Etingof, Pavel; **Gelaki, Shlomo**; **Nikshych, Dmitri**; **Ostrik, Victor**
Tensor categories. (English) Zbl 1365.18001
Mathematical Surveys and Monographs 205. Providence, RI: American Mathematical Society (AMS) (ISBN 978-1-4704-2024-6/hbk). xvi, 343 p. (2015).

Tensor categories, which are ubiquitous in noncommutative algebra and representation theory and which play a significant role in such various areas as algebraic geometry, algebraic topology, number theory, mathematical physics and theoretical computer science, occupy the position of rings in classical algebra within the realm of categories. The notion of a monoidal category dates back as far as [*S. MacLane*, Rice Univ. Stud. 49, No. 4, 28–46 (1963; Zbl 0244.18008)] and first appeared in book form in [*S. Mac Lane*, Categories for the working mathematician. New York-Heidelberg-Berlin: Springer- Verlag (1971; Zbl 0232.18001)]. A bit later, Saavedra Rivano, motivated by the needs of the theory of motives, developed a theory of Tannakian categories (symmetric monoidal structures on abelian categories) in his thesis under the direction of Grothendieck [*N. Saavedra Rivano*, Bull. Soc. Math. Fr. 100, 417–430 (1972; Zbl 0246.14003); Categories tannakiennes. Berlin-Heidelberg-New York: Springer-Verlag (1972; Zbl 0241.14008)], which was simplified and elaborated in [*P. Deligne* and *J. S. Milne*, Lect. Notes Math. 900, 101–228 (1982; Zbl 0477.14004)]. Shortly afterwards, the theory of tensor categories (monoidal abelian categories) became a vital topic with flamboyant connections to representation theory, quantum groups, infinite dimensional Lie algebras, conformal field theory, vertex operator algebras, operator algebras, invariants of knots and 3-dimensional manifolds, number theory and so on. Another major source of inspiration for the theory of tensor categories comes from Hopf algebras. The authors want to emphasize that many important results about Hopf algebras are better understood through the prism of tensor categories, deducing many of the most important results about Hopf algebras (the fundamental theorem on Hopf modules and bimodules, Nichols-Zoeller theorem, Larson-Radford theorems, Radford's S^4 formula, the Kac-Zhu theorem, and so on) as corollaries of the general theory of tensor categories. This excellent book, concerned with tensor categories and intended as an accessible introduction to them, consists of 9 chapters. Chapters 7–9 occupy the core of the book. The book consists of a bit more than 300 pages, and the three chapters occupy a bit less than 200 pages.

Chapter 7 first develops the theory of module categories over monoidal categories and passes to the abelian setting, introducing the key notion of an exact module category over a finite tensor category. It is shown that module categories arise as categories of modules over algebras in tensor categories, putting algebras in tensor categories in focus. The category of module functors between module categories is then investigated with the Drinfeld center construction as a significant special case. Dual categories and categorical Morita equivalence of tensor categories are then discussed together with the fundamental theorem for Hopf modules and bimodules over a Hopf algebra as well as the categorical version of *D. E. Radford*'s [Am. J. Math. 98, 333–355 (1976; Zbl 0332.16007)] S^4 formula. The theory of categorical dimensions of fusion categories and that of Davydov-Yetter cohomology [*D. N. Yetter*, Adv. Math. 174, No. 2, 266–309 (2003; Zbl 1017.18006); Contemp. Math. 230, 117–134 (1998; Zbl 0927.18003)] and deformations of tensor categories are finally developed.

Chapter 8 is concerned with the theory of braided categories. Discussing point braided categories and quasitriangular Hopf algebras, it is shown that the center of a tensor category is a braided category. The theory of commutative algebras in braided categories is developed, and it is established that the modules over such an algebra form a tensor category. The theory of factorizable, ribbon and modular categories, the S-matrix, Gauss sums is also developed together with the Verlinde formula and the existence of an $\mathrm{SL}_2(\mathbb{Z})$-action. The Anderson-Moore-Vafa theorem, claiming that the central charge and twists of a modular category are roots of unity, is established. Centralizers and projective centralizers in braided categories, de-equivariantization of braided categories and braided G-crossed categories are disucussed at last.

Chapter 9 is mostly concerned with fusion categories. This chapter is oriented towards applications of various tools from the previous chapters to bear on concrete problems about fusion categories. The chapter begins with Ocneanu rigidity, discussing vanishing of the Davydov-Yetter cohomology and lacking of deformations for multifusion categories. The theory of dual categories and that of pseudo-unitary categories are then developed. The authors next study integral and weakly integral fusion categories as well as group-theoretical and weakly group-theoretical fusion categories. They then study symmetric and Tannakian fusion categories, establishing Deligne's theorem on classification. A criterion of group-theoreticity of a fusion category is then given, and it is shown that any integral fusion category of prime power dimension is indeed group theoretical. Introducing the notion of a solvable fusion category, a categorical analog of Burnside's theorem, claiming that a fusion category of dimension $p^a q^b$ with p, q being primes, is solvable, is established. Lifting theory for fusion categories from characteristic p to characteristic zero is finally discussed.

Chapter 1 discusses the basics about abelian categories, usually presenting results without proofs and giving instead a more detailed discussion about the theory of locally finite categories, coalgebras, the coend construction and the reconstruction theory for coalgebras. Chapter 2 is a hasty course on monoidal categories. Chapter 3 deals with the theory of $\mathbb{Z}_+$-rings (rings with a basis in which the structure constants are nonnegative integers), serving as Grothendieck rings of tensor categories. This chapter has nothing to do with categories. Chapter 4 is concerned with the general theory of multitensor categories, introducing a few key notions and constructions such as pivotal and spherical structures, Frobenius-Perron dimensions of objects and categories, categorification, and so on. In Chapter 5 the authors consider tensor categories with a fiber functor (a tensor functor to the category of vector spaces), leading to the notion of Hopf algebras. Chapter 6 develops the theory of finite tensor categories (tensor categories which are equivalent, as an abelian category, to the category of representations of a finite dimensional Hopf algebra).

A minor complaint of the reviewer is that in the bibliography Deligne's four papers are listed with headers [De1]–[De4], but the work of Deligne and Milne on Tannakian categories is headed by [DelM], which is somewhat irritating.

Reviewer: Hirokazu Nishimura (Tsukuba)

MSC:

18-02	Research monographs (category theory)	Cited in **4** Reviews
18D10	Monoidal, symmetric monoidal and braided categories	Cited in **118** Documents
16T05	Hopf algebras and their applications	

Full Text: DOI