A Study on Combinatorial Games

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Abstract

In this thesis, we study combinatorial games, in particular, a class of impartial games.

First, we study a combination (called the generalized cyclic Nimhoff) of the cyclic Nimhoff and subtraction games. We give the $\mathcal{G}$-value of the game when all the $\mathcal{G}$-value sequences of the component subtraction games have a common h-stair structure.

Next, we study a game (called Delete Nim) which requires the OR operation to calculate the $\mathcal{G}$-values of its positions. In addition, the concept called 2-adic valuation, which is described in number theory, is utilized. This is very rare in analysis of impartial games, while the XOR operation is commonly used for calculations of the $\mathcal{G}$-values. Therefore, the research is expected to expand the potential strategies for analysis of impartial games.

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Contents

1 Introduction

Combinatorial Game Theory is a theory devised by Conway, Berlekamp and others in 1970's. It mainly aims at algebraic analysis of the strategy for a competition game. For the comprehensive knowledge on the theory, the reader should refer to the literature ([2], [3], [5], [9]). Here we describe the knowledge (especially on the impartial game) required to read this thesis.

1.1 Impartial Games

Definition 1.1. Combinatorial games have the following characteristics:

- No chance elements (the possible moves in any given position is determined from the beginning).
- No hidden information (both players have complete knowledge of the game states).

Moreover, we assume the following:

- Two players alternately make a move.
- The player who makes the last move wins. (normal rule)
- All game positions are "short" (namely there are finitely many positions that can be reached from a position, and any position cannot appear twice in a play).

In addition to the characteristics above, "impartial" combinatorial games have the following characteristics:

- Both players have the same set of possible moves in any position.

Definition 1.2 (outcome classes)**.** All impartial game positions are classified into two groups:

- $\mathcal{N}$-position $\cdots$ the first player (the $\mathcal{N}$ext player) has a winning strategy.
- $\mathcal{P}$-position $\cdots$ the second player (the $\mathcal{P}$revious player) has a winning strategy.

Let G be an impartial game position. If G is an $\mathcal{N}$-position, there exists a move from G to a $\mathcal{P}$-position. If G is a $\mathcal{P}$-position, there exists no move from G to a $\mathcal{P}$-position.

1.1.1 Nim

Nim is a typical impartial game. The rules are as follows:

- It is played with several heaps of tokens.
- The legal move is to remove any number of tokens (but at least one token) from any single heap.
- The end position is the state of no heaps of tokens.

We denote by $\mathbb{N}_0$ the set of all non-negative integers.

Definition 1.3 (nim-sum)**.** The value obtained by adding numbers in binary form without carry is called nim-sum. The nim-sum of $m_1, \ldots, m_n \in \mathbb{N}_0$ is written by

$$m_1 \oplus \cdots \oplus m_n.$$

Example 1.4. $3 \oplus 5 = (011)_2 \oplus (101)_2 = (110)_2 = 6.$

Proposition 1.5. Let a, b, $c \in \mathbb{N}_0$. Nim-sum $\oplus$ satisfies the following properties:

(1) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$

(2) $a \oplus 0 = 0 \oplus a = a$

(3) $a \oplus a = 0$

(4) $a \oplus b = b \oplus a$

Thus, the set $\mathbb{N}_0$ forms an Abelian group with respect to nim-sum. Moreover, $(\mathbb{N}_0, \oplus)$ is isomorphic to the direct sum of countably many $\mathbb{Z}/2\mathbb{Z}$'s.

Theorem 1.6 (Bouton [4])**.** We denote the Nim position with heaps of size $m_1, \ldots, m_n$ by $\langle m_1, \ldots, m_n \rangle$. Then,

$$m_1 \oplus \cdots \oplus m_n \neq 0 \iff \langle m_1, \ldots, m_n \rangle \text{ is an } \mathcal{N}\text{-position.}$$
$$m_1 \oplus \cdots \oplus m_n = 0 \iff \langle m_1, \ldots, m_n \rangle \text{ is a } \mathcal{P}\text{-position.}$$

Definition 1.7. Let G and G' be game positions. The notation $G \to G'$ means that G' can be reached from G by a single move.

Example 1.8. In the case of Nim position $\langle 7, 8, 10 \rangle$: Since we have

$$7 \oplus 8 \oplus 10 = 5 \neq 0,$$

this position is an $\mathcal{N}$-position. Therefore, the first player has a winning strategy. The first player should make a move to a position with total Nim-sum 0. We have

$$(7 \oplus 8 \oplus 10) \oplus 5 = 5 \oplus 5 = 0.$$

Therefore, 5 should be Nim-sumed to the total. Since we have

$$7 \oplus 5 = 2, \;\; 8 \oplus 5 = 13 \text{ and } 10 \oplus 5 = 15,$$

the good moves are $7 \to 2$, $8 \to 13$ and $10 \to 15$. However, the only legal good move is $7 \to 2$.

1.1.2 $\mathcal{G}$-values

The notion of the $\mathcal{G}$-values was introduced in an attempt to develop the theory about general impartial games. It is a basic tool to classify positions of impartial games.

Definition 1.9 (minimum excluded number)**.** Let T be a proper subset of $\mathbb{N}_0$. Then mex T is defined to be the least non-negative integer not contained in T, namely

$$\operatorname{mex} T = \min(\mathbb{N}_0 \setminus T).$$

Example 1.10. $\operatorname{mex}\{0, 1, 2, 4, 5, 7\} = 3$, $\operatorname{mex}\{1, 2, 4, 5, 7\} = 0$, $\operatorname{mex} \emptyset = 0$.

Definition 1.11 ($\mathcal{G}$-value)**.** The value $\mathcal{G}(G)$ called the $\mathcal{G}$-value (or nim value or Grundy value or SG-value, depending on authors) of G is defined as follows:

$$\mathcal{G}(G) = \operatorname{mex}\{\mathcal{G}(G') \mid G \to G'\}.$$

In particular, the $\mathcal{G}$-value of the end position is 0.

Example 1.12 (The $\mathcal{G}$-value of a single heap in Nim)**.** Let $\langle m \rangle$ be the Nim position with a single heap of size m. Then

$$\mathcal{G}(\langle m \rangle) = m.$$

Thus, all impartial game positions can be reduced to single heaps of Nim.

The following theorem is important.

Theorem 1.13 (Grundy [7], Sprague [10]). Let G be an impartial game position. Then

$$\mathcal{G}(G) \neq 0 \iff G \text{ is an } \mathcal{N}\text{-position.}$$
$$\mathcal{G}(G) = 0 \iff G \text{ is a } \mathcal{P}\text{-position.}$$

The $\mathcal{G}$-value is also useful for analysis of the disjunctive sum of games. If G and H are any positions of (possibly different) impartial games, the disjunctive sum of G and H (written as $G + H$) is defined as follows: each player must make a move in either G or H (but not both) on his turn.

Theorem 1.14 (Grundy [7], Sprague [10]). Let G and H be two game positions. Then

$$\mathcal{G}(G + H) = \mathcal{G}(G) \oplus \mathcal{G}(H).$$

Corollary 1.15 (Grundy [7], Sprague [10]). The $\mathcal{G}$-value of Nim position $\langle m_1, \ldots, m_n \rangle$ is the following:

$$\begin{aligned}\mathcal{G}(\langle m_1, \ldots, m_n \rangle) &= \mathcal{G}(\langle m_1 \rangle) \oplus \cdots \oplus \mathcal{G}(\langle m_n \rangle)\\ &= m_1 \oplus \cdots \oplus m_n.\end{aligned}$$

In summary, if we know the $\mathcal{G}$-values of the game positions:

- We can know the winning strategy.
- We can also know the winning strategy for the disjunctive sum of games.
- Computation of the $\mathcal{G}$-value by definition generally causes combinatorial explosion, but an explicit expression for the $\mathcal{G}$-value may reduce the complexity.

Therefore, the purpose of our research is to find an explicit formula of the $\mathcal{G}$-value for each game.

1.2 Example of Impartial Games

1.2.1 Wythoff's Nim

Shortly after Bouton published the studies on Nim [4], Wythoff conducted research on the $\mathcal{P}$-positions of a game which is nowadays called Wythoff's Nim [13]. Wythoff's Nim is a well-known impartial game with two heaps of tokens. The moves are of two types:

- Remove any positive number of tokens from a single heap.
- Remove the same number of tokens from both heaps.

Note that Wythoff's Nim is not a disjunctive sum of games.

The $\mathcal{G}$-value of a general Wythoff's Nim position is not known. It is one of the open problems in Combinatorial Game Theory, but the following theorem is well-known about the $\mathcal{P}$-positions of Wythoff's Nim.

Theorem 1.16 (Wythoff [13]). Let $\langle m, n \rangle$ be a Wythoff's Nim position. If $|n - m| = k$, the $\mathcal{P}$-positions of Wythoff's Nim are given by

$$\langle \lfloor k\Phi \rfloor, \lfloor k\Phi \rfloor + k \rangle \text{ or } \langle \lfloor k\Phi \rfloor + k, \lfloor k\Phi \rfloor \rangle,$$

where Φ is the golden ratio, i.e. $\Phi = \frac{1+\sqrt{5}}{2}$.

Wythoff's work was one of the earliest researches on heap games which permit the players to remove tokens from more than one heaps at the same time.

1.2.2 Cyclic Nimhoff

Cyclic Nimhoff was researched by Fraenkel and Lorberbom [6]. Let h be a fixed positive integer. The positions of cyclic Nimhoff are the same as those of Nim. The moves are of two types:

- Remove any positive number of tokens from a single heap.
- Remove s_i tokens from each ith heap such that $0 < \sum_{i=1}^{n} s_i < h$.

Theorem 1.17 (Fraenkel and Lorberbom [6])**.** The $\mathcal{G}$-value of position $\langle x_1, x_2, \ldots, x_n \rangle$ in the cyclic Nimhoff is

$$\mathcal{G}(\langle x_1, x_2, \ldots, x_n \rangle) = \left(\bigoplus_i \left\lfloor \frac{x_i}{h} \right\rfloor \right) h + \left(\left(\sum_i x_i \right) \bmod h \right),$$

where $\bigoplus_i a_i$ denotes the nim-sum of all a_i's and mod h means the remainder in the division by h.

1.2.3 Subtraction Games

Definition 1.18. Let S be a set of positive integers. In the subtraction game Subtraction(S), the legal moves are to remove s tokens from a heap for some $s \in S$.

In particular, Nim is Subtraction($\mathbb{N}_+$), where $\mathbb{N}_+$ is the set of all positive integers. There are a lot of preceding studies on subtraction games [3].

We denote the $\mathcal{G}$-value of the position with a single heap of size n by $G(n)$.

Example 1.19. Let n be a size of a single heap in Subtraction(S). Let $S = \{2, 3\}$. Then the $\mathcal{G}$-value of a single heap in Subtraction({2,3}) is shown in the below Table 1.

n	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$G(n)$	0	0	1	1	2	0	0	1	1	2	0	$\cdots$

Table 1: The $\mathcal{G}$-value of a single heap in Subtraction({2,3})

In fact, we obtain

$$\{G(n)\}_{n=0}^{\infty} = \dot{0},0,1,1,\dot{2},$$

where the dots above numbers indicate the beginning and the end of recursion of numbers.

We call sequence $G(0), G(1), \ldots$ the $\mathcal{G}$-value sequence.

Definition 1.20 (periodic)**.** Let $\mathcal{A} = \{\mathcal{A}(x)\}_{x=0}^{\infty}$ be a sequence of integers. We say that $\mathcal{A}$ is periodic with period p and preperiod n_0, if

$$\mathcal{A}(n + p) = \mathcal{A}(n) \text{ for all } n \geq n_0.$$

We say that $\mathcal{A}$ is purely periodic if it is periodic with preperiod $n_0 = 0$.

Definition 1.21 (arithmetic periodic)**.** Let $\mathcal{A} = \{\mathcal{A}(x)\}_{x=0}^{\infty}$ be a sequence of integers. We say that $\mathcal{A}$ is arithmetic periodic with period p, preperiod n_0, and saltus s, if

$$\mathcal{A}(n + p) = \mathcal{A}(n) + s \text{ for all } n \geq n_0.$$

1.2.4 All-but Subtraction Games

All-but subtraction games All-but(S) (i.e. Subtraction($\mathbb{N}_+ \setminus S$) such that S is a finite set) were studied in detail by Angela Siegel [8]. She proved that the $\mathcal{G}$-value sequence is arithmetic periodic and characterized some cases in which the sequence is purely periodic.

2 On a Combination of the Cyclic Nimhoff and Subtraction Games

2.1 Generalized Cyclic Nimhoff

We define generalized cyclic Nimhoff as a combination of cyclic Nimhoff and subtraction games as follows.

Definition 2.1. Let h be a fixed positive integer and $S_1, S_2, \ldots, S_n$ sets of positive integers. Let $\langle x_1, x_2, \ldots, x_n \rangle$ be the game position with n heaps of sizes $x_1, \ldots, x_n$. We define subsets X_1, X_2, $\cdots$, X_n, Y of game positions with n heaps as follows:

$$\begin{aligned}
X_1 &= \{\langle x_1 - s_1, x_2, \ldots, x_n \rangle \mid s_1 \in S_1\} \\
X_2 &= \{\langle x_1, x_2 - s_2, \ldots, x_n \rangle \mid s_2 \in S_2\} \\
&\vdots \\
X_n &= \{\langle x_1, x_2, \ldots, x_n - s_n \rangle \mid s_n \in S_n\} \\
Y &= \{\langle x_1 - s_1, x_2 - s_2, \ldots, x_n - s_n \rangle \mid 0 < \sum_{i=1}^{n} s_i < h\}.
\end{aligned}$$

In generalized cyclic Nimhoff $\mathrm{GCN}(h; S_1, S_2, \ldots, S_n)$, the set of legal moves from position $\langle x_1, x_2, \ldots, x_n \rangle$ is $X_1 \cup X_2 \cup \cdots \cup X_n \cup Y$.

Definition 2.2. Let h be a fixed positive integer and $\mathcal{A} = \{\mathcal{A}(x)\}_{x=0}^{\infty}$ an arbitrary sequence of non-negative integers. The h-stair $\mathcal{B} = \{\mathcal{B}(x)\}_{x=0}^{\infty}$ of sequence $\mathcal{A}$ is defined by the following:

$$\mathcal{B}(xh + r) = \mathcal{A}(x)h + r$$

for all $x \in \mathbb{N}$ and for all $r = 0, 1, \cdots, h-1$.

Example 2.3. If $\mathcal{A} = \underline{0}, \underline{0}, \underline{1}, \underline{5}, \underline{4}, \ldots$, then the 3-stair of sequence $\mathcal{A}$ is

$$\mathcal{B} = \underline{0,1,2}, \underline{0,1,2}, \underline{3,4,5}, \underline{15,16,17}, \underline{12,13,14}, \ldots.$$

2.2 Main Results

Let us denote the $\mathcal{G}$-value sequence of Subtraction(S) by $G_S = \{G_S(x)\}_{x=0}^{\infty}$.

Theorem 2.4. Let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ be arbitrary sequences of non-negative integers and $\langle x_1, x_2, ..., x_n \rangle$ a game position of the generalized cyclic Nimhoff $\mathrm{GCN}(h; S_1, S_2, \ldots, S_n)$. If G_{S_i} is the h-stair of sequence $\mathcal{A}_i$ for all i $(1 \le i \le n)$, then

$$\mathcal{G}(\langle x_1, x_2, \ldots, x_n \rangle) = \left(\bigoplus_i \left\lfloor \frac{G_{S_i}(x_i)}{h} \right\rfloor \right) h + \left(\left(\sum_i x_i \right) \bmod h \right).$$

Proof. For each $i = 1, \ldots, n$, let $x_i = q_i h + r_i$ where $0 \le r_i < h$. Since G_{S_i} is the h-stair of sequence $\mathcal{A}_i$, $G_{S_i}(x_i) = \mathcal{A}_i(q_i)h + r_i$. In other words, note that

$$\left\lfloor \frac{x_i}{h} \right\rfloor = q_i, \quad \left\lfloor \frac{G_{S_i}(x_i)}{h} \right\rfloor = \mathcal{A}_i(q_i), \quad x_i \equiv G_{S_i}(x_i) \equiv r_i \pmod{h}.$$

The proof is by induction. Let

$$\begin{aligned}
Q(x_1, x_2, \ldots, x_n) &= \bigoplus_i \left\lfloor \frac{G_{S_i}(x_i)}{h} \right\rfloor = \bigoplus_i \mathcal{A}_i(q_i), \\
R(x_1, x_2, \ldots, x_n) &= \sum_i x_i \bmod h = \sum_i G_{S_i}(x_i) \bmod h = \sum_i r_i \bmod h.
\end{aligned}$$

Then, it is sufficient to prove that

$$\mathcal{G}(\langle x_1, x_2, \ldots, x_n\rangle) = Q(x_1, x_2, \ldots, x_n)h + R(x_1, x_2, \ldots, x_n).$$

First, we show that for any $k < Q(x_1, x_2, \ldots, x_n)h + R(x_1, x_2, \ldots, x_n)$, there exists a position $\langle x'_1, x'_2, \ldots, x'_n\rangle \in X_1 \cup X_2 \cup \cdots \cup X_n \cup Y$ such that $\mathcal{G}(\langle x'_1, x'_2, \ldots, x'_n\rangle) = k$. There are two cases.

Case that $Q(x_1, x_2, \ldots, x_n)h \le k < Q(x_1, x_2, \ldots, x_n)h + R(x_1, x_2, \ldots, x_n)$:

In this case, k can be written in form $Q(x_1, x_2, \ldots, x_n)h+k'$ by k' such that $0 \le k' < R(x_1, x_2, \ldots, x_n)$. Since $0 < R(x_1, x_2, \ldots, x_n)-k' \le R(x_1, x_2, \ldots, x_n) = \sum_i r_i \bmod h$ and $0 < R(x_1, x_2, \ldots, x_n)-k' < h$, there exist $(k_1, k_2, \ldots, k_n)$ such that $k_1 + k_2 + \cdots + k_n = R(x_1, x_2, \ldots, x_n) - k'$ and $k_j \le r_j$ for each j. Then $\langle x_1-k_1, x_2-k_2, \ldots x_n-k_n\rangle \in Y$. In addition, $Q(x_1-k_1, x_2-k_2, \ldots, x_n-k_n) = Q(x_1, x_2, \ldots, x_n)$ and $R(x_1 - k_1, x_2 - k_2, \ldots, x_n - k_n) = R(x_1, x_2, \ldots, x_n) - (k_1 + k_2 + \cdots + k_n) = k'$. Therefore, $\mathcal{G}(\langle x_1 - k_1, x_2 - k_2, \ldots, x_n - k_n\rangle) = Q(x_1, x_2, \ldots, x_n)h + k' = k$ from induction hypothesis.

Case that $k < Q(x_1, x_2, \ldots, x_n)h$:

In this case, k can be written in form $Q'h + k'$ by Q' and k' such that $Q' < Q(x_1, x_2, \ldots, x_n) = \bigoplus_i \mathcal{A}_i(q_i)$ and $0 \le k' < h$.

According to the nature of nim-sum, there exists j and g which satisfy $Q' = \mathcal{A}_1(q_1)\oplus\mathcal{A}_2(q_2)\oplus\cdots\oplus \mathcal{A}_{j-1}(q_{j-1})\oplus g\oplus\mathcal{A}_{j+1}(q_{j+1})\oplus\cdots\oplus\mathcal{A}_n(q_n)$ and $g < \mathcal{A}_j(q_j)$. Without loss of generality, we assume $j = 1$. That is, there exist $g < \mathcal{A}_1(q_1)$ which satisfies $Q' = g \oplus \mathcal{A}_2(q_2) \oplus \cdots \oplus \mathcal{A}_n(q_n)$. If we put r'_1 to satisfy that $(r'_1 + r_2 + r_3 + \cdots + r_n) \bmod h = k'$ and $0 \le r'_1 < h$, then $gh + r'_1 < \mathcal{A}_1(q_1)h + r_1 = G_{S_1}(x_1)$, and therefore, there exists x'_1 such that $G_{S_1}(x'_1) = gh + r'_1$ and $x_1 - x'_1 \in S_1$. Thus, we have $\langle x'_1, x_2, \ldots, x_n\rangle \in X_1$. Therefore,

$$\begin{aligned}\mathcal{G}(\langle x'_1, x_2, \ldots, x_n\rangle) &= \left(\left\lfloor\frac{G_{S_1}(x'_1)}{h}\right\rfloor \oplus \left\lfloor\frac{G_{S_2}(x_2)}{h}\right\rfloor \oplus\cdots\oplus \left\lfloor\frac{G_{S_n}(x_n)}{h}\right\rfloor\right) h\\ &\quad + (x'_1 + x_2 + \cdots + x_n) \bmod h\\ &= (g \oplus \mathcal{A}_2(q_2) \oplus \cdots \oplus \mathcal{A}_n(q_n))h + k' = Q'h + k' = k\end{aligned}$$

from induction hypothesis.

Next, we show that, if $\langle x_1, x_2, \ldots, x_n\rangle \to \langle x'_1, x'_2, \ldots, x'_n\rangle$, then

$$Q(x_1, x_2, \ldots, x_n)h + R(x_1, x_2, \ldots, x_n) \ne Q(x'_1, x'_2, \ldots, x'_n)h + R(x'_1, x'_2, \ldots, x'_n).$$

Clearly, the claim is true if $\langle x'_1, x'_2, \ldots, x'_n\rangle$ is in Y, since $R(x'_1, x'_2, \ldots, x'_n) \ne R(x_1, x_2, \ldots, x_n)$. Therefore, we assume that $\langle x'_1, x'_2, \ldots, x'_n\rangle$ is in X_1 without loss of generality, namely $x'_j = x_j$ $(j > 1)$ and $x_1 - x'_1 \in S_1$. Let $x'_1 = q'_1h + r'_1$ $(0 \le r'_1 < h)$.

If $Q(x_1, x_2, \ldots, x_n)h + R(x_1, x_2, \ldots, x_n) = Q(x'_1, x_2, \ldots, x_n)h + R(x'_1, x_2, \ldots, x_n)$, then we have $Q(x_1, x_2, \ldots, x_n) = Q(x'_1, x_2, \ldots, x_n)$ and $R(x_1, x_2, \ldots, x_n) = R(x'_1, x_2, \ldots, x_n)$. Then, $r_1 = r'_1$ since $R(x_1, x_2, \ldots, x_n) = R(x'_1, x_2, \ldots, x_n)$, and $\lfloor G_{S_1}(x_1)/h\rfloor = \lfloor G_{S_1}(x'_1)/h\rfloor$ since $Q(x_1, x_2, \ldots, x_n) = Q(x'_1, x_2, \ldots, x_n)$. Therefore $G_{S_1}(x_1) = G_{S_1}(x'_1)$, but it is impossible because $x_1 - x'_1 \in S_1$. □

There are a variety of subtraction games with the h-stair of a simple integer sequence as their $\mathcal{G}$-value sequence.

Example 2.5 (Nim)**.** For any h, $G_{\mathbb{N}_+}(x) = x = \left(\lfloor \frac{x}{h} \rfloor\right) h + (x \bmod h)$.

Example 2.6 (Subtraction($\{1, \ldots, l-1\}$) and its variants)**.** If $\{1, \ldots, l-1\} \subset S \subset \mathbb{N}_+ \setminus \{kl \mid l \in \mathbb{N}_+\}$ and $h \mid l$, then

$$G_S(x) = x \bmod l = \left(\left\lfloor\frac{x \bmod l}{h}\right\rfloor\right) h + ((x \bmod l) \bmod h).$$

Example 2.7 (All-but($\{h, 2h, \ldots, kh\}$))**.** If $S = \mathbb{N}_+ \setminus \{h, 2h, \ldots, kh\}$, then $G_S(x)$ is the h-stair of $\{\underbrace{0, 0, \ldots, 0}_{k+1}, \underbrace{1, 1, \ldots, 1}_{k+1}, \underbrace{2, 2, \ldots, 2}_{k+1}, \ldots\}$.

Example 2.8 (All-but($\{s_1, s_2\}$) [8]). If $s_2 > s_1$, then $G_S(x)$ is the s_1-stair of a sequence of positive integers.

Theorem 2.4 allows us to combine several subtraction games which have the $\mathcal{G}$-value sequences of form h-stair for common h. For example, for GCN(4; $\mathbb{N}_+$, $\{1, 2, 3, 4, 5, 6, 7\}$, $\mathbb{N}_+ \setminus \{4, 8\}$), we have the following:

$$\mathcal{G}(\langle x_1, x_2, x_3\rangle) = \left(\left\lfloor \frac{x_1}{4} \right\rfloor \oplus \left\lfloor \frac{x_2 \bmod 8}{4} \right\rfloor \oplus \left\lfloor \frac{x_3}{12} \right\rfloor\right) \times 4 + (x_1 + x_2 + x_3) \bmod 4.$$

Suppose that a subtraction set S is given. Then we can define a new subtraction set S' such that the $\mathcal{G}$-value sequence of Subtraction(S') is the h-stair of the $\mathcal{G}$-value sequence of Subtraction(S).

Theorem 2.9. Let S be an arbitrary subtraction set and let $S' = \mathbb{N}_+ \setminus \{(\mathbb{N}_+ \setminus S)h\}$. Then

$$G_{S'}(n) = G_S\left(\left\lfloor \frac{n}{h} \right\rfloor\right) h + (n \bmod h).$$

Proof. Let $n = qh + i$ where $0 \leq i < h$. Then the formula to be shown is $G_{S'}(qh + i) = G_S(q)h + i$. The proof is by induction on n $(= qh + i)$.

First, we show that there exists a move to a position with any smaller $\mathcal{G}$-value $rh + k$ than $G_S(q)h + i$. There are two cases.

Case that $r = G_S(q)$ and $k < i$: Since $0 < i - k < h$, there exists a move $gh + i \to gh + k$ and we have

$$G_{S'}(gh + k) = G_S(q)h + k = rh + k$$

by induction hypothesis. Case that $r < G_S(q)$ and $0 \leq k < h$:

By the definition of $G_S(q)$, there exists q' such that $q - q' \in S$, $G_S(q') = r$ and

$$G_{S'}(q'h + k) = G_S(q')h + k = rh + k$$

by induction hypothesis. So we only need to prove that there is move to $q'h + k$. If $i \neq k$, clearly there exists a move $qh + i \to q'h + k$.

Assume that $i = k$ and that there does not exist a move $qh + i \to q'h + k$. Then

$$(q - q')h \notin S' \Rightarrow (q - q')h \in (\mathbb{N}_+ \setminus S)h \Rightarrow q - q' \in (\mathbb{N}_+ \setminus S) \Rightarrow q - q' \notin S,$$

which is a contradiction. Next, we show that, if $n = qh + i \to n' = q'h + k$, then

$$G_{S'}(n') \neq G_S(q)h + i.$$

If $G_{S'}(n') = G_S(q)h + i$, then we have $G_S(q) = G_S(q')$ and $k = i$ by induction hypothesis, but it is impossible by the definition of $G_S(q)$. Because

$$q - q' \notin S \Rightarrow q - q' \in (\mathbb{N}_+ \setminus S) \Rightarrow (q - q')h \in (\mathbb{N}_+ \setminus S)h \Rightarrow (q - q')h \notin \mathbb{N}_+ \setminus \{(\mathbb{N}_+ \setminus S)h\} \Rightarrow n - n' \notin S'.$$

□

Therefore, Theorem 2.4 has sufficiently wide application.

3 Delete Nim

3.1 Rules of the Delete Nim

The rules of Delete Nim are as follows:

- There are two heaps of tokens.
- The player selects a non-empty heap and deletes the other heap and removes 1 token from the selected heap and splits the heap into two (possibly empty heaps).

Example 3.1. $\langle 11, \underline{9}\rangle \to \langle \underline{5}, 3\rangle \to \langle \underline{4}, 0\rangle \to \langle \underline{2}, 1\rangle \to \langle \underline{1}, 0\rangle \to \langle 0, 0\rangle$.

3.2 OR operation

Definition 3.2. We denote by $\vee$ the usual OR operation of two numbers in binary notation.

Example 3.3. $3 \vee 5 = 11_2 \vee 101_2 = 111_2 = 7$.

Example 3.4. $9 \vee 12 = 1001_2 \vee 1100_2 = 1101_2 = 13$.

3.3 Main Results

Theorem 3.5. We denote the position of Delete Nim with two heaps of x tokens and y tokens by $\langle x, y\rangle$. Then,

$$\mathcal{G}(\langle x, y\rangle) \quad = \quad v_2((x \vee y) + 1),$$

where $v_p(n)$ is the p-adic valuation of n; that is,

$$v_p(n) = \begin{cases} \max\{l \in \mathbb{N} : p^l \mid n\} & (n \neq 0) \\ \infty & (n = 0). \end{cases}$$

Proof. Let $x = \sum_i 2^i x_i$, $y = \sum_i 2^i y_i$ $(x_i, y_i \in \{0, 1\})$ and $h = v_2((x \vee y) + 1)$.

First, we show that $\langle x, y\rangle$ has no next position $\langle x', y'\rangle$ such that $h = v_2((x' \vee y') + 1)$ by contradiction. Note that $x' + y' = x - 1$ or $x' + y' = y - 1$.

If $h = 0$, then x and y are even. Therefore, $x' + y'$ is an odd number and $v_2((x' \vee y') + 1) \neq 0$, which is a contradiction.

Let $x' = \sum_i 2^i x'_i$, $y' = \sum_i 2^i y'_i$ $(x'_i, y'_i \in \{0, 1\})$. If $h > 0$, then $x'_h = y'_h = 0$ and for any $k < h$, $x'_k = 1$ or $y'_k = 1$. Therefore, $2^h - 1 \leq ((x' + y') \bmod 2^{h+1}) \leq 2^{h+1} - 2$, and thus, $2^h \leq ((x' + y' + 1) \bmod 2^{h+1}) \leq 2^{h+1} - 1$. Then, $x_h = 1$ or $y_h = 1$, which is a contradiction.

Next, we show that for any $h' < h$, $\langle x, y\rangle$ has a next position such that $h' = v_2((x' \vee y') + 1)$. Since $h = v_2((x \vee y) + 1)$, without loss of generality, $x_{h'} = 1$. Let $x' = x - 2^{h'}$ and $y' = 2^{h'} - 1$. Evidently, $x'_{h'} = 0$, $y'_{h'} = 0$, $y'_k = 1$ $(k < h')$, and $x' + y' = x - 1$. Therefore, $\langle x', y'\rangle$ is a next position of $\langle x, y\rangle$ and $h' = v_2((x' \vee y') + 1)$. ◻

A game similar to Delete Nim was introduced in [11]. The rules are as follows:

- There are two (non-empty) heaps of tokens.
- The player selects one of the heaps and deletes it, and splits the other heap into two (non-empty) heaps.

We call this game Variant of Delete Nim (VDN).

Example 3.6. $\langle \underline{8}, 5\rangle \to \langle \underline{7}, 1\rangle \to \langle \underline{4}, 3\rangle \to \langle \underline{3}, 1\rangle \to \langle \underline{2}, 1\rangle \to \langle 1, 1\rangle$.

The $\mathcal{N}$-positions and $\mathcal{P}$-positions of VDN have been already shown [11]. However, the $\mathcal{G}$-values of the positions of the game have not been discussed.

Theorem 3.7 ([11])**.** Let G be a VDN position.

$$\text{At least one of the heaps has an even numbers of tokens} \iff G \text{ is an } \mathcal{N}\text{-position.}$$
$$\text{Both heaps have an odd number of tokens} \iff G \text{ is a } \mathcal{P}\text{-position.}$$

VDN can be shown to be isomorphic to Delete Nim, in the following sense.

Definition 3.8. Let G and H be game positions. We say that G is isomorphic to H, if G and H have the same game tree.

Theorem 3.9. Let F be the function denoted by

$$F\langle x, y\rangle = \langle x-1, y-1\rangle.$$

Then, F is an isomorphism from the set of all positions of VDN to that of Delete Nim. Namely,

$$\langle x, y\rangle \to \langle x', y'\rangle \iff F\langle x, y\rangle \to F\langle x', y'\rangle.$$

Proof. Evidently, the end positions of the games hold this isomorphism. Let $\langle x'-1, y'-1\rangle$ be a next position of $\langle x-1, y-1\rangle$ in Delete Nim. Then, without loss of generality, $(x'-1)+(y'-1) = (x-1)-1$. In contrast, $\langle x, y\rangle$ of VDN has the next position $\langle x', y'\rangle$ because $x'+y' = x$. For the other side, let $\langle x', y'\rangle$ be a next position of $\langle x, y\rangle$ in VDN. Then, without loss of generality, $x'+y' = x$. In contrast, $\langle x-1, y-1\rangle$ of Delete Nim has the next position $\langle x'-1, y'-1\rangle$ because $(x'-1)+(y'-1) = (x-1)-1$. Therefore, there is a one-to-one correspondence between $\langle x-1, y-1\rangle$ of Delete Nim and $\langle x, y\rangle$ of VDN. □

Corollary 3.10. By Theorem 3.9, we can compute the $\mathcal{G}$-value of position $\langle x, y\rangle$ of VDN as

$$v_2(((x-1) \vee (y-1)) + 1).$$

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