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An Application of Two-Constraint Consumption Model: Two goods case with time constraint Revised Version

by

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An Application of Two-Constraint Consumption Model: Two goods case with time constraint Revised Version*

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CONTENTS

0 Abstract

This paper conducts microeconomic analyses in which not only budget but also another constraint can affect consumer's behavior. We construct a model in which two constraints exist. One of the constraints is budget and the other is time. We conduct comparative statics analyses involving two goods: time-intensive good and money-intensive good. In the context of transportation, an example of the two good is local train and high-speed railway. We also study consumption behavior when paid work is introduced, whereby the consumer faces the trade-off between time and money. Furthermore, we modify the model to analyze the saving behavior of the consumer.

Keywords:
consumption behavior, modal choice, "time-Giffen good", time-intensive good, time saving

1 Introduction

Most economics models incorporate only the budget constraint, parameterized by prices. In a large class of environments, however, other constraints, such as time or space, bind the behavior of economic agents. For example, Hummels and Schaur (2013) estimate that time cost of trade has an effect on the demand which is equivalent to the certain amount of an advalorem tariff ('*each day in transit is equivalent to an advalorem tariff of 0.6 to 2.1 percent*').

There are similar cases in policy problems: space for nuclear waste or landfill, or resources such as water and electricity. They constrain consumption behavior, producer behavior, or government's policy.

In the context of modal choice, Small (2012) and Behrens and Pels (2012) focus on time as important aspect of economic behavior.

In this paper, we construct a model that includes the time constraint. The results of the model analysis can be extrapolated to other types of the constraint by applying.

Amongst preceding works incorporating the concept of time, Becker (1965) analyzes the optimal time allocation supposing utility is decided by the time allocation for leisure and work. DeSerpa (1971) and Carpio and Wohlgenant (2010) regard time as a good which is required to consume with the consumption good at the same time.

Applied studies regarding time include Fujii, Kitamura and Kumada (1998), Kono and Morisugi (2001), Kato and Imai (2005) and Jara-Díaz (2007), *inter alia.* Fujii, Kitamura and Kumada (1998) conduct the empirical studies to estimate the individual traffic demand. Kono and Morisugi (2001) theoretically examines the change of the value of time depending on the economic environment. Kato and Imai (2005) formulate the private travel behavior to valuate the travel-time savings of the private trip by applying the DeSerpa's definition of the travel time savings. Jara-Díaz (2007) summarizes and compares the theories on time allocation and the concepts of value of time, and suggests the sources of improvement in the modeling of the value of time.

This paper mainly refers to Carpio and Wohlgenant (2010) and DeSerpa (1971). In their models, utility depends on both goods and time. And both budget and time constraints bind. The difference between the two models is that the amount of time to consume each good is expressed by an equality in Carpio and Wohlgenant (2010) while it is expressed by an inequality in DeSerpa (1971). The advantage of the equality expression is that it facilitates the calculation. Using this advantage, Carpio and Wohlgenant (2010) derive the basic mathematical features of the model similar to Slutky equation and Roy identity. On the other hand, the advantage of the inequality is that it expresses the minimum amount of time consumption for a certain amount of good consumption. It lets us analyze the case that additional marginal time input without the increase in good improves the utility level, although it has drawbacks: complicated calculation and the difficulty of interpreting the values of time.

In my model, we assume that both the budget constraint and the time constraint are inequalities. We also assume that the relationship between a good and its time consumption is equality for each good, and the input of the utility function is expressed only by the amount of time.

The model involves two goods, good 1 being the "time-intensive good" and good 2 the "money-intensive good". The word "time-intensive" means that the good requires more time proportionately to money than the other good. Conversely, "money-intensive" means that the consumption needs relatively more money than time.

We analyze the case that only one of the constraints binds, which is precluded in Carpio and Wohlgenant (2010) and DeSerpa (1971), and Evans (1972) mentioned the possibility of inequality budget but the time constraint is equality. The proportional relationship between a good and its consumption time and the form of the utility function allow me to figure the two-good analysis in two dimensions while DeSerpa (1971) had to deal with four dimensions.

Using the advantages of the model, we conduct the analyses of two goods which can be contrasted as "fast but expensive" and "slow but cheap".

The results are then extrapolated to further analyses. First is the case that the consumer can contemporaneously exchange her disposable time for income, and *vice versa*. Second is an intertemporal extension, the simplest of which is a two-period model.

2 The Model

We construct the two-good model as follows:

$$
\begin{aligned}
&max\; U(T_1, T_2) \\
&s.t. \\
&\quad T^0 \geq T_1 + T_2 \qquad (1) \\
&\quad T_i = a_i X_i \ (\forall i = 1, 2) \qquad (2) \\
&\quad Y \geq P_1 X_1 + P_2 X_2 \qquad (3)
\end{aligned}
$$

Note that

- T_i : the amount of time input to consume the ith good $\left(T_i \in [0, T^0]\right)$
- $T^0 > 0$: exogenous disposable time
- $X_i \geq 0$: amount of consumption of ith good
- $P_i > 0$: price of ith good
- $Y > 0$: exogenous income
- $U(T_1, T_2)$: utility function, continuous, twice-differentiable and, $\frac{\partial U}{\partial T_i} > 0$ and $\frac{\partial^2 U}{\partial T_i^2} < 0$ $(\forall i = 1, 2)$
- $a_i > 0$: the parameter which shows the necessary amount of time to consume one unit of ith good [*1]

(1) and (3) allow the switching of the binding constraint as discussed later. From the other point of view, it is the switch of the idle resource.

[*1] The higher the technology level, either the larger or the smaller a_i, depending on the type of the good.

(2) represents the assumption that the time required to consume each unit of good is constant, while DeSerpa (1971) supposes only the minimum amount of time for the good consumption. T_i and X_i must be in some interval. And we assume that the interval consists of only one element for simplicity.

We also assume that good 2 is comparatively pricey whereas good 1 is time consuming. More specifically,

$$0 < \frac{Y}{P_2/a_2} < T^0 < \frac{Y}{P_1/a_1} \tag{4}$$

which can be shown by the figure as follows.

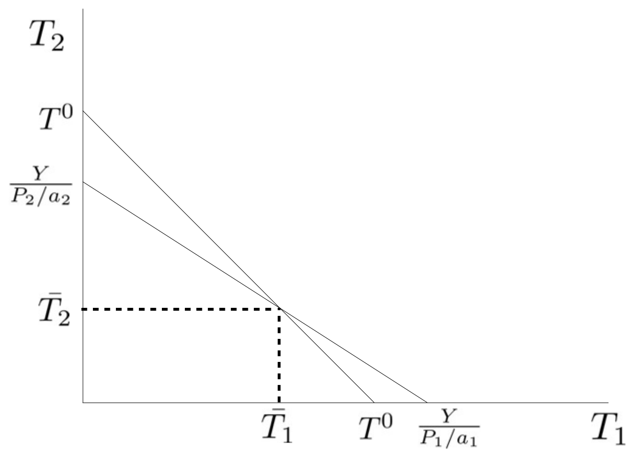


Fig. 2-1

The two lines in the figure are the budget constraint line and the time constraint line. The line, sloped -1 with intercepts T^0 is the time constraint. The other line, sloped flatter, is the budget constraint, of which the intercepts are expressed by the relative prices P_i and a_i.

The intersection point of these two lines is:

$$(\bar{T}_1, \bar{T}_2) = \left(\frac{T^0 P_2/a_2 - Y}{P_2/a_2 - P_1/a_1}, \frac{Y - T^0 P_1/a_1}{P_2/a_2 - P_1/a_1} \right) \tag{5}$$

The consumption feasibility set is the shaded area in Fig. 2-2.

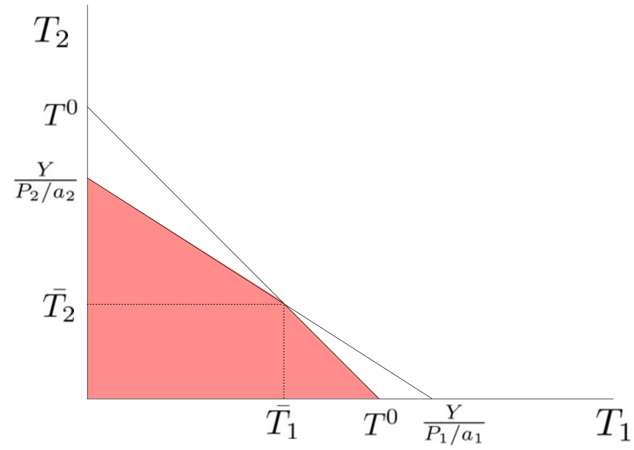


Fig. 2-2

2.1 MRS and Optimal Point

We separate the optimisation into three cases. The first case is when the slope of the indifference curve which goes through the intersection point, $(\bar{T}_1, \bar{T}_2)$, is flatter than the slope of the budget constraint line at the intersection point. That is $MRS(\bar{T}_1, \bar{T}_2) = -\frac{dU(\bar{T}_1, \bar{T}_2)/dT_2}{dU(\bar{T}_1, \bar{T}_2)/dT_1} < \frac{P_1/a_1}{P_2/a_2}$. Then the optimal point (T_1^*, T_2^*) is the tangency point of the budget line and the indifference curve. And it locates closer to the vertical axis than the intersection point. It is shown in Fig. 2-3.

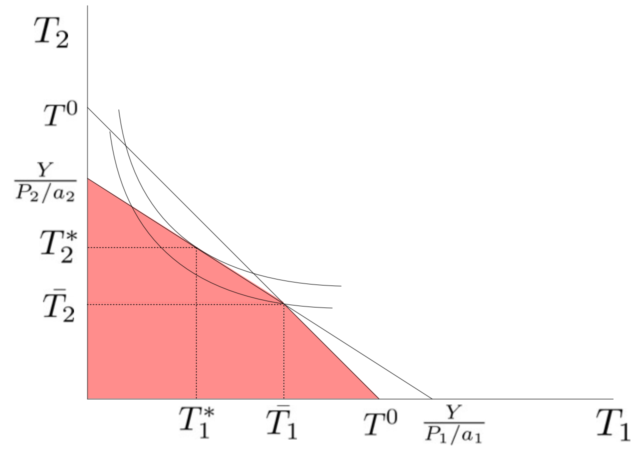


Fig. 2-3

Second, when the slope is steeper than -1, the optimal point is the tangency point of the time constraint line and the indifference curve. In this case, the optimal point (T_1^*, T_2^*) locates left of the intersection point and closer to the horizontal axis, as illustrated:

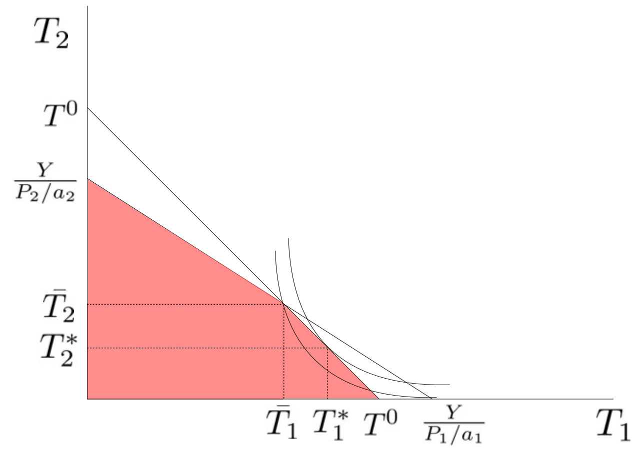


Fig. 2-4

Finally, in the case the slope lies within $\left[-1, -\frac{P_1/a_1}{P_2/a_2}\right]$, the optimal point is the intersection point of the two constraints. ($T_1^* = \bar{T}_1$, $T_2^* = \bar{T}_2$)

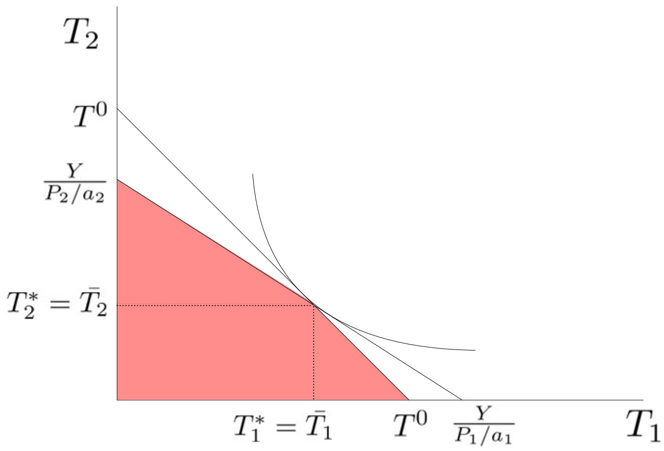


Fig. 2-5

3 Comparative Statics

3.1 Price Change

3.1.1 The price of the "time-intensive good"

When the price of good 1 rises ($P_1 < P_1'$), the possible consumption set is shown by the lower shaded area in Fig. 3-1.

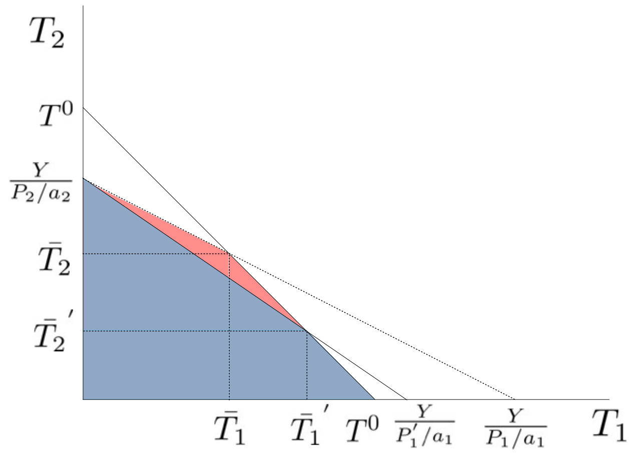


Fig. 3-1

The set can be partitioned into three areas as Fig. 3-2 shows. The partition depends on how the slope of the binding constraint line changes according to the price change.

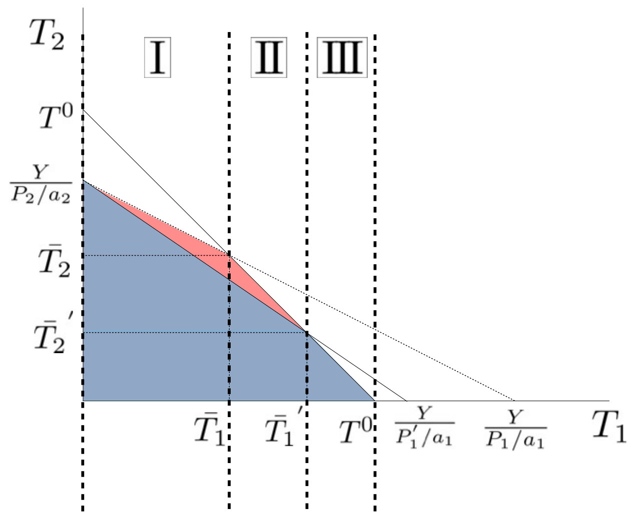


Fig. 3-2

Note that the binding constraint switches from the time constraint to the budget constraint in area II. Consequently, the slope of the binding constraint becomes flatter by the price change.

To explain the "switching effect", the following two provisional demand functions of good 1 are defined.

$$\begin{aligned} T_1^T = \; & arg\ max\ U(T_1, T_2) \\ & \qquad s.t.\ \ T^0 = T_1 + T_2 \end{aligned} \tag{6}$$

$$\begin{aligned} T_1^Y = \; & arg\ max\ U(T_1, T_2) \\ & \qquad s.t.\ \ T_i = a_i X_i\ (\forall i = 1, 2) \\ & \qquad\qquad Y = P_1 X_1 + P_2 X_2 \end{aligned} \tag{7}$$

The former shows the time demand for good 1 if the consumer should face the time constraint only. Conversely, the later is the time demand for good 1 if the consumer should face the budget constraint only.

The effect of the price change when $T_1^* \in \left[\bar{T}_1, \bar{T}_1'\right]$ can be divided into two parts:

$$\frac{dT_1^*}{dP_1} = \frac{\partial T_1^T}{\partial T^0}\left(\frac{Y}{P_1'/a_1} - T^0\right) + \frac{\partial T_1^Y}{\partial\left(P_2/a_2\right)}\left(\frac{Y}{P_1'/a_1} - \frac{Y}{P_2/a_2}\right) \tag{8}$$

The first and the second terms respectively correspond to (a) and (b) in the following figure:

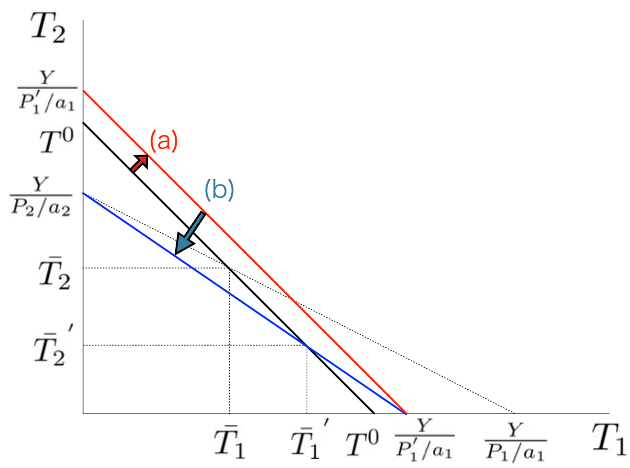


Fig. 3-3

Conditions under which the optimal T_1 increases when P_1 rises, can be summarized as follows.

Proposition 1

Suppose $T_1^* \in \left[\bar{T}_1, \bar{T}_1'\right]$.

If $\frac{\partial T_1^T}{\partial T^0}\left(\frac{Y}{P_1'/a_1} - T^0\right) + \frac{\partial T_1^Y}{\partial (P_2/a_2)}\left(\frac{Y}{P_1'/a_1} - \frac{Y}{P_2/a_2}\right) > 0$, then $\frac{dT_1^*}{dP_1} > 0$.

Definition 1 ("time-Giffen good")

Good 1 is a time-Giffen good if $T_1^* \in \left[\bar{T}_1, \bar{T}_1'\right]$ and $\frac{dT_1^*}{dP_1} > 0$.

3.1.2 Example of a "time-Giffen good": local train and express railway

The more money and time we have, the more frequently we afford to travel. Realistically, however, we face the shortage in at least one of these two resources; one of the constraints has a binding effect on the travel demand.

Suppose that the local train is "time-intensive transportation" and the express railway is "money-intensive transportation". Then, some travelers may increase the frequency of using the local train, when the fee of the local train rises. They will afford less frequent journeys by the express railway because of the effective budget shrink. And the decrease of the express railway use relaxes the time constraint, and then, the consumer utilizes the saved time by increasing the use of the local train. This will happen because of the time constraint.

3.1.3 The price of the "money-intensive good"

The case that the price of good 2 increases ($P_2 < P_2'$) is drawn in Fig. 3-4.

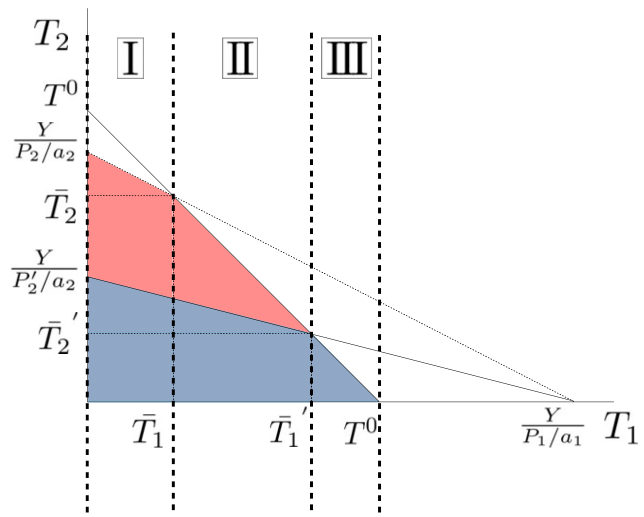


Fig. 3-4

The effect of the price change when $T_1^* \in [\bar{T_1}, \bar{T_1}']$ is:

$$\frac{dT_1^*}{dP_2} = \frac{\partial T_1^T}{\partial T^0}\left(\frac{Y}{P_1/a_1} - T^0\right) + \frac{\partial T_1^Y}{\partial (P_2/a_2)}\left(\frac{Y}{P_1/a_1} - \frac{Y}{P_2'/a_2}\right) \tag{9}$$

The first and the second terms respectively correspond to (a) and (b) in the following figure:

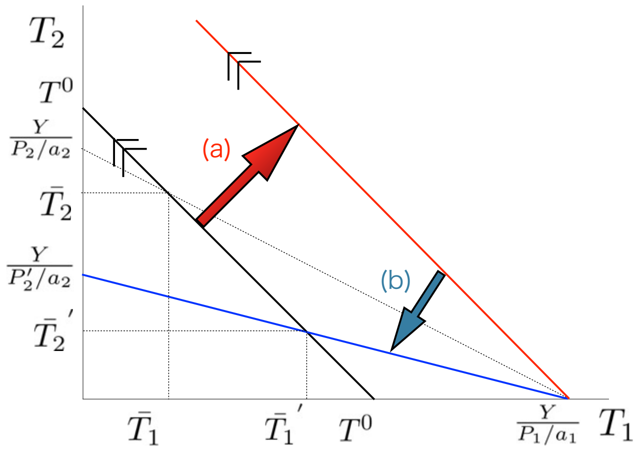


Fig. 3-5

Definition 2 ("time-substitution(/time-complementary) good" (good 1))

(i) Good 1 is the time-substitution good for good 2

if $T_1^* \in \left[\bar{T}_1, \bar{T}_1'\right]$ and $\frac{dT_1^*}{dP_2} > 0$.

(ii) Good 1 is the time-complementary good for good 2

if $T_1^* \in \left[\bar{T}_1, \bar{T}_1'\right]$ and $\frac{dT_1^*}{dP_2} < 0$.

(iii)Good 1 is the time-neutral good for good 2

if $T_1^* \in \left[\bar{T}_1, \bar{T}_1'\right]$ and $\frac{dT_1^*}{dP_2} = 0$.

Proposition 2

Suppose $T_1^* \in \left[\bar{T}_1, \bar{T}_1'\right]$.

(i) If $\frac{\partial T_1^T}{\partial T^0}\left(\frac{Y}{P_1/a_1} - T^0\right) + \frac{\partial T_1^Y}{\partial (P_2/a_2)}\left(\frac{Y}{P_1/a_1} - \frac{Y}{P_2'/a_2}\right) > 0$, then $\frac{dT_1^*}{dP_2} > 0$

(good 1 is the time-substitution good of good 2).

(ii)If $\frac{\partial T_1^T}{\partial T^0}\left(\frac{Y}{P_1/a_1} - T^0\right) + \frac{\partial T_1^Y}{\partial (P_2/a_2)}\left(\frac{Y}{P_1/a_1} - \frac{Y}{P_2'/a_2}\right) < 0$, then $\frac{dT_1^*}{dP_2} < 0$

(good 1 is the time-complementary good of good 2).

(iii)If $\frac{\partial T_1^T}{\partial T^0}\left(\frac{Y}{P_1/a_1} - T^0\right) + \frac{\partial T_1^Y}{\partial (P_2/a_2)}\left(\frac{Y}{P_1/a_1} - \frac{Y}{P_2'/a_2}\right) = 0$,then $\frac{dT_1^*}{dP_2} = 0$

(good 1 is the time-neutral good of good 2).

Similarly, the demands for good 2 can be defined as follows:

$$\begin{aligned} T_2^T &= \ arg\ max\ U(T_1, T_2) \\ & \qquad\quad s.t.\ \ T^0 = T_1 + T_2 \end{aligned} \tag{10}$$

$$\begin{aligned} T_2^Y &= \ arg\ max\ U(T_1, T_2) \\ & \qquad\quad s.t.\ \ T_i = a_i X_i\ (\forall i = 1, 2) \\ & \qquad\qquad\quad Y = P_1 X_1 + P_2 X_2 \end{aligned} \tag{11}$$

And

$$\frac{dT_2^*}{dP_2} = \frac{\partial T_2^T}{\partial T^0}\left(\frac{Y}{P_1/a_1} - T^0\right) + \frac{\partial T_2^Y}{\partial\left(P_2/a_2\right)}\left(\frac{Y}{P_1/a_1} - \frac{Y}{P_2'/a_2}\right) \tag{12}$$

Also note that the effect of P_1 change on T_2^*:

$$\frac{dT_2^*}{dP_1} = \frac{\partial T_2^T}{\partial T^0}\left(\frac{Y}{P_1'/a_1} - T^0\right) + \frac{\partial T_2^Y}{\partial\left(P_2/a_2\right)}\left(\frac{Y}{P_1'/a_1} - \frac{Y}{P_2/a_2}\right) \tag{13}$$

Definition 3 **(**"time-substitution(/time-complementary) good" (good 2)**)**

(i) Good 2 is the time-substitution good for good 1
if $T_1^* \in \left[\bar{T}_1, \bar{T}_1'\right]$ and $\frac{dT_2^*}{dP_1} > 0$.

(ii) Good 1 is the time-complementary good for good 2
if $T_1^* \in \left[\bar{T}_1, \bar{T}_1'\right]$ and $\frac{dT_2^*}{dP_1} < 0$.

(iii)Good 1 is the time-neutral good for good 2
if $T_1^* \in \left[\bar{T}_1, \bar{T}_1'\right]$ and $\frac{dT_2^*}{dP_1} = 0$.

Proposition 3

Suppose $T_1^* \in \left[\bar{T}_1, \bar{T}_1'\right]$.

(i) If $\frac{\partial T_2^T}{\partial T^0}\left(\frac{Y}{P_1'/a_1} - T^0\right) + \frac{\partial T_2^Y}{\partial (P_2/a_2)}\left(\frac{Y}{P_1'/a_1} - \frac{Y}{P_2/a_2}\right) > 0$, then $\frac{dT_2^*}{dP_1} > 0$ (good 2 is the time-substitution good of good 1).

(ii)If $\frac{\partial T_2^T}{\partial T^0}\left(\frac{Y}{P_1'/a_1} - T^0\right) + \frac{\partial T_2^Y}{\partial (P_2/a_2)}\left(\frac{Y}{P_1'/a_1} - \frac{Y}{P_2/a_2}\right) < 0$, then $\frac{dT_2^*}{dP_1} < 0$ (good 2 is the time-complementary good of good 1).

(iii)If $\frac{\partial T_2^T}{\partial T^0}\left(\frac{Y}{P_1'/a_1} - T^0\right) + \frac{\partial T_2^Y}{\partial (P_2/a_2)}\left(\frac{Y}{P_1'/a_1} - \frac{Y}{P_2/a_2}\right) = 0$,then $\frac{dT_2^*}{dP_1} = 0$ (good 2 is the time-neutral good of good 1).

The following $E^Y (\leq Y)$ is defined as the expenditure at (T_1^*, T_2^*):

$$\begin{aligned} E^Y &= X_1^* P_1 + X_2^* P_2 \\ &= T_1^* \frac{P_1}{a_1} + T_2^* \frac{P_2}{a_2} \end{aligned} \tag{14}$$

and

$$\frac{dE^Y}{dP_1} = \frac{dT_1^*}{dP_1}\frac{P_1}{a_1} + \frac{T_1^*}{a_1} + \frac{dT_2^*}{dP_1}\frac{P_2}{a_2} \tag{15}$$

Because $\frac{P_1}{a_1}$, $\frac{P_2}{a_2}$, and $\frac{T_1^*}{a_1}$ are positive, if good 1 is the time-Giffen good; (T_1^*, T_2^*) is in area II and $\frac{dE^Y}{dP_1} \leq 0$,[*2] then $\frac{dT_2^*}{dP_1}$ must be negative; good 2 is the time-complementary good for good 1. It is summarized as the following proposition.

Proposition 4

If good 1 is the time-Giffen good,

then good 2 is the time-complementary good for good 1.

[*2]This is because of the switching from the time constraint to the budget constraint.

3.1.4 Time Consumption Parameter

The result of the change in a_i is inverse to P_i. We can confirm it easily because a_i and P_i are expressed by the relative price (P_i/a_i) in the figures. It can be expressed as follows:

$$\frac{dT_i^*}{da_1} = \frac{\partial T_i^T}{\partial T^0}\left(\frac{Y}{P_1/a_1'} - T^0\right) + \frac{\partial T_i^Y}{\partial\left(P_2/a_2\right)}\left(\frac{Y}{P_1/a_1'} - \frac{Y}{P_2/a_2}\right) \tag{16}$$

and

$$\frac{dT_i^*}{da_2} = \frac{\partial T_i^T}{\partial T^0}\left(\frac{Y}{P_1/a_1} - T^0\right) + \frac{\partial T_i^Y}{\partial\left(P_2/a_2\right)}\left(\frac{Y}{P_1/a_1} - \frac{Y}{P_2/a_2'}\right) \tag{17}$$

Proposition 5

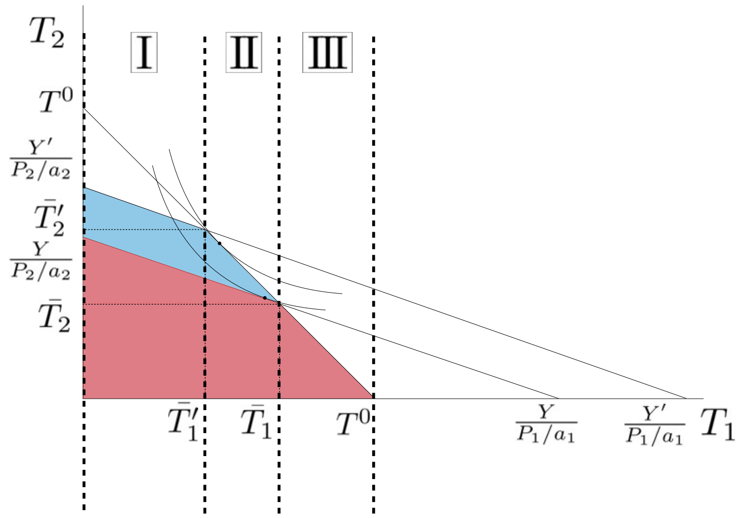
Suppose $T_1^* \in \left[\bar{T}_1, \bar{T}_1'\right]$, then $\frac{dT_l^*}{da_i} = -\frac{dT_l^*}{dP_i}$ $(i = 1, 2 \text{ and } l = 1, 2)$.

3.2 Income

Fig. 3-6 shows the case that an increase in income shifts the budget line outward $(Y \to Y')$. The constraint line shifts outward without rotating in area I. It is the same as the case without the time constraint. In area II, the slope steepens. And there is no change in III. The effect of the change in area II is separated as follows:

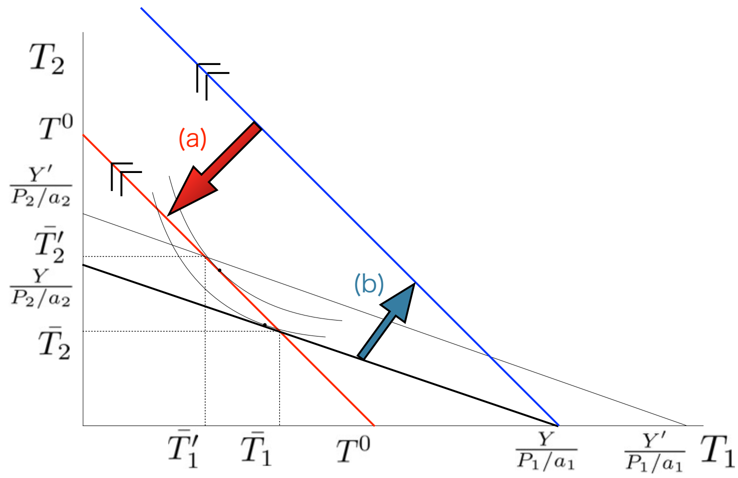
$$\frac{dT_i^*}{dY} = -\frac{\partial T_i^T}{\partial T^0}\left(\frac{Y}{P_1/a_1} - T^0\right) - \frac{\partial T_i^Y}{\partial\left(P_2/a_2\right)}\left(\frac{Y}{P_1/a_1} - \frac{Y}{P_2/a_2}\right) (i = 1, 2) \tag{18}$$

The first and the second term respectively correspond to (a) and (b) in Fig. 3-7.



T_2
I
II
III
T^0
$\frac{Y'}{P_2/a_2}$
$\bar{T}_2'$
$\frac{Y}{P_2/a_2}$
$\bar{T}_2$
$\bar{T}_1'$
$\bar{T}_1$
T^0
$\frac{Y}{P_1/a_1}$
$\frac{Y'}{P_1/a_1}$
T_1

Fig. 3-6



T_2
T^0
(a)
(b)
$\frac{Y'}{P_2/a_2}$
$\bar{T}_2'$
$\frac{Y}{P_2/a_2}$
$\bar{T}_2$
$\bar{T}_1'$
$\bar{T}_1$
T^0
$\frac{Y}{P_1/a_1}$
$\frac{Y'}{P_1/a_1}$
T_1

Fig. 3-7

3.3 Disposable Time

Fig. 3-8 shows the case that an increase in the disposable time shifts the time constraint line ($T^0 \rightarrow T^{0\prime}$).

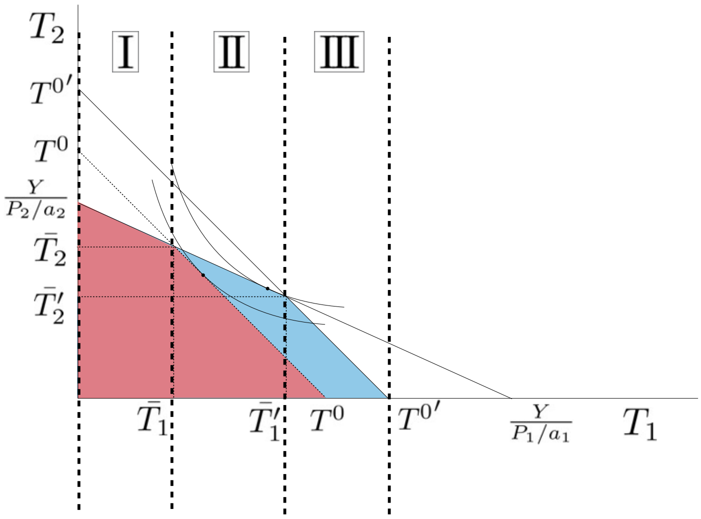


Fig. 3-8

There is no change in area I. The slope of the constraint flattens in area II. It is contrary to the case of income change. In area III, the change shifts the constraint outward. In area II, the effect of the disposable time change can be shown by the two effects:

$$\frac{dT_i^*}{dT^0} = \frac{\partial T_i^T}{\partial T^0}\left(\frac{Y}{P_1/a_1} - T^0\right) + \frac{\partial T_i^Y}{\partial\left(P_2/a_2\right)}\left(\frac{Y}{P_1/a_1} - \frac{Y}{P_2/a_2}\right)(i = 1, 2) \tag{19}$$

And the figure is:

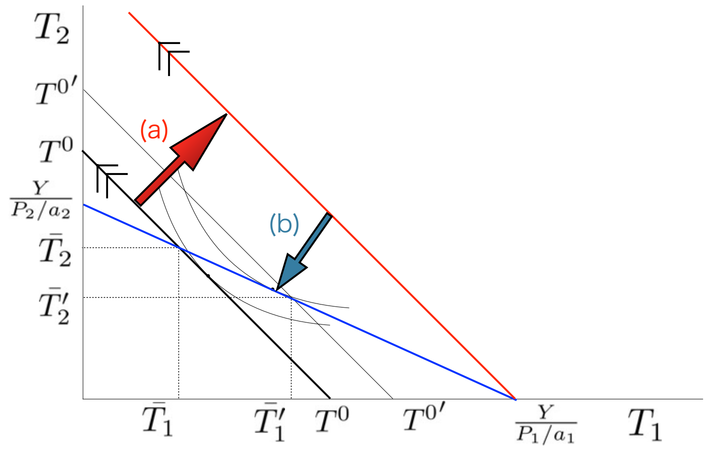


Fig. 3-9

Also note that the difference between the income change and the disposable time is in their directions.

Proposition 6
Suppose $T_1^* \in \left[\bar{T}_1, \bar{T}_1'\right]$, then $\frac{dT_i^*}{dT^0} = -\frac{dT_i^*}{dY}$ $(i = 1, 2)$.

4 Working Time and Income

In this section we analyze the case in which the consumer can transfer her income and disposable time. The consumer can earn more money by sacrificing her time to enjoy consumption goods, or increase the disposable time by reduction of income(/working time). And the consumer adjusts the allocation of time as far as it improves her utility level.

Here we introduce working time T_w and modify the time constraint as follows:

$$T^0 = T_w + T_c \tag{20}$$
$$T_c = T_1 + T_2 \tag{21}$$

Increasing the income by reduction of time for consumption can be expressed by the combination of an inner shift of the time constraint line and an outer shift of the budget line. In this case, the intersection point of the constraint lines moves leftward $\left((\bar{T_1}, \bar{T_2}) \to (\bar{T_1}', \bar{T_2}')\right)$. The intersection point is rewritten as $(\bar{T_1}, \bar{T_2}) = \left(\frac{T_c P_2/a_2 - Y(T_w)}{P_2/a_2 - P_1/a_1}, \frac{Y(T_w) - T_c P_1/a_1}{P_2/a_2 - P_1/a_1}\right)$. $Y(T_w)$ means the income is a function of the working time, T_w. Then, $\frac{\partial \bar{T_1}}{\partial T_w} = \frac{-P_2/a_2 - dY/dT_w}{P_2/a_2 - P_1/a_1} < 0$ and $\frac{\partial \bar{T_2}}{\partial T_w} = \frac{dY/dT_w + P_1/a_1}{P_2/a_2 - P_1/a_1} > 0$. It is shown in Fig. 4-1.

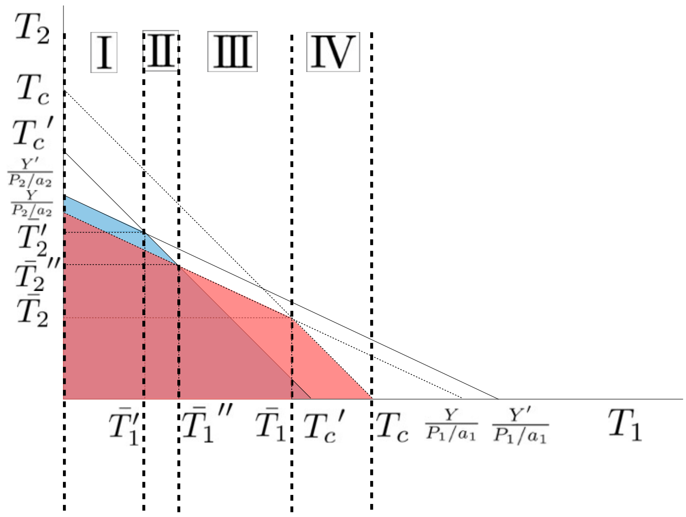


Fig. 4-1

We also show the pictures before and after the change respectively, in Fig. 4-2 and Fig. 4-3.

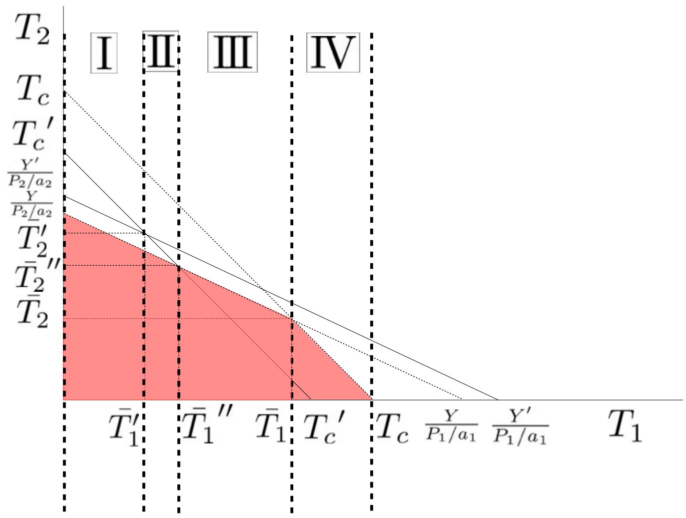


Fig. 4-2

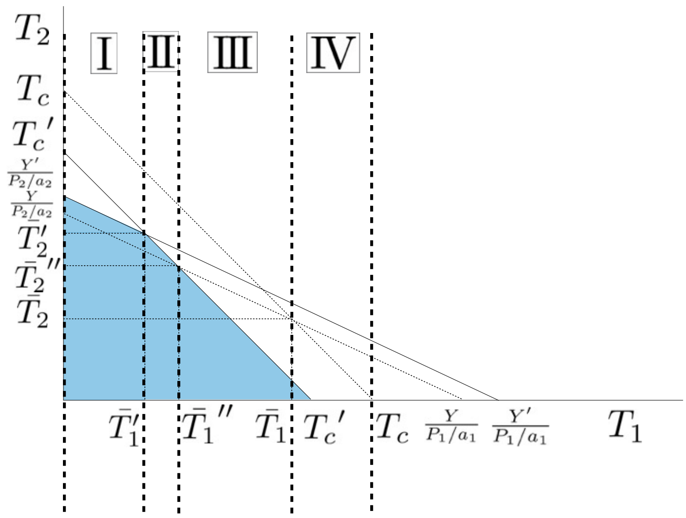


Fig. 4-3

The change of the binding constraint line is categorized into the following four cases:

1. The constraint shifts outward in I, same as in area I when income increases.
2. The effect of the change can be shown by
$$-\frac{\partial T_1^T}{\partial T^0}\left(\frac{Y}{P_1/a_1}-T_c'\right)-\frac{\partial T_1^Y}{\partial\left(P_2/a_2\right)}\left(\frac{Y}{P_1/a_1}-\frac{Y}{P_2/a_2}\right),$$
same as in II when income increases.
3. The effect of the change can be shown by
$$\frac{\partial T_1^T}{\partial T^0}\left(\frac{Y}{P_1/a_1}-T_c'\right)+\frac{\partial T_1^Y}{\partial\left(P_2/a_2\right)}\left(\frac{Y}{P_1/a_1}-\frac{Y}{P_2/a_2}\right),$$
same as in II when disposable time increases.
4. The constraint shifts inward in IV, same as in III when disposable time decreases.

The switching happens in both II and III; time to budget, and budget to time, respectively.

The boundary between II and III is the intersection point of the budget line before the change and the time constraint line after the change:

$$\left(\bar{T_1}'',\bar{T_2}''\right)=\left(\frac{T_c'P_2/a_2-Y}{P_2/a_2-P_1/a_1},\frac{Y-T_c'P_1/a_1}{P_2/a_2-P_1/a_1}\right)\tag{22}$$

We have summarized the conditions that the individual changes her working time by locating the optimal point before the change. Note that we have precluded the corner solution cases in I and IV for simplicity.

The condition that the individual increases or decreases working time in each area is as follows:

(I) if the optimal point before the working adjustment locates on the budget line; the optimal point exists in area I.

(II) if the switching from the budget constraint to the time constraint improves the utility level in area II.

(III) if the switching from the time constraint to the budget constraint improves the utility level in area III.

(IV) if the optimal point before the working adjustment locates on the budget line; the optimal point exists in area IV.

If the adjustment is complete, the optimal point after the adjustment is the intersection point of the constraint lines after the change:

$$(\bar{T_1}^*, \bar{T_2}^*) = \left(\bar{T_1}', \bar{T_2}'\right) \tag{23}$$

The reason is because, the individual can improve her utility level by reducing the working time and increasing the disposable time if the consumption point is in III or IV; the slope of the indifference curve which expresses the utility level at the consumption point is larger than one. Contrarily, the consumer can improve her utility by sacrificing the disposable time and working longer if the consumption point is in I or II; if the slope is less than $\frac{P_1/a_1}{P_2/a_2}$. It is summarized by the following condition:

$$MRS(T_1^*, T_2^*) = MRS(\bar{T_1}', \bar{T_2}') \in \left[\frac{P_1/a_1}{P_2/a_2}, 1\right] \tag{24}$$

where

$$MRS(T_1, T_2) = \frac{\partial U(T_1, T_2)/\partial T_1}{\partial U(T_1, T_2)/\partial T_2} \tag{25}$$

The optimal solution including the optimal working time is as follows:

$$
\begin{aligned}
(T_1^*, T_2^*, T_w^*) &= (\bar{T_1}', \bar{T_2}', T_w^*) \\
&= \left(\frac{T_c' P_2/a_2 - Y'}{P_2/a_2 - P_1/a_1}, \ \frac{Y' - T_c' P_1/a_1}{P_2/a_2 - P_1/a_1}, \ T^0 - T_c' \right) \\
& s.t. \quad MRS(T_1^*, T_2^*) = MRS(\bar{T_1}', \bar{T_2}') \in \left[\frac{P_1/a_1}{P_2/a_2}, 1 \right]
\end{aligned} \tag{26}
$$

5 Saving

Now we consider saving by augmenting my model to two periods. Suppose $T_1^0 = T_2^0$(; the exogenous time resources are the same in the two period) and the consumer does not work. We modify the utility function to express the two periods. The utility function of each period is consistently $u(\cdot)$, and shown by

$$
U(T_{11}, T_{21}, T_{21}, T_{22}) = u(T_{11}, T_{21}) + \delta u(T_{12}, T_{22}) \tag{27}
$$

where T_{ij} represents the time for good i in period j ($i = 1, 2$ and $j = 1, 2$). δ is the discount factor between the two periods. If there is no revenue in the second period, then the budget constraint through the two periods is expressed by the income in each period, Y_j ($j = 1, 2$), saving, S, and interest rate, r:

$$
Y_2 = (1+r)S = (1+r)\left(Y_1 - \frac{P_1}{a_1} T_{11} - \frac{P_2}{a_2} T_{21} \right) \tag{28}
$$

and

$$
(1+r)\left(Y_1 - \frac{P_1}{a_1} T_{11} - \frac{P_2}{a_2} T_{21} \right) \geq \frac{P_1}{a_1} T_{12} + \frac{P_2}{a_2} T_{22} \tag{29}
$$

Note that time cannot be saved although the constraint is expressed in terms of times, T_{ij}. The following condition must hold.

$$
T_j^0 \geq T_{1j} + T_{2j}, \ \forall j = 1, 2 \tag{30}
$$

Note also that income and time are asymmetric; only income can be saved.

5.1 Shortage of Budget

Suppose that $\delta = 1$ and $r = 0$ for simplicity. Then

$$Y_2 = Y_1 - \frac{P_1}{a_1} T_{11} - \frac{P_2}{a_2} T_{21} \tag{31}$$

and

$$Y_1 - \frac{P_1}{a_1} T_{11} - \frac{P_2}{a_2} T_{21} \geq \frac{P_1}{a_1} T_{12} + \frac{P_2}{a_2} T_{22} \tag{32}$$

The optimization for the consumer is to achieve (T^*_{1j}, T^*_{2j}) such that $MRS(T^*_{1j}, T^*_{2j}) \in \left[\frac{P_1/a_1}{P_2/a_2}, 1\right]$ $(\forall j = 1, 2)$. The optimal point is not necessarily the intersection point after the adjustment by saving or borrowing.

Obviously, the more disposable time the individual has, the more money is required to achieve the ideal(/possible/potential) time consumption. In this case, the ratio of disposable time, (T_{cj}), and required income, $(\underline{Y}_j)$, of each period is the same. The following equality shows the relationship.

$$\frac{T_{c1}}{T_{c2}} = \frac{\underline{Y}_1}{\underline{Y}_2} \tag{33}$$

If she has enough money that makes the budget constraints in each period slack, then she spends time to each good by the same ratio in each period:

$$\frac{T^*_{11}}{T^*_{21}} = \frac{T^*_{12}}{T^*_{22}} \tag{34}$$

In other words, the optimal point divides the time constraint line by a constant ratio in each period.

However, if money is scarce in at least one of the periods, the optimal point is the tangent point of the indifferent curve and the budget constraint line, which locates left of the intersection point in each period. Then $Y_j = \frac{P_1}{a_1}T_{1j} + \frac{P_2}{a_2}T_{2j}$ $(j = 1, 2)$ holds, and the marginal utility from budget increasing is positive in each period after the consumption smoothing. The optimal solution is decided by the equalization of marginal utility with marginal costs:

$$\frac{\partial u\left(T_{11}(Y_1), T_{21}(Y_1)\right)}{\partial Y_1} + \frac{\partial u\left(T_{21}(Y_2), T_{22}(Y_2)\right)}{\partial Y_2} = 0 \tag{35}$$

If $\delta \in [0, 1]$ and $r \geq 0$, then the consumption smoothing cannot always achieve $\frac{T^*_{11}}{T^*_{21}} = \frac{T^*_{12}}{T^*_{22}}$. [*3] And the condition of the optimization is:

$$\frac{\partial u\left(T_{11}(Y_1), T_{21}(Y_1)\right)}{\partial Y_1} + \delta \frac{\partial u\left(T_{21}(Y_2), T_{22}(Y_2)\right)}{\partial Y_2}\frac{dY_2}{dY_1} = 0 \tag{36}$$

5.2 Liquidity Constraint

Under the liquidity constraint on saving and borrowing, the consumer cannot save money enough to carry out the consumption smoothing, and the optimal point in the second period locates on the budget line, left of the intersection point, and some amount of the time resource is left unused. It can be shown by the following

[*3] There are some exceptions. For example, if the preferences are contemporaneously homothetic in each period, then $\frac{T^*_{11}}{T^*_{21}} = \frac{T^*_{12}}{T^*_{22}}$ always holds.

conditions.

$$MRS(T_{11}^*, T_{21}^*) \in \left[\frac{P_1/a_1}{P_2/a_2}, 1\right] \tag{37}$$

and

$$MRS(T_{12}^*, T_{22}^*) < \frac{P_1/a_1}{P_2/a_2} \tag{38}$$

where

$$MRS(T_{1j}, T_{2j}) = \frac{\partial U(T_{1j}, T_{2j})/\partial T_{1j}}{\partial U(T_{1j}, T_{2j})/\partial T_{2j}}, \quad j = 1, 2 \tag{39}$$

5.2.1 Example: age-based wage

If income is fixed by age-based wage system, the increase in income by aging can be shown by $Y_1 < Y_2$. Suppose that the preference of the consumer, the prices of the goods, (a_1, a_2) and working time for the consumer are consistent through the two periods. Then the time constraint and the slope of the budget constraint remain unchanged through the two periods. And the difference between the two periods is caused only by the difference of Y_j. In this case, because the consumer cannot adjust the working time, she tries to improve her utility level by borrowing money in the first period and pay back from the excess budget in the second period, if the optimal consumption without the borrowing is the tangency point on the budget line(; the consumer face the budget constraint) in the first period. In other words, the consumer has a chance to improve her utility by utilizing the liquidity. On the other hand, if the optimal point is on the time constraint line(; the consumer face the time constraint) in the first period, she cannot conduct any improvement. And the liquidity constraint has effect only in the former case.

5.3 Other Applications

- The cases in which each period is long enough for the prices or the time consumption parameter of each good to change from the first to the second periods. For example, the price of only one of the goods changes by inflation, or the parameter change, $a_{i1} \neq a_{i2}$, by technological innovation.
- The cases that T_j has an effect on a_{j+1}, by job training or R&D investment.
- The cases that the necessary time input for each good is a function, $T_i = T_i(X_i)$, instead of $T_i = a_i X_i$.

6 Concluding Remarks

In this paper we have constructed a model which includes time constraint in addition to budget constraint, and conducted the comparative statics analyses. The main result in the comparative statics is that the switching of the binding constraint can cause an increase in the demand of the "time-intensive" good when its price rises. We have further analyzed the case that the consumer can adjust working time. The consumer faces the trade-off between working(/disposable time) and income. The adjustment of the budget constraint and the time constraint, via exchanging time for money, improves the consumer's utility level. Finally we extend the model to two periods. The consumer improves her total utility through the two periods by borrowing or saving. We also find that the improvement is possible only if the consumer has time left unconsumed in that period when the budget constraint binds and this can be relaxed by liquidity. As further research, application to producer or government behavior is expected.

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