $\mathcal{Z}\left(\mathcal{C}\right)$.

If $\mathcal{C}$ is simply the category $\mathcal{C}_r$ of $\mathbb{Z}_r$-graded $\mathbb{C}$-vector spaces for some $r>0$ and Σ_g is a surface of genus g without boundary, then we have (Proposition 5.2)

$$\dim\,\mathrm{SN}\left(\Sigma_g,\mathcal{C}_r\right)=\left\{\begin{array}{c}r^{2g};\ 2-2g\text{ divisible by }r\\ 0;\ \text{otherwise}\end{array}\right\}$$

Denoting the set of isomorphism classes of *r-spin structures* on Σ_g by $\mathcal{R}^r\left(\Sigma_g\right)$, we have

$$\dim\,\mathrm{SN}\left(\Sigma_g,\mathcal{C}_r\right)=\left|\mathcal{R}^r\left(\Sigma_g\right)\right|$$

for all $g\geq 0$. There is a natural action of the mapping class group $\mathrm{MCG}\left(\Sigma_g\right)$ of Σ_g by push-forward on r-spin structures and on graphs on Σ_g, it turning out that these actions agree (Theorem 6.1).

The $\mathrm{MCG}\left(\Sigma_g\right)$-representations $\mathrm{span}_{\mathbb{C}}\left(\mathcal{R}^r\left(\Sigma_g\right)\right)$ and $\mathrm{SN}\left(\Sigma_g,\mathcal{C}_r\right)$ are isomorphic.

Given a modular fusion category $\mathcal{M}$, which is taken as $\mathcal{C}$, the possible pivotal structures on $\mathcal{M}$ are a torsor over the group of natural monoidal isomorphisms of the identity functor on $\mathcal{M}$, which is in turn isomorphic to the group of isoclasses of invertible simple objects in $\mathcal{M}$ [*V. Drinfeld* et al., Sel. Math., New Ser. 16, No. 1, 1–119 (2010; Zbl 1201.18005)]. Let $J\in\mathcal{M}$ be an invertible simple object and equip $\mathcal{C}$ with the pivotal structure induced by J. It turns out that $\mathcal{C}$ is spherical iff the order of J is 1 or 2. Since

$\mathcal{M}$ is modular, we have

$$\mathcal{Z}(\mathcal{C}) \cong \mathcal{C} \boxtimes \mathcal{C}^{\mathrm{rev}}$$

so that we can label simple objects in $\mathcal{Z}(\mathcal{C})$ by pairs (U, V) with $U, V \in \mathcal{C}$ simple. Denoting by $S^2(U, V)$ a sphere with one marked point labeled by $(U, V) \in \mathcal{Z}(\mathcal{C})$, we have (Proposition 7.1)

$$\dim \mathrm{SN}\left(S^2(U,V), \mathcal{C}\right) = \left\{ \begin{array}{c} 1;\ U \cong V^\vee \cong J \otimes J \text{ holds} \\ 0;\ \text{otherwise} \end{array} \right\}$$

Some comments on works related to this paper are now in order.

- The appearance of nonspherical pivotal categories in topologically twisted $N = 2$ supersymmetric conformal field theories has been observed in [*N. Carqueville* and *I. Runkel*, Commun. Math. Phys. 310, No. 1, 135–179 (2012; Zbl 1242.81121)]
- The potential applications to conformal field theory mentioned above rest on the construction of consistent systems of correlators in the sense of [*J. Fjelstad* et al., Theory Appl. Categ. 16, 342–433 (2006; Zbl 1151.81038); *J. Fuchs* and *C. Schweigert*, Adv. Math. 307, 598–639 (2017; Zbl 1355.81034)]. The description of such systems in terms of string-net spaces is being developed in [*C. Schweigert* and *Y. Yang*, "CFT correlators for Cardy bulk fields via string-net models", Preprint, arXiv:1911.10147].
- String-net spaces for fusion categories with $\mathbb{Z}_r$ fusion rules have been considered in [*L.-Y. Hung* and *Y. Wan*, "String-net models with Z_N fusion algebra", Phys. Rev. B 86, No. 23, Article ID 235132, 14 p. (2012; doi:10.1103/PhysRevB.86.235132); *C.-H. Lin* and *M. Levin*, "Generalizations and limitations of string-net models", ibid 89, No. 19, Article ID 195130, 33 p. (2014; doi:10.1103/PhysRevB.89.195130)], where the pivotal structures used are always spherical, but the associators are more general than those considered in this paper.
- The action of surface mapping class groups was investigated for generalizations of Kitaev models defined in terms of pivotal Hopf algebras [*C. Meusburger* and *T. Voß*, "Mapping class group actions from Hopf monoids and ribbon graphs", Preprint, arXiv:2002.04089].
- It was demonstrated in [*C. L. Douglas* et al., "Dualizable tensor categories", Preprint, arXiv:1312.7188] that fusion categories are fully dualizable in the 3-category of finite tensor categories, defining three-dimensional TQFTs on framed manifolds. It would be interesting to understand the precise relation to the string-net construction for nonspherical pivotal fusion categories.
- A construction of TQFT state spaces in terms of so-called fields, being applied to pivotal fusion categories and being equivalent to the string-net construction, was described in [*S. Morrison* and *K. Walker*, Geom. Topol. 16, No. 3, 1481–1607 (2012; Zbl 1280.57026)], where the state space is obtained via a quotient by local relations.
- In the context of Turaev-Viro TQFT [*V. Turaev* and *A. Virelizier*, "On two approaches to 3-dimensional TQFTs", Preprint, arXiv:1006.3501] vector spaces assigned to colored graphs on surfaces were considered for pivotal fusion categories, but only before the projection to the state space.

Now a synopsis of the paper is in order. §2 addresses some notation and properties of pivotal fusion categories, while §3 is a review of the string-net construction and §4 explores some of their general properties, both after [*A. Kirillov jun.*, "String-net model of Turaev-Viro invariants", Preprint, arXiv:1106.6033]. §5 is concerned with computations of the string-net spaces for $\mathbb{Z}_r$-graded vector spaces. §6 and §7 are applications to r-spin structures and to spheres with one marked point, respectively. The slightly lengthy proof of Lemma 4.7 is relegated to an appendix.

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MSC:

18M20	Fusion categories, modular tensor categories, modular functors
57R15	Specialized structures on manifolds (spin manifolds, framed manifolds, etc.)
57K20	2-dimensional topology (including mapping class groups of surfaces, Teichmüller theory, curve complexes, etc.)
81T45	Topological field theories in quantum mechanics

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pivotal fusion categories; string net models; surface mapping class groups; r-spin structures; topological phases of matter

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