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Spectral analysis of hypoelliptic Višik–Ventcel' boundary value problems

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Dedicated to Professor Bernard Helffer on the occasion of his 70th birthday

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Abstract This paper is devoted to the study of a class of *hypoelliptic* Višik–Ventcel' boundary value problems for second order, uniformly elliptic differential operators. Our boundary conditions are supposed to correspond to the diffusion phenomenon along the boundary, the absorption and reflection phenomena at the boundary in probability. If the absorbing boundary portion is not a *trap* for Markovian particles, then we can prove two existence and uniqueness theorems of the non-homogeneous Višik–Ventcel' boundary value problem in the framework of L^2 Sobolev spaces. Moreover, if the absorbing boundary portion is *empty*, then we can prove a generation theorem of analytic semigroups for the closed realization of the uniformly elliptic differential operator associated with the hypoelliptic Višik–Ventcel' boundary condition in the L^2 topology. As a by-product, this paper is the first time to prove the angular distribution of eigenvalues, the asymptotic eigenvalue distribution and the completeness of generalized eigenfunctions of the closed realization, similar to the elliptic (non-degenerate) case.

Keywords Višik–Ventcel' boundary value problem · hypoelliptic operator · analytic semigroup · asymptotic eigenvalue distribution · strong Markov process

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1 Formulation of the Višik–Ventcel' boundary value problem

Let Ω be a bounded domain of Euclidean space $\mathbf{R}^n$, $n \geq 2$, with smooth boundary $\Gamma = \partial\Omega$; its closure $\overline{\Omega} = \Omega \cup \Gamma$ is an n-dimensional, compact smooth manifold with boundary.

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Let $A = A(x, D)$ be a second order, *uniformly elliptic* differential operator with real coefficients on the closure $\overline{\Omega}$ such that

$$Au = \sum_{i=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u. \tag{1.1}$$

Here:

(1) $a^{ij} \in C^\infty(\overline{\Omega})$ and $a^{ij}(x) = a^{ji}(x)$ for all $x \in \overline{\Omega}$ and $1 \leq i, j \leq n$, and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq a_0 \, |\xi|^2 \quad \text{for all } (x, \xi) \in \overline{\Omega} \times \mathbf{R}^n.$$

(2) $b^i \in C^\infty(\overline{\Omega})$ for all $1 \leq i \leq n$.
(3) $c \in C^\infty(\overline{\Omega})$ and $c(x) \leq 0$ in Ω.

In this paper, we consider a boundary condition $B\boldsymbol{\gamma}$ of the form

$$B\boldsymbol{\gamma}u = \mu(x') \frac{\partial u}{\partial \boldsymbol{\nu}}\bigg|_{\Gamma} + Q\left(x', D_{x'}\right)(u|_{\Gamma}). \tag{1.2}$$

Here:

(4) $\mu \in C^\infty(\Gamma)$ and $\mu(x') \geq 0$ on Γ.
(5) $\partial/\partial\boldsymbol{\nu}$ is the conormal derivative associated with the operator A:

$$\frac{\partial}{\partial \boldsymbol{\nu}} = \sum_{i,j=1}^{n} a^{ij}(x') n_j \frac{\partial}{\partial x_i},$$

where $\mathbf{n} = (n_1, n_2, \ldots, n_n)$ is the unit *inward* normal to the boundary Γ.
(6) $Q = Q\left(x', D_{x'}\right)$ is a second order, differential operator with real coefficients defined on Γ such that, in terms of local coordinate systems

$$x' = (x_1, x_2, \ldots, x_{n-1})$$

of Γ, we have the formula

$$Q\varphi = \sum_{i,j=1}^{n-1} \alpha^{ij}(x') \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{i=1}^{n-1} \beta^i(x') \frac{\partial \varphi}{\partial x_i} + \gamma(x')\varphi, \tag{1.3}$$

where the coefficients satisfy the following conditions:
(a) The $\alpha^{ij}(x')$ are the components of a C^∞ symmetric contravariant tensor of type $\binom{2}{0}$ on Γ and satisfy the condition

$$\sum_{i,j=1}^{n-1} \alpha^{ij}(x')\, \xi_i \, \xi_j \geq 0 \quad \text{on the cotangent bundle } T^*(\Gamma) = \textstyle\bigsqcup_{x' \in \Gamma} T_{x'}(\Gamma),$$

where $T^*_{x'}(\Gamma)$ is the cotangent space of Γ at x'.
(b) $\beta\left(x', D_{x'}\right) = \sum_{i=1}^{n-1} \beta^i(x')\partial/\partial x_i$ is a real C^∞ vector field on Γ.
(c) $\gamma = Q1 \in C^\infty(\Gamma)$ and $\gamma(x') \leq 0$ on Γ.

The boundary condition $B\boldsymbol{\gamma}$ is called a *Višik–Ventcel' boundary condition* (see [59], [60], [7]). The three terms of the boundary condition $B\boldsymbol{\gamma}$

$$\sum_{i,j=1}^{n-1} \alpha^{ij}(x') \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n-1} \beta^i(x') \frac{\partial u}{\partial x_i}, \quad \gamma(x')u, \quad \mu(x') \frac{\partial u}{\partial \boldsymbol{\nu}}$$

are supposed to correspond to the diffusion phenomenon along the boundary, the absorption phenomenon and the reflection phenomenon, respectively (see [50]).

The first purpose of this paper is to study the following *non-homogeneous* Višik–Ventcel' boundary value problem: Given functions $f(x)$ defined in Ω and $\psi(x')$ defined on Γ, respectively, find a function $u(x)$ in Ω such that

$$\begin{cases} Au = f & \text{in } \Omega, \\ B\boldsymbol{\gamma} u = \psi & \text{on } \Gamma. \end{cases} \tag{1.4}$$

We remark that the Višik–Ventcel' boundary value problem (1.4) is non-degenerate or coercive if and only if the differential operator $Q(x', D_{x'})$ is *elliptic* on Γ, that is, there exists a constant $\alpha_0 > 0$ such that

$$\sum_{i,j=1}^{n-1} \alpha^{ij}(x')\, \xi_i \, \xi_j \geq \alpha_0 |\xi'|^2 \quad \text{on the cotangent bundle } T^*(\Gamma).$$

The non-degenerate case is studied by Višik [59, Section 8], Hörmander [21, p. 264, problem (10.5.13)], Agranovich–Vishik [4, p. 69, formula (3.11)] and Bony–Courrège–Priouret [7, p. 436, formula (II.2.1)].

In this paper, if the boundary portion

$$\Gamma_0 := \{x' \in \Gamma : \mu(x') = 0\}$$

is not a *trap* for Markovian particles, then we can prove two existence and uniqueness theorems of the non-homogeneous Višik–Ventcel' boundary value problem (1.4) in the framework of L^2 Sobolev spaces (Theorems 2.2 and 2.3).

The second purpose of this paper is to study the following *homogeneous* Višik–Ventcel' boundary value problem in the framework of L^2 Sobolev spaces: Given a function $f(x)$ defined in Ω, find a function $u(x)$ in Ω such that

$$\begin{cases} (A - \lambda)\, u = f & \text{in } \Omega, \\ B\boldsymbol{\gamma} u = 0 & \text{on } \Gamma, \end{cases} \tag{1.5}$$

where λ is a *complex spectral parameter*.

In this paper, if the boundary portion Γ_0 is *empty*, then we can prove a generation theorem of *analytic semigroups* for the closed realization associated with the Višik–Ventcel' boundary value problem (1.5) in the L^2 topology (Theorem 2.4). Moreover, this paper is the first time to prove the angular distribution of eigenvalues, the asymptotic eigenvalue distribution and the completeness of generalized eigenfunctions of the closed realization (Theorem 2.5), similar to the non-degenerate case. These rather surprising results for a degenerate problem work, since the degeneracy occurs only for the boundary data.

Our approach here is distinguished by the extensive use of the ideas and techniques characteristic of the recent developments in the theory of pseudo-differential operators, which will lead to a deep insight into the study of three interrelated subjects in analysis: Semigroups, elliptic boundary value problems and Markov processes.

2 Statement of main results

The purpose of this section is to formulate two existence and uniqueness theorems of the Višik–Ventcel' boundary value problem (1.4) in the framework of L^2 Sobolev spaces (Theorems 2.2 and 2.3). As an application of Theorems 2.2 and 2.3, we state a generation theorem of *analytic semigroups* for the Višik–Ventcel' boundary value problem (1.5) in the L^2 topology (Theorem 2.4), generalizing earlier results due to Agranovich–Vishik [4] to the hypoelliptic case. Finally, we state five spectral properties of the Višik–Ventcel' boundary value problem (1.5), similar to the elliptic case (Theorem 2.5).

2.1 Subelliptic estimates for second order differential operators in terms of subunit trajectories

In this subsection, following Fefferman–Phong [15] we formulate subelliptic estimates for the second order, differential operator $Q = Q\left(x', D_{x'}\right)$ with real coefficients given by formula (1.3).

A tangent vector

$$X = \sum_{j=1}^{n-1} \gamma^j \frac{\partial}{\partial x_j} \in T_{x'}(\varGamma)$$

at $x' \in \varGamma$ is said to be *subunit* for the differential operator Q if it satisfies the condition

$$\left(\sum_{j=1}^{n-1} \gamma^j \eta_j \right)^2 \leq \sum_{i,j=1}^{n-1} \alpha^{ij}(x') \eta_i \eta_j \quad \text{for all } \eta = \textstyle\sum_{j=1}^{n-1} \eta_j dx_j \in T^*_{x'}(\varGamma).$$

Note that this notion is coordinate-free.

A *subunit trajectory* is a Lipschitz path

$$\gamma\colon\ [t_1, t_2] \longrightarrow \varGamma$$

such that the tangent vector

$$\dot{\gamma}(t) = \frac{d}{dt}\left(\gamma(t)\right)$$

is subunit for Q at $\gamma(t)$ for almost every t. We remark that if $\dot{\gamma}(t)$ is subunit for Q, so is $-\dot{\gamma}(t)$. This implies that subunit trajectories are not oriented.

If x' is a point of $\varGamma$ and $\rho > 0$, then we associate a "non-Euclidean" ball $B_Q(x', \rho)$ of radius ρ about x' by the following formula:

$$\begin{aligned} B_Q(x', \rho) = {} & \text{the set of all points } y' \in \varGamma \text{ that can be joined to } x' \text{ by} \\ & \text{a Lipschitz path } v\colon [0, \rho] \to \varGamma, \text{ for which the tangent} \\ & \text{vector } \dot{v}(t) \text{ is subunit for } Q \text{ at } v(t) \text{ for almost every } t. \end{aligned}$$

If Q is the Laplace–Beltrami operator $\varDelta'$ on the boundary $\varGamma$, then we find that $B_{\varDelta'}(x', \rho)$ coincides with the usual ball of radius ρ about x' with respect to the Riemannian metric of $\varGamma$ induced by the natural metric of $\mathbf{R}^n$.

This paper is based on the following criterion for *subellipticity* due to Fefferman–Phong [15, Theorem 1] (see Sections 3 and 4):

Theorem 2.1 (Fefferman–Phong) *Let $Q = Q\left(x', D_{x'}\right)$ be a second order, differential operator given by formula* (1.3)*. Then the following three conditions are equivalent:*

(i) There exist constants $0 < \varepsilon \leq 1$ and $C' > 0$ such that we have, for $\rho > 0$ sufficiently small,
$$B_E(x', \rho) \subset B_Q\left(x', C' \rho^{\varepsilon}\right) \quad \text{for every } x' \in \Gamma. \tag{2.1}$$
Here $B_E(x', \rho)$ is an ordinary Euclidean ball of radius ρ about x'.

(ii) There exist constants $c_0 > 0$ and $C_0 > 0$ such that
$$-\mathrm{Re}\left(Q(x', D_{x'})\varphi, \varphi\right)_{L^2(\Gamma)} \geq c_0 \|\varphi\|^2_{H^{\varepsilon}(\Gamma)} - C_0 \|\varphi\|^2_{L^2(\Gamma)} \quad \text{for all } \varphi \in C^{\infty}(\Gamma). \tag{2.2}$$

(iii) There exist constants $c_1 > 0$ and $C_1 > 0$ such that
$$\left\|Q(x', D_{x'})\varphi\right\|_{L^2(\Gamma)} + C_1 \|\varphi\|^2_{L^2(\Gamma)} \geq c_1 \|\varphi\|^2_{H^{2\varepsilon}(\Gamma)} \text{ for all } \varphi \in C^{\infty}(\Gamma).$$

Here $(\cdot, \cdot)_{L^2(\Gamma)}$ is the inner product of the Hilbert space $L^2(\Gamma) = H^0(\Gamma)$ and $\|\cdot\|_{H^s(\Gamma)}$ is the norm of the Sobolev space $H^s(\Gamma)$ of order s on Γ, respectively.

2.2 Existence and uniqueness theorems for the Višik–Ventcel' boundary problem (1.4)

(I) The first purpose of this paper is to prove the following existence and uniqueness theorem for the Višik–Ventcel' boundary value problem (1.4) in the framework of L^2 Sobolev spaces:

Theorem 2.2 *Assume that the following two conditions (H.1) and (H.2) are satisfied:*

(H.1) $\mu(x') \geq 0$, $\gamma(x') \leq 0$ and $\mu(x') - \gamma(x') > 0$ on Γ.

(H.2) There exists an open neighborhood V of the boundary portion $\Gamma_0 = \{x' \in \Gamma : \mu(x') = 0\}$ such that we have, for $\rho > 0$ sufficiently small,
$$B_E(x', \rho) \subset B_Q\left(x', C' \rho^{1/2}\right) \quad \text{for every } x' \in V. \tag{2.3}$$

If the condition
$$c(x) \leq 0 \text{ and } c(x) \not\equiv 0 \quad \text{in } \Omega \tag{2.4}$$
is satisfied, then the mapping
$$\boxed{\mathcal{A} = (A, B\boldsymbol{\gamma}) : H^{s+2}(\Omega) \longrightarrow H^s(\Omega) \oplus H^{s+1/2}(\Gamma)} \tag{2.5}$$
is bijective *for every $s > -1/2$. In other words, the Višik–Ventcel' boundary value problem* (1.4) *has a unique solution $u \in H^{s+2}(\Omega)$ for any $f \in H^s(\Omega)$ and any $\varphi \in H^{s+1/2}(\Gamma)$. Here $H^s(\Omega)$ denotes the Sobolev space of order s in Ω.*

It should be emphasized that every solution u of the Višik–Ventcel' boundary value problem (1.4) has the elliptic gain of 2 derivatives from f. This rather surprising result for a degenerate problem works, since the degeneracy occurs only for the boundary data φ.

Remark 2.1 Some remarks are in order.

1° The intuitive meaning of condition (H.1) is that either the absorption phenomenon or the reflection phenomenon occurs at each point of the boundary Γ. More precisely, condition (H.1) implies that the absorption phenomenon may occur at each point of the boundary portion Γ_0, while the reflection phenomenon may occur at each point of the boundary portion $\Gamma \setminus \Gamma_0 = \{x' \in \Gamma : \mu(x') > 0\}$ (cf. [50]).
2° Condition (2.3) is just condition (2.1) with $\varepsilon := 1/2$, and the constant C' depends on the open neighborhood V of Γ_0.
3° Condition (H.2) implies that a Markovian particle goes out of the absorbing barrier Γ_0 in *finite time*, so that Γ_0 is not a *trap* for Markovian particles (see [47, Remark 5.2] and [49, Theorem 7.2.2], [54, Section 6]).

(II) Secondly, we assume that the differential operator $Q\left(x', D_{x'}\right)$ is of the generalized Kolmogorov form (see [27])

$$Q\left(x', D_{x'}\right) = \sum_{j=1}^{r} X_j\left(x', D_{x'}\right)^2 + X_0\left(x', D_{x'}\right) + \gamma(x'), \tag{2.6}$$

where the $X_j\left(x', D_{x'}\right)$ are *real* C^∞ vector fields on the boundary Γ.

Then the above-mentioned condition (H.2) may be replaced by a simple condition in terms of the Lie algebra $\mathcal{L}\left(X_1, X_2, \ldots, X_r\right)$ generated by the vector fields $\{X_1, X_2, \ldots, X_r\}$ (see [23]). More precisely, we can prove the following theorem:

Theorem 2.3 *Assume that the following two conditions (H.1) and (H.3) are satisfied:*

(H.1) $\mu(x') \geq 0$, $\gamma(x') \leq 0$ *and* $\mu(x') - \gamma(x') > 0$ *on* Γ.
(H.3) The differential operator $Q\left(x', D_{x'}\right)$ *is of the form* (2.6), *and the vector fields*

$$\begin{cases} X_j & \text{for } 1 \leq j \leq r, \\ [X_j,\, X_k] = X_j\, X_k - X_k\, X_j & \text{for } 1 \leq j, k \leq r, \end{cases}$$

span *all vector fields at every point of some open neighborhood* V *of the boundary portion* $\Gamma_0 = \{x' \in \Gamma : \mu(x') = 0\}$.

If condition (2.4) *is satisfied, then the Višik–Ventcel' boundary value problem* (1.4) *has a unique solution* $u \in H^{s+2}(\Omega)$ *for any* $f \in H^s(\Omega)$ *and any* $\varphi \in H^{s+1/2}(\Gamma)$ *with* $s > -1/2$.

Remark 2.2 Condition (H.3) implies that a Markovian particle goes out of the absorbing barrier Γ_0 in *finite time* (see [13, Satz C]).

We give a simple example of conditions (H.1) and (H.3) in the space $\mathbf{R}^3$:

Example 2.1 Let

$$\Omega = \left\{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 < 1\right\}$$

be the unit ball with the boundary (unit sphere)

$$\Gamma = \left\{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\right\}.$$

For a local coordinate system (spherical coordinate system)

$$\begin{cases} x_1 = r\,\cos\theta\,\cos\omega, \\ x_2 = r\,\cos\theta\,\sin\omega, \\ x_3 = r\,\sin\theta, \end{cases}$$

where

$$0 \le r \le 1, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad 0 \le \omega \le 2\pi,$$

we define the Višik–Ventcel' boundary value condition $B_1\boldsymbol{\gamma}$ by the formula

$$B_1\boldsymbol{\gamma} u = -\theta^2 \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} + \theta^2 \frac{\partial^2 u}{\partial \omega^2} + \left(\theta^2 - \frac{\pi^2}{4}\right) u$$

on the unit sphere $\varGamma = \{r = 1\}$. Here:

$$\begin{cases} \frac{\partial}{\partial \boldsymbol{\nu}} = -\frac{\partial}{\partial r}, \\ \mu(\theta,\omega) = \theta^2, \quad \gamma(\theta,\omega) = \theta^2 - \frac{\pi^2}{4}. \end{cases}$$

We remark that the set

$$\varGamma_0 = \{r = 1,\, \theta = 0\}$$

is the *equator* and further that the vector fields

$$\begin{cases} X = \frac{\partial}{\partial \theta}, \quad Y = \theta\,\frac{\partial}{\partial \omega}, \\ [X,Y] = \frac{\partial}{\partial \omega} \end{cases}$$

span all vector fields at every point of $\varGamma$.

Table 2.1 below gives a bird's-eye view of unique solvability theorems for *hypoelliptic* boundary value problems in the framework of Sobolev spaces.

Order of $B\boldsymbol{\gamma}$	Conditions on $B\boldsymbol{\gamma}$	proved by
1	Subelliptic oblique derivative case	[53, Theorem 2.1]
1	Hypoelliptic Robin case	[55, Theorem 1.1]
2	(H.1), (H.2)	Theorem 2.2
2	(H.1), (H.3)	Theorem 2.3

Table 2.1 A bird's-eye view of unique solvability theorems for hypoelliptic boundary value problems

Both Theorem 2.2 and Theorem 2.3 are a generalization of Agranovich–Vishik [4, Theorem 5.1] to the hypoelliptic case. See also Bony–Courrège–Priouret [7, p. 484, Théorème XVIII] in the framework of Hölder spaces.

2.3 Generation of analytic semigroups for the Višik–Ventcel' boundary value problem (1.5)

The second purpose of this paper is to study the Višik–Ventcel' boundary value problem (1.5) when $|\lambda| \to \infty$ from the point of view of the Hille–Yosida theory of semigroups in functional analysis ([62]). The generation theorem for analytic semigroups is well established in the non-degenerate case in the L^2 topology (see [4], [16], [35], [57]). We generalize this generation theorem for analytic semigroups to the *hypoelliptic* case (Theorem 2.4).

To do so, we associate with the homogeneous Višik–Ventcel' boundary value problem (1.5) a densely defined, closed linear operator

$$\boxed{\mathfrak{A}\colon L^2(\Omega) \longrightarrow L^2(\Omega)}$$

in the Hilbert space $L^2(\Omega)$ as follows (see Proposition 7.1):

(1) The domain $\mathcal{D}(\mathfrak{A})$ of definition is the space

$$\mathcal{D}(\mathfrak{A}) = \left\{u \in L^2(\Omega) : Au \in L^2(\Omega),\ B\gamma u = 0 \text{ on } \Gamma\right\} \tag{2.7a}$$

$$= \left\{u \in H^2(\Omega) : B\gamma u = 0 \text{ on } \Gamma\right\}. \tag{2.7b}$$

(2) $\mathfrak{A}u = Au$ for every $u \in \mathcal{D}(\mathfrak{A})$.

It should be emphasized that the Višik–Ventcel' boundary condition $B\gamma u$ can be defined as an element of the Sobolev space $H^{-5/2}(\Gamma)$ (see Theorem 5.1).

Then, by arguing just as in the proof of [52, Theorem 2.2] we can obtain the following generation theorem of analytic semigroups for the closed realization $\mathfrak{A}$ associated with the Višik–Ventcel' boundary value problem (1.5):

Theorem 2.4 *Assume that the following condition (G) is satisfied:*

(G) $\mu(x') > 0$ *and* $\gamma(x') \leq 0$ *on* Γ.

Then we have the following two assertions (i) and (ii):

(i) For every $0 < \varepsilon < \pi/2$, *there exists a constant* $r(\varepsilon) > 0$ *such that the resolvent set of the closed realization* $\mathfrak{A}$ *contains the set*

$$\Sigma(\varepsilon) = \left\{\lambda = r^2\, e^{i\theta} : r \geq r(\varepsilon), -\pi + \varepsilon \leq \theta \leq \pi - \varepsilon\right\},$$

and further that the resolvent $(\mathfrak{A} - \lambda I)^{-1}$ *satisfies the estimate*

$$\left\|(\mathfrak{A} - \lambda I)^{-1}\right\| \leq \frac{c(\varepsilon)}{|\lambda|} \quad \text{for all } \lambda \in \Sigma(\varepsilon), \tag{2.8}$$

where $c(\varepsilon) > 0$ *is a constant depending on* ε.

(ii) The operator $\mathfrak{A}$ *generates a semigroup* $U(z) = e^{z\,\mathfrak{A}}$ *on the space* $L^2(\Omega)$ *that is* analytic *in the sector*

$$\Delta_\varepsilon = \{z = t + is : z \neq 0, |\arg z| < \pi/2 - \varepsilon\}$$

for any $0 < \varepsilon < \pi/2$.

Remark 2.3 Some remarks are in order.

1° Conditions (H.1), (H.2) and (H.3) are trivially satisfied under condition (G), since $\Gamma_0 = \emptyset$.

2° The intuitive meaning of condition (G) is that the reflection phenomenon occurs at *every* point of the boundary Γ. In other words, a Markovian particle goes out of the boundary Γ *instantaneously*.

3° We obtain from Theorem 2.4 that there exists a uniformly stochastically continuous Feller function $p_t(x, dy)$ on the state space $\overline{\Omega} = \Omega \cup \Gamma$ such that $p_t(x, dy)$ is the transition function of some *strong Markov process* $\mathcal{X} = \{x_t\}_{t \geq 0}$ whose paths are right-continuous and have no discontinuities other than jumps (see [50, Section 12.3]).

We give a simple example of condition (G) in the space $\mathbf{R}^3$:

Example 2.2 As in the same situation in Example 2.1, we define the Višik–Ventcel' boundary value condition $B_2\boldsymbol{\gamma}$ by the formula

$$B_2\boldsymbol{\gamma}u = -\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} + \exp\left[-\frac{2}{\theta^2}\right]\frac{\partial^2 u}{\partial \omega^2}$$

on the unit sphere $\Gamma = \{r = 1\}$. Here:

$$\begin{cases} \mu(\theta, \omega) \equiv 1, \quad \gamma(\theta, \omega) \equiv 0, \\ \Gamma_0 = \emptyset. \end{cases}$$

It should be noticed that the *Lie algebra* generated by the vector fields

$$X = \frac{\partial}{\partial \theta}, \quad Z = \exp\left[-\frac{1}{\theta^2}\right]\frac{\partial}{\partial \omega}$$

does *not* contain the vector field $\frac{\partial}{\partial \omega}$ at every point of the equator $\{r = 1, \ \theta = 0\}$.

Table 2.2 below gives a bird's-eye view of generation theorems for analytic semigroups of the closed realization $\mathfrak{A}$:

Order of $B\boldsymbol{\gamma}$	Conditions on $B\boldsymbol{\gamma}$	Semigroup $U(z) = e^{z\,\mathfrak{A}}$	proved by
1	Subelliptic oblique derivative case	analytic	[52, Theorem 2.4]
1	Hypoelliptic Robin case	analytic	[55, Theorem 1.2]
2	(G)	analytic	Theorem 2.4

Table 2.2 A bird's-eye view of generation theorems for semigroups

Theorem 2.4 is a generalization of Agranovich–Vishik [4, Theorems 4.1 and 5.1] to the *hypoelliptic* case. See also Bony–Courrège–Priouret [7, p. 492, Théorème XIX] in the framework of Hölder spaces.

2.4 Spectral analysis of the Višik–Ventcel' boundary value problem (1.5)

The third purpose of this paper is devoted to the spectral analysis of the closed realization $\mathfrak{A}$ associated with the Višik–Ventcel' boundary value problem (1.5). By combining Agmon [2, Theorems 14.4 and 15.1] with Theorem 2.4, we can obtain the following five spectral properties of $\mathfrak{A}$:

Theorem 2.5 *Assume that condition (G) is satisfied. Then the closed realization $\mathfrak{A}$ enjoys the following five spectral properties:*

(i) The spectrum of $\mathfrak{A}$ is discrete and the eigenvalues λ_j of $\mathfrak{A}$ have finite multiplicities.

(ii) All rays $\arg \lambda = \theta$ different from the negative axis are rays of minimal growth *of the resolvent $(\mathfrak{A} - \lambda I)^{-1}$. More precisely, for every $-\pi < \theta < \pi$ there exists a constant $R(\theta) > 0$ depending on θ such that if $\lambda = r^2 e^{i\theta}$ and $|\lambda| = r^2 \geq R(\theta)$, then we have the resolvent estimate*

$$\left\| (\mathfrak{A} - \lambda I)^{-1} \right\| \leq \frac{C(\theta)}{|\lambda|}, \tag{2.9}$$

where $C(\theta) > 0$ is a constant depending on θ.

(iii) The negative axis is a direction of condensation *of eigenvalues of $\mathfrak{A}$. More precisely, for each $\varepsilon > 0$ there are only a finite number of eigenvalues inside the angle: $-\pi + \varepsilon < \theta < \pi - \varepsilon$.*

(iv) Let

$$N(t) := \sum_{\operatorname{Re} \lambda_j \geq -t} 1$$

be the number of eigenvalues λ_j such that $\operatorname{Re} \lambda_j \geq -t$, where each λ_j is repeated according to its multiplicity. Then the asymptotic eigenvalue distribution *formula*

$$N(t) = \frac{1}{(2\pi)^n} \int_\Omega |A(x)|\, dx \cdot t^{n/2} + o(t^{n/2}) \quad \text{as } t \to +\infty \tag{2.10}$$

holds true. Here $|A(x)|$ denotes the volume of the subset

$$A(x) = \left\{ \xi \in \mathbf{R}^n : \sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j < 1 \right\}.$$

(v) The generalized eigenfunctions of $\mathfrak{A}$ are complete in the Hilbert space $L^2(\Omega)$; they are also complete in the domain $\mathcal{D}(\mathfrak{A})$ in the $H^2(\Omega)$-norm.

We give a simple example of Theorem 2.5 in the space $\mathbf{R}^3$:

Example 2.3 As in the same situation in Example 2.1, we define the Višik–Ventcel' boundary value condition $B_3\gamma$ by the formula (see [27])

$$B_3 \gamma u = -\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} + \theta \frac{\partial u}{\partial \omega} + \left(\theta^2 - \frac{\pi^2}{4} \right) u$$

on the unit sphere $\Gamma = \{r = 1\}$. Here:

$$\begin{cases} \mu(\theta,\omega) \equiv 1, \quad \gamma(\theta,\omega) = \theta^2 - \frac{\pi^2}{4}, \\ \Gamma_0 = \emptyset. \end{cases}$$

We remark that the vector fields

$$\begin{cases} X = \frac{\partial}{\partial\theta}, \quad W = \theta\,\frac{\partial}{\partial\omega}, \\ [X, W] = \frac{\partial}{\partial\omega} \end{cases}$$

span all vector fields at every point of Γ.

We consider the Višik–Ventcel' eigenvalue problem

$$\begin{cases} \Delta u = \lambda\, u & \text{in } \Omega, \\ B_3\boldsymbol{\gamma} u = 0, & \text{on } \Gamma. \end{cases}$$

Then we have the asymptotic eigenvalue distribution formula

$$N(t) = \sum_{\mathrm{Re}\,\lambda_j \geq -t} 1 = \frac{2}{9\pi}\, t^{3/2} + o(t^{3/2}) \quad \text{as } t \to +\infty.$$

Table 2.3 below gives a bird's-eye view of asymptotic eigenvalue distributions and eigenfunction expansions of the closed realization $\mathfrak{A}$.

Order of $B\boldsymbol{\gamma}$	Conditions on $B\boldsymbol{\gamma}$	proved by
1	Subelliptic oblique derivative case	[53, Theorem 2.3]
1	Hypoelliptic Robin case	[55, Theorem 2.2]
2	(G)	Theorem 2.5

Table 2.3 A bird's-eye view of spectral properties of $\mathfrak{A}$

Theorem 2.5 is the first time to prove the angular distribution of eigenvalues, the asymptotic eigenvalue distribution (2.10) and the completeness of generalized eigenfunctions of the closed realization $\mathfrak{A}$ associated with the hypoelliptic Višik–Ventcel' boundary value problem (1.5).

2.5 Outline of the paper

The rest of this paper is organized as follows. Section 3 is devoted to a brief review of variants of Gårding's inequality (Theorems 3.2 and 3.3). In Section 4 we formulate a characterization of classical hypoelliptic pseudo-differential operators due to Radkevič [38] and Hörmander [24] (Theorem 4.1) which plays a crucial role in this paper. Section 5 is devoted to the study of the non-homogeneous Dirichlet problem

$$\begin{cases} Au = f & \text{in } \Omega, \\ \gamma_0 u = u|_\Gamma = \varphi & \text{on } \Gamma \end{cases} \tag{D}$$

from the viewpoint of the theory of pseudo-differential operators essentially due to Hörmander [22], [25] and Seeley [43], [44] based on Calderón [10] (Theorems 5.1, 5.2 and 5.3),

Section 6 is devoted to the formulation of the non-homogeneous Višik–Ventcel' boundary value problem (1.4). This section is the heart of the subject. In Subsection 6.1, we prove the trace theorem which plays an essential role in the definition of the Višik–Ventcel' boundary condition $B\boldsymbol{\gamma}$ (Theorem 6.1 and Corollary 6.1). By using Corollary 6.1, we can associate with problem (1.4) a densely defined, closed linear operator $\mathcal{A} = (A, B\boldsymbol{\gamma})$ defined by formula (2.5). In Subsection 6.2, we show that the Višik–Ventcel' problem (1.4) can be reduced to the study of the pseudo-differential operator

$$T = \mu(x')\, \Pi + Q\left(x', D_{x'}\right) \quad \text{on } \Gamma \tag{6.17}$$

in the framework of Sobolev spaces (Proposition 6.1 and formula (6.17)). Here Π is called the *Dirichlet-to-Neumann operator* that is a first order, elliptic pseudo-differential operator on Γ (see formula (6.18)). The virtue of this reduction is that there is no difficulty in taking adjoints or transposes after restricting the attention to the boundary, whereas boundary value problems in general do not have adjoints or transposes. In Subsection 6.3, we prove that if condition (H.2) is satisfied, then both the pseudo-differential operators T and T^* are *hypoelliptic* with loss of one derivative on Γ (Propositions 6.3 and 6.4). Moreover, by using Peetre's lemma (Lemma 6.1) we prove that the index of the Višik–Ventcel' boundary value problem (1.4) is equal to *zero* for $s > -1/2$ (Proposition 6.6 and Theorem 6.3):

$$\operatorname{ind} \mathcal{A} = \dim \mathcal{N}(\mathcal{A}) - \operatorname{codim} \mathcal{R}(\mathcal{A}) = 0. \tag{6.49}$$

In Section 7, by using Propositions 6.1 and 6.3 we prove a regularity theorem for the Višik–Ventcel' boundary value problem (1.4) under condition (H.2) (Theorem 7.1). In particular, by applying Sobolev's imbedding theorem we obtain a regularity result for the null space of the mapping $\mathcal{A}$ under condition (H.2) for $s > -1/2$ (Corollary 7.1):

$$\mathcal{N}(\mathcal{A}) \subset C^\infty(\overline{\Omega}).$$

Moreover, we prove the closedness of $\mathfrak{A}$ defined by formula (2.7a) and also the regularity property (2.7b) (Proposition 7.1).

Section 8 is devoted to the proof of Theorems 2.2 and 2.3. In Subsection 8.1, we prove Theorem 2.2. First, we prove a uniqueness theorem for the Višik–Ventcel' boundary value problem (1.4) in the framework of C^2 functions under conditions (2.4) and (H.2) (Theorem 8.1). The proof of Theorem 8.1 is based on the strong maximum principle and Hopf's boundary point lemma. By combining Corollary 7.1 and Theorem 8.1, we find that if conditions (2.4), (H.1) and (H.2) are satisfied, then the mapping $\mathcal{A}$ is *injective* for $s > -1/2$:

$$\mathcal{N}(\mathcal{A}) = \{0\}.$$

However, we have, by formula (6.49),

$$\operatorname{codim} \mathcal{R}(\mathcal{A}) = \dim \mathcal{N}(\mathcal{A}) = 0.$$

This proves the *surjectivity* of the mapping $\mathcal{A}$ for $s > -1/2$.

In this way, we can prove that if conditions (2.4), (H.1) and (H.2) are satisfied, then the mapping $\mathcal{A} = (A, B\boldsymbol{\gamma})$ defined by formula (2.5) is *bijective* for $s > -1/2$. In other words, the Višik–Ventcel' boundary value problem (1.4) is *uniquely solvable* in the framework of Sobolev spaces for every $s > -1/2$ if conditions (2.4), (H.1) and (H.2) are satisfied (Theorem 2.2).

In Subsection 8.2, the proof of Theorem 2.3 is essentially the same as that of Theorem 2.2 if we replace condition (H.2) by condition (H.3). In fact, Propositions 6.3, 6.4, 6.5 and 6.6 and Theorem 6.3 remain valid for the pseudo-differential operator $T = \mu(x')\Pi + Q(x', D_{x'})$ when we replace formula (1.3) by formula (2.6) and condition (H.2) by condition (H.3), respectively. Therefore, the proof of Theorem 2.3 goes through just as in Section 7 and Subsection 8.1 if conditions (2.4), (H.1) and (H.3) are satisfied.

In Section 9, in order to prove an existence and uniqueness theorem for the homogeneous Višik–Ventcel' boundary value problem (1.5) in the framework of Sobolev spaces when $|\lambda| \to \infty$ (Theorem 2.4), we make use of a method essentially due to Agmon ([2], [29]), just as in Taira [52], [53]. This is a technique of treating a spectral parameter λ as a second order, elliptic differential operator of an extra variable y on the *unit circle* S, and relating the old problem to a new one with the additional variable. Our presentation of this technique is due to Fujiwara [17] and Taira [48]. More precisely, if we express the complex parameter λ in the form

$$\lambda = r^2 e^{i\theta} \quad \text{for } r \geq 0 \text{ and } -\pi < \theta < \pi,$$

then we replace the uniformly elliptic differential operator

$$A - \lambda = A - r^2 e^{i\theta}$$

defined in the original domain Ω by the second order, *strongly uniform elliptic* differential operator

$$\widetilde{\Lambda}(\theta) = A + e^{i\theta} \frac{\partial^2}{\partial y^2} \quad \text{for } -\pi < \theta < \pi \tag{9.1}$$

defined in the product domain $\Omega \times S$. We consider instead of the original problem (1.5) the following homogeneous Višik–Ventcel' boundary value problem in the product domain $\Omega \times S$:

$$\begin{cases} \widetilde{\Lambda}(\theta)\widetilde{u} = \left(A + e^{i\theta} \dfrac{\partial^2}{\partial y^2} \right) \widetilde{u} = \widetilde{f} & \text{in } \Omega \times S, \\ B\boldsymbol{\gamma}\widetilde{u} = \mu(x') \left.\dfrac{\partial \widetilde{u}}{\partial \boldsymbol{\nu}}\right|_{\Gamma \times S} + Q\left(\widetilde{u}|_{\Gamma \times S}\right) = 0 & \text{on } \Gamma \times S. \end{cases} \tag{9.2}$$

We prove that the Višik–Ventcel' boundary value problem (9.2) in $\Omega \times S$ has a *finite index* if condition (G) is satisfied (Theorem 9.1). Theorem 9.1 is an essential step in the proof of Theorem 2.4 and its proof will be given in Section 13, due to its length.

In Section 10, by using the theory of pseudo-differential operators we consider the Dirichlet problem for the second order, differential operator $\widetilde{\Lambda}(\theta)$ in the framework of Sobolev spaces on the product domain $\Omega \times S$ (Theorem 10.1).

In Section 11, we reduce the homogeneous Višik–Ventcel' boundary value problem (9.2) to the study of a second order, pseudo-differential operator

$$\widetilde{T}(\theta) = \mu(x')\, \widetilde{\Pi}(\theta) + Q\left(x', D_{x'}\right) \quad \text{on } \Gamma \times S \tag{11.3}$$

(Proposition 11.1 and formula (11.3)). Here $\widetilde{\Pi}(\theta)$ is called the *Dirichlet-to-Neumann operator* that is a first order, elliptic pseudo-differential operator on $\Gamma \times S$ (see formula (11.4)).

The purpose of Section 12 is to prove that if condition (G) is satisfied, then both the pseudo-differential operators $\widetilde{T}(\theta)$ and $\widetilde{T}(\theta)^*$ are *hypoelliptic* with loss of one derivative on $\Gamma \times S$ (Proposition 12.2 and Remark 12.1). Moreover, by using Peetre's lemma (Lemma 6.1) we can prove that the closed realization $\widetilde{\mathcal{T}}(\theta)$ is a Fredholm operator, analogous to Proposition 6.6 (Proposition 12.3).

Section 13 is devoted to the proof of Theorem 9.1. More precisely, we show how Theorem 9.1 follows from Propositions 11.1, 12.2 and 12.3 if condition (G) is satisfied.

Section 14 is devoted to the proof of Theorem 2.4. By using Theorem 2.2 with $\varphi := 0$ under condition (G) we prove the index formula (Theorem 14.1)

$$\operatorname{ind}(\mathfrak{A} - \lambda I) = 0 \quad \text{for all complex number } \lambda \in \mathbf{C}. \tag{14.2}$$

Furthermore, by using Theorem 9.1 and the index formula (14.2) we can derive the resolvent estimate (2.8). We remark that the resolvent estimate (2.9) is a special case of the resolvent estimate (2.8). In this way, we can prove Theorem 2.4.

Both Section 15 and Section 16 are devoted to the proof of Theorem 2.5. Our proof of Theorem 2.5 is based on Agmon [2, Theorems 14.4 and 15.1] which are summarized in [55, Section 4].

In Section 15, for some large number $\lambda_0 > 0$ (see condition (5.2) with $A := A - \lambda_0$) we study the homogeneous Višik–Ventcel' boundary value problem

$$\begin{cases} (A - \lambda_0)\, u = f & \text{in } \Omega, \\ B\gamma u = 0 & \text{on } \Gamma \end{cases} \tag{15.1}$$

in the framework of Sobolev spaces if condition (G) is satisfied. However, in the hypoelliptic case, we cannot use Green's formula to characterize the adjoint operator $\mathfrak{A}^* - \lambda_0 I$ of the boundary value problem (15.1), as in Schechter [40, Theorem 4.1] and Browder [9, Theorem 5]. Therefore, we shift our attention to the Green operator (the resolvent) $(\mathfrak{A} - \lambda_0 I)^{-1}$ and its adjoint operator $(\mathfrak{A}^* - \lambda_0 I)^{-1}$ from the viewpoint of the *Boutet de Monvel calculus* [8]. More precisely, we make use of the Boutet de Monvel calculus in order to study the mapping properties of the resolvent and its adjoint operator in the framework of L^2 Sobolev spaces. In this way, we can verify all the conditions of [55, Theorem 4.1] and [55, Remark 4.1] in Sections 15 and 16 (Theorems 15.2 and 16.1).

In Section 15, by a *homotopy argument* we consider instead of the original boundary value problem (15.1) the homogeneous Višik–Ventcel' boundary value problem:

$$\begin{cases} (\Delta - 1)\, u = f & \text{in } \Omega, \\ B\boldsymbol{\gamma} u = 0 & \text{on } \Gamma \end{cases} \tag{15.7}$$

under condition (G), where Δ is the usual Laplacian

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}.$$

Indeed, there is a homotopy in the class of second order, uniformly elliptic symbols between $A - \lambda_0$ and $\Delta - 1$ such that

$$A_t := (1-t)\,(A - \lambda_0) + t\,(\Delta - 1) \quad \text{for } 0 \leq t \leq 1.$$

First, we characterize the Green operator $\mathcal{G}_D$ for the homogeneous Dirichlet problem

$$\begin{cases} A_1 v = (\Delta - 1)\, v = f & \text{in } \Omega, \\ \gamma_0 v = v|_\Gamma = 0 & \text{on } \Gamma \end{cases} \tag{15.8}$$

in terms of the Boutet de Monvel calculus (Theorem 15.1). Secondly, we characterize the mapping property of the Green operator (the resolvent) $\mathcal{G}$ of the homogeneous Višik–Ventcel' boundary value problem (15.7) in the framework of Sobolev spaces (Theorem 15.2). In the proof of Theorems 15.1 and 15.2, following Rempel–Schulze [39], Schrohe [41] and [50, Appendix B] we calculate explicitly various symbols of trace, potential and boundary operators in terms of the Boutet de Monvel calculus if condition (G) is satisfied (see assertions (15.23) and (15.24)).

In Section 16, by using Theorem 15.2 we can characterize the *adjoint operator* $\mathcal{G}^*$ of the Green operator $\mathcal{G}$ in the framework of Sobolev spaces if condition (G) is satisfied (formula (16.1)), and prove that the adjoint operator $\mathcal{G}^*$ has the same mapping property as $\mathcal{G}$ (Theorem 16.1).

Finally, Theorem 2.5 follows from an application of Agmon [2, Theorems 14.4 and 15.1] by combining part (i) of Theorem 2.4, Theorem 15.2 and Theorem 16.1, just as in [55].

In the last Section 17, as concluding remarks, we state a brief history of the *stochastic analysis* methods for Višik–Ventcel' boundary value problems.

3 Variants of Gårding's inequality

Let Ω be an open subset of $\mathbf{R}^n$ and let $A = A(x, D)$ be a properly supported, classical pseudo-differential operator of order m on Ω with the principal symbol $a_m(x,\xi) \in S^m_{1,0}(\Omega \times \mathbf{R}^n)$. In this subsection we are concerned with inequalities from below for $A(x,D)$ of the form

$$\mathrm{Re}\,(Au, u)_{L^2(\Omega)} \geq C_K\, \|u\|^2_{H^s(\Omega)} \quad \text{for all } u \in C^\infty_K(\Omega), \tag{3.1}$$

Here:

(1) s is a real number.
(2) K is a compact subset of Ω.

(3) $C_K^\infty(\Omega)$ is a function space defined by the formula

$$C_K^\infty(\Omega) = \{u : u \in C^\infty(\Omega), \operatorname{supp} u \subset K\},$$

and $(\cdot,\cdot)_{L^2(\Omega)}$ is the inner product of the Hilbert space $L^2(\Omega)$.

We remark that inequality (3.1) holds true for $s \geq m/2$, since we have the inequality

$$\left|(Au,u)_{L^2(\Omega)}\right| \leq C'_K \|u\|^2_{H^{m/2}(\Omega)} \quad \text{for all } u \in C_K^\infty(\Omega),$$

with another constant $C'_K > 0$.

In what follows we give sufficient conditions on $A(x,D)$ for inequality (3.1) to hold true for $s < m/2$. These results play an important role in deriving *a priori* estimates for (non-)elliptic boundary value problems .

(I) The next result, first proved by Gårding for differential operators, is a milestone in the theory of elliptic boundary value problems (see [12], [25], [28], [58], [61]):

Theorem 3.1 (Gårding's inequality) *Let $A = A(x,D)$ be a properly supported, classical pseudo-differential operator of order m on Ω having the principal symbol $a_m(x,\xi)$. Assume that there exists a constant $a_0 > 0$ such that*

$$\operatorname{Re} a_m(x,\xi) \geq a_0 |\xi|^m \quad \textit{for all } (x,\xi) \in T^*(\Omega) = \Omega \times \mathbf{R}^n.$$

Then, for every compact $K \subset \Omega$ and $s < m/2$ there exist constants $c_{K,s} > 0$ and $C_{K,s} > 0$ such that

$$\operatorname{Re}(Au,u)_{L^2(\Omega)} \geq c_{K,s} \|u\|^2_{H^{m/2}(\Omega)} - C_{K,s} \|u\|^2_{H^s(\Omega)} \quad \textit{for all } u \in C_K^\infty(\Omega). \tag{3.2}$$

The inequality (3.2) is called *Gårding's inequality.*

(II) A sharpened form of Gårding's inequality is given by Hörmander [22, Theorem 1.3.3], [25, Theorem 18.1.14] and also by Melin [30, Theorem 3.1]:

Theorem 3.2 (the sharp Gårding inequality) *Let $A = A(x,D) \in L^m(\Omega)$ be as in Theorem 3.1. Assume that*

$$\operatorname{Re} a_m(x,\xi) \geq 0 \quad \textit{for all } (x,\xi) \in T^*(\Omega) = \Omega \times \mathbf{R}^n.$$

Then, for every compact $K \subset \Omega$ and $s < (m-1)/2$, there exist constants $c_{K,s} > 0$ and $C_{K,s} > 0$ such that

$$\operatorname{Re}(Au,u)_{L^2(\Omega)} \geq -c_{K,s} \|u\|^2_{H^{(m-1)/2}(\Omega)} - C_{K,s} \|u\|^2_{H^s(\Omega)} \quad \textit{for all } u \in C_K^\infty(\Omega). \tag{3.3}$$

It should be emphasized that Fefferman–Phong [14] proved the following precise inequality for $m = 2$ (see [14, Theorem]; [25, Corollary 18.6.11]):

Theorem 3.3 (the Fefferman–Phong inequality) *Let $A(x,D)$ be a second order, pseudo-differential operator having the complete symbol $a(x,\xi) \in S^2_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$ such that*

$$a(x,\xi) \geq 0 \quad \textit{for all } (x,\xi) \in T^*(\mathbf{R}^n) = \mathbf{R}^n \times \mathbf{R}^n.$$

Then there exists a constant $C > 0$ such that we have, for all $u \in C_0^\infty(\mathbf{R}^n)$,

$$\operatorname{Re}(A(x,D)u,u)_{L^2(\mathbf{R}^n)} \geq -C\|u\|^2_{L^2(\mathbf{R}^n)}. \tag{3.4}$$

Here the constant C in inequality (3.4) may be chosen uniformly in the $a(x,\xi)$ in a bounded subset of the symbol class $S^2_{1,0}(\mathbf{R}^n \times \mathbf{R}^n)$.

4 Hypoelliptic pseudo-differential operators

Let Ω be an open subset of $\mathbf{R}^n$. A properly supported, pseudo-differential operator $P = P(x, D)$ on Ω is said to be *hypoelliptic* if it satisfies the condition

$$\operatorname{sing\,supp} u = \operatorname{sing\,supp} Pu \quad \text{for every } u \in \mathcal{D}'(\Omega). \tag{4.1}$$

For example, elliptic operators are all hypoelliptic. It is easy to see that condition (4.1) is equivalent to the following condition: For any open subset Ω_1 of Ω, we have the assertion

$$u \in \mathcal{D}'(\Omega),\ Pu \in C^\infty(\Omega_1) \implies u \in C^\infty(\Omega_1).$$

It should be noticed that this notion may be transferred to manifolds.

In this section we describe a class of hypoelliptic pseudo-differential operators of [24] that arises in the study of elliptic boundary value problems.

Let $P = P(x, D)$ be a properly supported, classical pseudo-differential operator of order m on Ω such that the complete symbol $p(x, \xi)$ has an asymptotic expansion

$$p(x, \xi) \sim p_m(x, \xi) + p_{m-1}(x, \xi) + \dots,$$

where $p_j(x, \xi)$ is positively homogeneous of degree j in the variable ξ. For simplicity, we assume that there exists a constant $C > 0$ such that the principal symbol $p_m(x, \xi)$ satisfies the condition

$$|\operatorname{Im} p_m(x, \xi)| \leq C \operatorname{Re} p_m(x, \xi) \quad \text{on } T^*(\Omega) \setminus \{0\} = \Omega \times (\mathbf{R}^n \setminus \{0\}). \tag{4.2}$$

The following criterion for hypoellipticity is due to Radkevič [38, Theorem 7] and Hörmander [24, Theorem 5.2] (see also [25, Theorem 22.2.6]):

Theorem 4.1 *Let $P = P(x, D)$ be a pseudo-differential operator of order m such that $p_m(x, \xi)$ satisfies condition (4.2). Assume that, for some $s_0 \in \mathbf{R}$ and for every compact K of Ω and $s' < s_0 + m - 1$ there exists a constant $C_{K,s_0,s'} > 0$ such that*

$$\|u\|_{H^{s_0+m-1}(\Omega)} \leq C_{K,s_0,s'} \left(\|Pu\|_{H^{s_0}(\Omega)} + \|u\|_{H^{s'}(\Omega)} \right) \text{ for all } u \in C_K^\infty(\Omega). \tag{4.3}$$

Then it follows that P is hypoelliptic *with loss of one derivative in Ω. More precisely, we have, for every $s \in \mathbf{R}$,*

$$u \in \mathcal{D}'(\Omega),\ Pu \in H^s_{\mathrm{loc}}(\Omega) \implies u \in H^{s+m-1}_{\mathrm{loc}}(\Omega). \tag{4.4}$$

Here the localized Sobolev space $H^s_{\mathrm{loc}}(\Omega)$ is defined as follows:

$$\begin{aligned} H^s_{\mathrm{loc}}(\Omega) = &\text{the space of distributions } u \in \mathcal{D}'(\Omega) \text{ such that} \\ &\varphi u \in H^s(\mathbf{R}^n) \text{ for all } \varphi \in C_0^\infty(\Omega). \end{aligned}$$

We equip $H^s_{\mathrm{loc}}(\Omega)$ with the topology defined by the seminorms

$$u \longmapsto \|\varphi u\|_{H^s(\mathbf{R}^n)}$$

as φ ranges over $C_0^\infty(\Omega)$. It is easy to see that $H^s_{\mathrm{loc}}(\Omega)$ is a Fréchet space.

5 The Dirichlet problem

This section is devoted to the classical Dirichlet problem from the viewpoint of the theory of pseudo-differential operators due to Hörmander [22], [25] and Seeley [43], [44] based on Calderón [10] (see also [12], [28], [58]).

Let Ω be a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega = \Gamma$. Without loss of generality, we may assume the following (see [32]):

(a) The domain Ω is a relatively compact open subset of an n-dimensional, compact smooth manifold M without boundary.
(b) In a neighborhood W of Γ in M a normal coordinate t is chosen so that the points of W are represented as (x', t), $x' \in \Gamma$, $-1 < t < 1$; $t > 0$ in Ω, $t < 0$ in $M \setminus \overline{\Omega}$ and $t = 0$ only on Γ.
(c) The manifold M is equipped with a strictly positive density μ which, on W, is the product of a strictly positive density ω on Γ and the Lebesgue measure dt on $(-1, 1)$. This manifold $M = \widehat{\Omega}$ is called the *double* of Ω.

Let $A = A(x, D)$ be a second order, uniformly elliptic differential operator with real coefficients on the double $M = \widehat{\Omega}$ of Ω such that

$$Au = \sum_{i=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u.$$

Here:

(1) The $a^{ij}(x)$ are the components of a C^∞ symmetric contravariant tensor of type $\binom{2}{0}$ on M and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq a_0 \, |\xi|^2 \quad \text{for all } (x, \xi) \in T^*(M),$$

where $T^*(M)$ is the cotangent bundle of M.
(2) $b^i \in C^\infty(M)$ for all $1 \leq i \leq n$.
(3) $c \in C^\infty(M)$.

Following Seeley [43], [44], we let

$$\mathcal{N}_0(A) := \left\{ u \in C^\infty(M) : \operatorname{supp} u \subset \overline{\Omega}, \ Au = 0 \text{ in } \Omega \right\}. \tag{5.1}$$

It is known (see [43, Theorem 7]) that $\mathcal{N}_0(A)$ is *finite-dimensional*. We find from formula (5.1) that

$$\mathcal{N}_0(A) \subset \mathcal{N}(\mathfrak{A}),$$

since the operator $\mathfrak{A}$ is defined by formula (2.7) and the boundary condition $B\boldsymbol{\gamma}$ is defined by formula (1.2).

We remark that all the sufficiently large eigenvalues of the Dirichlet problem for the differential operator A and its formal adjoint A^* lie in the *parabolic* type region, as discussed in [2, pp. 274–277] and [31, Chapter 3]. Hence, by considering $A - \lambda_0$ and $A^* - \lambda_0$ for some large number $\lambda_0 > 0$ we may assume that

$$\mathcal{N}_0(A) = \mathcal{N}_0(A^*) = \{0\}. \tag{5.2}$$

Then we have the following theorem for *surface potentials* of A (see [43, Theorems 5 and 6] and [44, pp. 274–275]):

Theorem 5.1 (Seeley) *Assume that condition* (5.2) *is satisfied. If* $s \in \mathbf{R}$*, we define the null space for the operator* A *by the formula*

$$\mathcal{N}(A, s) := \left\{u \in H^s(\Omega) : Au = 0 \text{ in } \Omega\right\}. \tag{5.3}$$

Then we can construct a continuous map $\mathcal{P}$ *of* $H^{s-1/2}(\Gamma)$ *onto* $\mathcal{N}(A, s)$*. Moreover, the spaces* $\mathcal{N}(A, s)$ *and* $H^{s-1/2}(\Gamma)$ *are isomorphic in such a way that*

$$\mathcal{N}(A, s) \xrightarrow{\gamma_0} H^{s-1/2}(\Gamma), \tag{5.4a}$$

$$\mathcal{N}(A, s) \xleftarrow[\mathcal{P}]{} H^{s-1/2}(\Gamma), \tag{5.4b}$$

where γ_0 *is the* trace map *defined by the formula*

$$\gamma_0 u = u|_{\Gamma}\,.$$

The operator $\mathcal{P}$ *is called the* Poisson kernel *for the operator* A*.*

Especially, for every given function $\varphi \in H^{s-1/2}(\Gamma)$ the function $w = \mathcal{P}\varphi \in H^s(\Omega)$ is a unique solution of the Dirichlet problem

$$\begin{cases} Aw = 0 & \text{in } \Omega, \\ \gamma_0 w = \varphi & \text{on } \Gamma. \end{cases} \tag{5.5}$$

Furthermore, we have the following theorem for *volume potentials* of A (see [44, pp. 276–277]):

Theorem 5.2 (Seeley) *Assume that condition* (5.2) *is satisfied. Then there exists an elliptic, pseudo-differential operator* C *of order* -2 *on* M *such that*

$$A\,(CEf)|_{\Omega} = f \quad \text{in } \Omega\text{, for every } f \in H^s(\Omega) \text{ with } s \in \mathbf{R}. \tag{5.6}$$

Here the operator

$$E \colon H^s(\Omega) \longrightarrow H^s(M)$$

is Seeley's extension operator *for* $s \in \mathbf{R}$ *(see [42], [1, Theorems 5.21 and 5.22]).*

By using Theorems 5.1 and 5.2, we can prove the following *existence and uniqueness theorem* for the non-homogeneous Dirichlet problem:

Theorem 5.3 *Assume that condition* (5.2) *is satisfied. If* $s > -3/2$*, then the non-homogeneous Dirichlet problem*

$$\begin{cases} Au = f & \text{in } \Omega, \\ \gamma_0 u = \varphi & \text{on } \Gamma \end{cases} \tag{D}$$

has a unique solution u *in the space* $H^{s+2}(\Omega)$ *for any* $f \in H^s(\Omega)$ *and any* $\varphi \in H^{s+3/2}(\Gamma)$*.*

Proof Indeed, it suffices to note that the unique solution u of the Dirichlet problem (D) is given by the following formula:

$$u = (CEf)|_{\Omega} + \mathcal{P}\,(\varphi - (CEf)|_{\Gamma}) \quad \text{in } \Omega. \tag{5.7}$$

Here:

(a) $C \colon H^s(M) \to H^{s+2}(M)$ is the *right inverse* to A (see formula (5.6)).
(b) $\mathcal{P} \colon H^{s+3/2}(\Gamma) \to H^{s+2}(\Omega)$ is the *Poisson kernel* for A (see the Dirichlet problem (5.5)).

The proof of Theorem 5.3 is complete.

6 The non-homogeneous Višik–Ventcel' boundary value problem (1.4)

The purpose of this section is to study the *non-homogeneous* Višik-Ventcel' boundary value problem (1.4) for second order, uniformly elliptic differential operators A in the framework of Sobolev spaces. This section is the heart of the subject.

6.1 Definition of the Višik–Ventcel' boundary condition (1.2)

First, we introduce a *maximal domain* $H_A(\Omega)$ for the differential operator A in the Hilbert space $L^2(\Omega)$ as follows:

$$H_A(\Omega) := \left\{ u \in L^2(\Omega) : Au \in L^2(\Omega) \right\}. \tag{6.1}$$

We equip the space $H_A(\Omega)$ with the *graph norm*

$$\|u\|_{H_A(\Omega)} := \left(\|u\|^2_{L^2(\Omega)} + \|Au\|^2_{L^2(\Omega)} \right)^{1/2}. \tag{6.2}$$

The maximal domain $H_A(\Omega)$ is a Hilbert space.

Then we can prove the following *trace theorem* which plays an essential role in the study of the Višik–Ventcel' boundary condition $B\boldsymbol{\gamma}$ given by formula (1.2):

Theorem 6.1 *Assume that condition* (5.2) *is satisfied. For every* $u \in H_A(\Omega)$*, we can define the Višik–Ventcel' boundary condition* $B\boldsymbol{\gamma}u$ *as an element of the Sobolev space* $H^{-5/2}(\Gamma)$*. Moreover, the Višik–Ventcel' boundary operator*

$$B\boldsymbol{\gamma} \colon H_A(\Omega) \longrightarrow H^{-5/2}(\Gamma)$$

is continuous. Namely, there exists a constant $C > 0$ *such that*

$$\|B\boldsymbol{\gamma}u\|_{H^{-5/2}(\Gamma)} \leq C\, \|u\|_{H_A(\Omega)} \quad \text{for all } u \in H_A(\Omega). \tag{6.3}$$

Proof The proof is divided into three steps.

Step 1: For a given function $u \in H_A(\Omega)$ with $Au = f \in L^2(\Omega)$, we consider the homogeneous Dirichlet problem

$$\begin{cases} Av = f & \text{in } \Omega, \\ \gamma_0 v = 0 & \text{on } \Gamma. \end{cases} \tag{6.4}$$

By applying Theorem 5.3, we find from formula (5.7) with $\varphi := 0$ that the Dirichlet problem (6.4) has a unique solution $v \in H^{s+2}(\Omega)$ for every $f \in H^s(\Omega)$ with $s > -3/2$.

Therefore, we can define a continuous operator

$$\mathcal{G}_D \colon H^s(\Omega) \longrightarrow H^{s+2}(\Omega) \quad \text{for } s > -3/2$$

by the formula

$$v := \mathcal{G}_D f = (CEf)|_\Omega - \mathcal{P}\left(\gamma_0 \left(CEf\right)\right) \in H^{s+2}(\Omega) \quad \text{for } f \in H^s(\Omega). \tag{6.5}$$

The operator $\mathcal{G}_D$ is called the *Green operator* for the Dirichlet problem (6.4).

However, it follows from an application of the *trace theorem* ([1, Remarks 7.45], [12, p. 98, Section 4]) that the trace map

$$\boldsymbol{\gamma} = (\gamma_0, \gamma_1) : H^2(\Omega) \longrightarrow H^{3/2}(\Gamma) \oplus H^{1/2}(\Gamma)$$

is continuous, where

$$\begin{cases} \gamma_0 v = v|_{\Gamma}\,, \\ \gamma_1 v = \left.\frac{\partial v}{\partial \boldsymbol{\nu}}\right|_{\Gamma}\,. \end{cases}$$

Since $\gamma_0 \left(\mathcal{G}_D(Au)\right) = 0$ on Γ for $u \in H_A(\Omega)$, we have, by formula (6.5) with $f := Au$ and $s := 0$,

$$\begin{aligned} B\boldsymbol{\gamma} v &= B\boldsymbol{\gamma}\left(\mathcal{G}_D(Au)\right) = \mu(x') \frac{\partial}{\partial \boldsymbol{\nu}} \left(\mathcal{G}_D(Au)\right)\Big|_{\Gamma} + Q\left(x', D_{x'}\right)\left(\left(\mathcal{G}_D(Au)\right)|_{\Gamma}\right) \\ &= \mu(x')\gamma_1\left(\mathcal{G}_D(Au)\right) + Q\left(x', D_{x'}\right)\left(\gamma_0\left(\mathcal{G}_D(Au)\right)\right) \\ &= \mu(x')\gamma_1\left(\mathcal{G}_D(Au)\right) \in H^{1/2}(\Gamma) \quad \text{for every } u \in H_A(\Omega). \end{aligned}$$

This proves that

$$\begin{aligned} \|B\boldsymbol{\gamma} v\|_{H^{1/2}(\Gamma)} &= \|B\boldsymbol{\gamma}\left(\mathcal{G}_D(Au)\right)\|_{H^{1/2}(\Gamma)} \\ &= \left\|\mu(x')\gamma_1\left(\mathcal{G}_D(Au)\right)\right\|_{H^{1/2}(\Gamma)} \leq C_1 \|\mathcal{G}_D(Au)\|_{H^2(\Omega)} \\ &\leq C_2 \|Au\|_{L^2(\Omega)} \quad \text{for every } u \in H_A(\Omega). \end{aligned} \tag{6.6}$$

Here and in the following the letter C_j denotes a generic positive constant.

Step 2: On the other hand, if we let

$$w := u - v = u - \mathcal{G}_D(Au) \in L^2(\Omega) \quad \text{for } u \in H_A(\Omega),$$

then it follows that the function w satisfies the homogeneous equation

$$Aw = Au - Av = Au - Au = 0 \quad \text{in } \Omega.$$

By applying Seeley [43, Theorems 5 and 6] to the function w, we obtain that the trace maps

$$\begin{cases} \gamma_0 : \mathcal{N}(A, 0) \longrightarrow H^{-1/2}(\Gamma), \\ \gamma_1 : \mathcal{N}(A, 0) \longrightarrow H^{-3/2}(\Gamma) \end{cases}$$

are both continuous on the null space for A (see formula (5.3) with $s := 0$)

$$\mathcal{N}(A, 0) = \left\{ w \in L^2(\Omega) : Aw = 0 \text{ in } \Omega \right\}.$$

Hence, we have the inequalities

$$\|\gamma_0 w\|_{H^{-1/2}(\Gamma)} \leq C_3 \|w\|_{L^2(\Omega)}\,, \tag{6.7a}$$

$$\|\gamma_1 w\|_{H^{-3/2}(\Gamma)} \leq C_4 \|w\|_{L^2(\Omega)}\,. \tag{6.7b}$$

Therefore, we have the assertion

$$B\boldsymbol{\gamma} w = \mu(x')\gamma_1 w + Q\left(x', D_{x'}\right)(\gamma_0 w) \in H^{-5/2}(\Gamma),$$

and we have, by inequalities (6.7a) and (6.7b),

$$\begin{aligned} \|B\boldsymbol{\gamma} w\|_{H^{-5/2}(\Gamma)} &\leq \left\|\mu(x')\gamma_1 w\right\|_{H^{-5/2}(\Gamma)} + \left\|Q\left(x', D_{x'}\right)(\gamma_0 w)\right\|_{H^{-5/2}(\Gamma)} \\ &\leq C_5 \left(\left\|\mu(x')\gamma_1 w\right\|_{H^{-3/2}(\Gamma)} + \|\gamma_0 w\|_{H^{-1/2}(\Gamma)}\right) \\ &\leq C_6 \|w\|_{L^2(\Omega)} = C_6 \|u - v\|_{L^2(\Omega)} \\ &\leq C_6 \left(\|u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\right) = C_6 \left(\|u\|_{L^2(\Omega)} + \|\mathcal{G}_D(Au)\|_{L^2(\Omega)}\right) \\ &\leq C_7 \left(\|u\|_{L^2(\Omega)} + \|\mathcal{G}_D(Au)\|_{H^2(\Omega)}\right) \\ &\leq C_8 \left(\|u\|_{L^2(\Omega)} + \|Au\|_{L^2(\Omega)}\right). \end{aligned} \tag{6.8}$$

Step 3: In this way, we can express uniquely every function $u \in H_A(\Omega)$ in the form

$$u = v + w \tag{6.9}$$

where

$$v = \mathcal{G}_D(Au) \in H^2(\Omega), \tag{6.10a}$$
$$w = u - v \in \mathcal{N}(A, 0). \tag{6.10b}$$

Therefore, by using the decompositions (6.9) and (6.10) we can define the Višik–Ventcel' boundary condition $B\boldsymbol{\gamma} u$ by the formula

$$B\boldsymbol{\gamma} u := B\boldsymbol{\gamma} v + B\boldsymbol{\gamma} w \in H^{-5/2}(\Gamma) \quad \text{for } u \in H_A(\Omega).$$

Then we have, by inequalities (6.6) and (6.8) and formula (6.2),

$$\begin{aligned} \|B\boldsymbol{\gamma} u\|_{H^{-5/2}(\Gamma)} &\leq \|B\boldsymbol{\gamma} v\|_{H^{-5/2}(\Gamma)} + \|B\boldsymbol{\gamma} w\|_{H^{-5/2}(\Gamma)} \\ &\leq C_9 \|B\boldsymbol{\gamma} v\|_{H^{1/2}(\Gamma)} + \|B\boldsymbol{\gamma} w\|_{H^{-5/2}(\Gamma)} \\ &\leq C_2\, C_9 \|Au\|_{L^2(\Omega)} + C_8 \left(\|u\|_{L^2(\Omega)} + \|Au\|_{L^2(\Omega)}\right) \\ &\leq C \|u\|_{H_A(\Omega)} \quad \text{for all } u \in H_A(\Omega). \end{aligned} \tag{6.11}$$

This proves the desired inequality (6.3), with

$$C = \max\left\{C_2\, C_9,\, \sqrt{2}\, C_8\right\}.$$

The proof of Theorem 6.1 is complete.

Similarly, we can prove the following corollary:

Corollary 6.1 *Assume that condition* (5.2) *is satisfied. For every $u \in H^{s+2}(\Omega)$ with $s > -1/2$, we can define the Višik–Ventcel' boundary condition $B\boldsymbol{\gamma} u$ as an element of the Sobolev space $H^{s-1/2}(\Gamma)$. Moreover, the Višik–Ventcel' boundary operator*

$$B\boldsymbol{\gamma}\colon H^{s+2}(\Omega) \longrightarrow H^{s-1/2}(\Gamma) \tag{6.12}$$

is continuous for $s > -1/2$.

Proof As in Step 3 of the proof of Theorem 6.1, we can decompose the function $u \in H^{s+2}(\Omega)$ in the form

$$u = v + w \in H^{s+2}(\Omega), \tag{6.13}$$

where

$$v = \mathcal{G}_D(Au) \in H^{s+2}(\Omega), \tag{6.14a}$$

$$w = u - v \in \mathcal{N}(A, s+2). \tag{6.14b}$$

Hence, it follows from formulas (6.13) and (6.14) and the trace theorem ([1, Remarks 7.45], [12, p. 98, Section 4]) that

$$\begin{aligned} B\boldsymbol{\gamma} u &= B\boldsymbol{\gamma} v + B\boldsymbol{\gamma} w \\ &= \mu(x')\gamma_1 v + Q\left(x', D_{x'}\right)(\gamma_0 v) + \mu(x')\gamma_1 w + Q\left(x', D_{x'}\right)(\gamma_0 w) \\ &= \mu(x')\gamma_1 v + \mu(x')\gamma_1 w + Q\left(x', D_{x'}\right)(\gamma_0 w) \in H^{s-1/2}(\Gamma), \end{aligned}$$

since we have the assertions

$$\begin{cases} \gamma_0 v = 0, \quad \gamma_1 v \in H^{s+1/2}(\Gamma), \\ \gamma_0 w \in H^{s+3/2}(\Gamma), \quad \gamma_1 w \in H^{s+1/2}(\Gamma). \end{cases}$$

Therefore, the continuity (6.12) of $B\boldsymbol{\gamma}$ can be proved just as in the proof of inequality (6.11).

The proof of Corollary 6.1 is complete.

Now we can formulate the *non-homogeneous* Višik–Ventcel' boundary value problem in the framework of Sobolev spaces as follows: Given functions $f \in H^s(\Omega)$ and $\psi \in H^{s+1/2}(\Gamma)$ with $s > -1/2$, find a function $u \in H^{s+2}(\Omega)$ such that

$$\begin{cases} Au = f & \text{in } \Omega, \\ B\boldsymbol{\gamma} u = \psi & \text{on } \Gamma. \end{cases} \tag{1.4}$$

Remark 6.1 In this paper, we consider the Višik–Ventcel' boundary value problem (1.4) under the condition that the boundary data φ has one more regularity

$$s + \frac{1}{2} = \left(s - \frac{1}{2}\right) + 1,$$

compared with the Sobolev regularity stated in Corollary 6.1.

6.2 A special reduction to the boundary

For given functions $f \in H^s(\Omega)$ and $\varphi \in H^{s+1/2}(\Gamma)$ with $s > -1/2$, we assume that a function $u \in H^{s+2}(\Omega)$ is a solution of the non-homogeneous Višik-Ventcel' boundary value problem (1.4). Then, by using formulas (6.13) and (6.14) we can reduce the study of the boundary value problem (1.4) to that of a pseudo-differential equation on the boundary Γ, just as in the classical Fredholm integral equation in potential theory.

In fact, we can prove the following proposition (see [49, Theorem 8.3.3]):

Proposition 6.1 *Assume that condition* (5.2) *is satisfied. For given functions* $f \in H^s(\Omega)$ *and* $\psi \in H^{s+1/2}(\Gamma)$ *with* $s > -1/2$, *there exists a solution* $u \in H^{s+2}(\Omega)$ *of the non-homogeneous Višik-Ventcel' problem* (1.4) *if and only if there exists a solution* $\varphi \in H^{s+3/2}(\Gamma)$ *of the equation*

$$B\boldsymbol{\gamma}\left(\mathcal{P}\varphi\right) = \psi - \mu(x')\gamma_1\left(\mathcal{G}_D f\right) = \psi - \mu(x')\,\frac{\partial}{\partial\boldsymbol{\nu}}\left(\mathcal{G}_D f\right)\Big|_{\Gamma} \quad \text{on } \Gamma. \tag{6.15}$$

Moreover, the solutions u *and* φ *are related as follows:*

$$u = \mathcal{G}_D f + \mathcal{P}\varphi \in H^{s+2}(\Omega), \tag{6.16a}$$

$$\varphi = u|_{\Gamma} \in H^{s+3/2}(\Gamma). \tag{6.16b}$$

Proof Indeed, it suffices to note that

$$\begin{aligned}
\psi &= B\boldsymbol{\gamma}u = B\boldsymbol{\gamma}\left(\mathcal{G}_D f + \mathcal{P}\varphi\right) \\
&= B\boldsymbol{\gamma}\left(\mathcal{P}\varphi\right) + \mu(x')\gamma_1\left(\mathcal{G}_D f\right) + Q\left(x', D_{x'}\right)\left(\gamma_0\left(\mathcal{G}_D f\right)\right) \\
&= B\boldsymbol{\gamma}\left(\mathcal{P}\varphi\right) + \mu(x')\gamma_1\left(\mathcal{G}_D f\right) \in H^{s+1/2}(\Gamma),
\end{aligned}$$

since $\gamma_0\left(\mathcal{G}_D f\right) = 0$ on Γ.

The proof of Proposition 6.1 is complete.

If we introduce a boundary operator T by the formula

$$\begin{aligned}
T\colon\ C^\infty(\Gamma) &\longrightarrow C^\infty(\Gamma) \\
\varphi &\longmapsto B\boldsymbol{\gamma}\left(\mathcal{P}\varphi\right),
\end{aligned}$$

then we have the formula

$$T = B\boldsymbol{\gamma}\mathcal{P} = \mu(x')\gamma_1\mathcal{P} + Q\left(x', D_{x'}\right)\left(\gamma_0\mathcal{P}\right) = \mu(x')\,\Pi + Q\left(x', D_{x'}\right), \tag{6.17}$$

since $\gamma_0\mathcal{P} = I$ (formulas (5.4)). Here $\Pi = \gamma_1\mathcal{P}$ is called the *Dirichlet-to-Neumann operator* defined as follows:

$$\Pi\varphi = \gamma_1\left(\mathcal{P}\varphi\right) = \frac{\partial}{\partial\boldsymbol{\nu}}\left(\mathcal{P}\varphi\right)\Big|_{\Gamma} \quad \text{for all } \varphi \in C^\infty(\Gamma). \tag{6.18}$$

It is well known (see [12], [22], [25], [28], [43], [58]) that the Dirichlet-to-Neumann operator Π is a classical, *elliptic* pseudo-differential operator of first order on the boundary Γ.

More precisely, if A is the usual Laplacian Δ, then we can write down the complete symbol $p(x', \xi')$ of Π as follows (see [18], [50, Section 10.7]):

$$\begin{aligned}
p(x', \xi') &:= p_1(x', \xi') + p_0(x', \xi') + \sqrt{-1}\,q_0(x', \xi') + \text{ terms of order} \leq -1 \qquad (6.19) \\
&= -|\xi'| - \frac{1}{2}\left(\frac{\omega_{x'}(\widehat{\xi'}, \widehat{\xi'})}{|\xi'|^2} - (n-1)M(x')\right) \\
&\quad + \sqrt{-1}\,\frac{1}{2}\operatorname{div}\delta_{(\xi')}(x') + \text{ terms of order} \leq -1.
\end{aligned}$$

Here:

(a) $p_1(x', \xi') = -|\xi'|$, where $|\xi'|$ is the length of ξ' with respect to the Riemannian metric of Γ induced by the natural metric of $\mathbf{R}^n$.
(b) $M(x')$ is the *mean curvature* of the boundary Γ at x'.
(c) $\omega_{x'}(\widehat{\xi'}, \widehat{\xi'})$ is the *second fundamental form* of Γ at x', while $\widehat{\xi'} \in T_{x'}(\Gamma)$ is the tangent vector corresponding to the cotangent vector $\xi' \in T^*_{x'}(\Gamma)$ by the duality between $T_{x'}(\Gamma)$ and $T^*_{x'}(\Gamma)$ with respect to the Riemannian metric $(g_{ij}(x'))$ of Γ.
(d) $q_0(x', \xi') = \frac{1}{2} \operatorname{div} \delta_{(\xi')}$ where $\operatorname{div} \delta_{(\xi')}$ is the *divergence* of a real smooth vector field $\delta_{(\xi')}$ on Γ defined (in local coordinates) by the formula

$$\delta_{(\xi')} = \sum_{j=1}^{n-1} \frac{\partial |\xi'|}{\partial \xi_j} \frac{\partial}{\partial x_j} \quad \text{for } \xi' \neq 0.$$

Summing up, we have the following proposition:

Proposition 6.2 *The complete symbol $t(x', \xi')$ of the pseudo-differential operator $T = \mu(x')\,\Pi + Q\left(x', D_{x'}\right)$ is given by the formula*

$$\begin{aligned} t(x', \xi') &:= t_2(x', \xi') + t_1(x', \xi') + \text{ terms of order} \le 0 \qquad (6.20)\\ &= -\Big[\sum_{j,k=1}^{n-1} \alpha^{jk}(x')\xi_j\xi_k\Big] + \Big[\mu(x')\, p_1(x', \xi') + \sqrt{-1}\left(\mu(x')\, q_1(x', \xi') + \sum_{k=1}^{n-1} \beta^k(x')\xi_k\right)\Big] \\ &\quad + \text{ terms of order} \le 0. \end{aligned}$$

Here:

$$\sum_{i,j=1}^{n-1} \alpha^{ij}(x')\, \xi_i\, \xi_j \ge 0 \quad \text{on } T^*(\Gamma). \qquad (6.21a)$$

$$p_1(x', \xi') < 0 \quad \text{on } T^*(\Gamma) \setminus \{0\}. \qquad (6.21b)$$

By using Propositions 6.1 and 6.2, we can reduce the Višik–Ventcel' boundary value problem (1.4) to the study of the pseudo-differential equation (see equation (6.15))

$$T\varphi = \mu(x')\, \Pi\varphi + Q\left(x', D_{x'}\right)\varphi = \psi - \mu(x')\, \frac{\partial}{\partial \boldsymbol{\nu}} \left(\mathcal{G}_D f\right)\Big|_{\Gamma} \quad \text{on } \Gamma. \qquad (6.22)$$

This equation may be considered as a modern version of the classical *Fredholm integral equation* in terms of pseudo-differential operators. We shall formulate this fact more precisely in terms of functional analysis.

Now let X and Y be Banach spaces over the same scalar field. We recall that a densely defined, closed linear operator T from X into Y with domain $\mathcal{D}(T)$ is called a *Fredholm operator* if it satisfies the following three conditions (see [20]):

(i) The null space $\mathcal{N}(T) = \{x \in \mathcal{D}(T) : Tx = 0\}$ has finite dimension, that is, $\dim \mathcal{N}(T) < \infty$.
(ii) The range $\mathcal{R}(T) = \{Tx : x \in \mathcal{D}(T)\}$ is closed in Y.
(iii) The range $\mathcal{R}(T)$ has finite codimension, that is, $\operatorname{codim} \mathcal{R}(T) = \dim Y/\mathcal{R}(T) < \infty$.

Then the *index* of T is defined by the formula

$$\operatorname{ind} T = \dim \mathcal{N}(T) - \operatorname{codim} \mathcal{R}(T).$$

We give a very useful criterion for conditions (i) and (ii) made for Fredholm operators due to Peetre (see [36, Lemma 3], [51, Theorem 2.42]):

Lemma 6.1 (Peetre) *Let X, Y, Z be Banach spaces such that $X \hookrightarrow Z$ with* compact injection, *and let $T : X \to Y$ be a closed linear operator with dense domain $\mathcal{D}(T)$. Then the following two conditions are equivalent:*

(i) The null space $\mathcal{N}(T)$ of T has finite dimension and the range $\mathcal{R}(T)$ of T is closed in Y.
(ii) There is a constant $C > 0$ such that the a priori *estimate*

$$\|x\|_X \leq C\left(\|Tx\|_Y + \|x\|_Z\right)$$

holds true for all $x \in \mathcal{D}(T)$.

First, by using Corollary 6.1 we can associate with problem (1.4) a densely defined, closed linear operator

$$\boxed{\mathcal{A} = (A,\, B\boldsymbol{\gamma}) : H^{s+2}(\Omega) \longrightarrow H^s(\Omega) \oplus H^{s+1/2}(\Gamma)}$$

for $s > -1/2$ as follows.

(a) The domain $\mathcal{D}(\mathcal{A})$ of definition of $\mathcal{A}$ is the space

$$\mathcal{D}(\mathcal{A}) = \left\{u \in H^{s+2}(\Omega) :\, B\boldsymbol{\gamma} u \in H^{s+1/2}(\Gamma)\right\}. \tag{6.23}$$

(b) $\mathcal{A}u = (Au,\, B\boldsymbol{\gamma} u)$ for every $u \in \mathcal{D}(\mathcal{A})$.

Indeed, since $A\colon H^{s+2}(\Omega) \to H^s(\Omega)$ and $B\boldsymbol{\gamma} : H^{s+2}(\Omega) \to H^{s-1/2}(\Gamma)$ are both continuous, it follows that $\mathcal{A}$ is a closed operator (see the proof of Step (1) of Proposition 7.1). Furthermore, the operator $\mathcal{A}$ is densely defined, since the domain $\mathcal{D}(\mathcal{A})$ contains a dense subspace $C^\infty(\overline{\Omega})$ of $H^{s+2}(\Omega)$. The situation can be visualized in Table 6.1 below (see Remark 6.1).

$$\begin{tikzcd}
H^{s+2}(\Omega) \arrow[r, "{(A,\, B\boldsymbol{\gamma})}"] & H^s(\Omega) \oplus H^{s-1/2}(\Gamma) \\
\mathcal{D}(\mathcal{A}) \arrow[u] \arrow[r, "\mathcal{A}"] & H^s(\Omega) \oplus H^{s+1/2}(\Gamma) \arrow[u] \\
C^\infty(\overline{\Omega}) \arrow[u] \arrow[r, "{(A,\, B\boldsymbol{\gamma})}"'] & C^\infty(\overline{\Omega}) \oplus C^\infty(\Gamma) \arrow[u]
\end{tikzcd}$$

Table 6.1 The mapping property of the operator $\mathcal{A}$

Similarly, by using Proposition 6.1 we can associate with equations (6.15) and (6.17) a densely defined, closed linear operator

$$\boxed{\mathcal{T}\colon H^{s+3/2}(\Gamma) \longrightarrow H^{s+1/2}(\Gamma)}$$

for $s > -1/2$ as follows.

(α) The domain $\mathcal{D}(\mathcal{T})$ of definition of $\mathcal{T}$ is the space

$$\mathcal{D}(\mathcal{T}) = \left\{ \varphi \in H^{s+3/2}(\Gamma) : T\varphi \in H^{s+1/2}(\Gamma) \right\}. \tag{6.24}$$

(β) $\mathcal{T}\varphi = B\boldsymbol{\gamma}\left(\mathcal{P}\varphi\right) = \mu(x')\,\Pi\varphi + Q\left(x', D_{x'}\right)\varphi$ for every $\varphi \in \mathcal{D}(\mathcal{T})$.

Indeed, since $T\colon H^{s+3/2}(\Gamma) \to H^{s-1/2}(\Gamma)$ is continuous, it follows that $\mathcal{T}$ is a closed operator. Furthermore, the operator $\mathcal{T}$ is densely defined, since the domain $\mathcal{D}(\mathcal{T})$ contains a dense subspace $C^\infty(\Gamma)$ of $H^{s+3/2}(\Gamma)$. The situation can be visualized in Table 6.2 below.

$$\begin{tikzcd}
H^{s+3/2}(\Gamma) \arrow[r, "T"] & H^{s-1/2}(\Gamma) \\
\mathcal{D}(\mathcal{T}) \arrow[u] \arrow[r, "\mathcal{T}"] & H^{s+1/2}(\Gamma) \arrow[u] \\
C^\infty(\Gamma) \arrow[u] \arrow[r, "T"'] & C^\infty(\Gamma) \arrow[u]
\end{tikzcd}$$

Table 6.2 The mapping property of the operator $\mathcal{T}$

Then we obtain the following formula for the indices of the operators $\mathcal{A}$ and $\mathcal{T}$ (see [49, Theorem 8.3.8]):

Theorem 6.2 *Assume that condition* (5.2) *is satisfied. If the operator $\mathcal{T}$, defined by formula* (6.24)*, is a Fredholm operator, then the operator $\mathcal{A}$, defined by formula* (6.23)*, is a Fredholm operator. In this case, we have the formula*

$$\operatorname{ind} \mathcal{A} = \operatorname{ind} \mathcal{T}. \tag{6.25}$$

6.3 Index of the operator $\mathcal{T}$

The purpose of this subsection is to prove the assertion

$$\operatorname{ind} \mathcal{T} = \operatorname{ind} \mathcal{A} = 0 \quad \text{for every } s > -1/2$$

under condition (H.2) (Proposition 6.6 and Theorem 6.3).

First, since the pseudo-differential operator

$$T = \mu(x')\,\Pi + Q\left(x', D_{x'}\right) \tag{6.17}$$

has the complete symbol $t(x', \xi')$ given in formula (6.20), we can obtain the following fundamental proposition for T (see [24, Theorem 5.2] and [38, Theorem 7]):

Proposition 6.3 *Assume that condition (H.2) is satisfied. Then we have the following three assertions:*

(i) There exist constants $c > 0$ and $C > 0$ such that

$$-\text{Re}\,(T\varphi, \varphi)_{L^2(\Gamma)} \geq c\, \|\varphi\|^2_{H^{1/2}(\Gamma)} - C\, \|\varphi\|^2_{L^2(\Gamma)} \quad \text{for all } \varphi \in C^\infty(\Gamma). \tag{6.26}$$

(ii) There exists a constant $C_1 > 0$ such that

$$\|\varphi\|^2_{H^{1/2}(\Gamma)} \leq C_1 \left(\|T\varphi\|^2_{H^{-1/2}(\Gamma)} + \|\varphi\|^2_{L^2(\Gamma)} \right) \quad \text{for all } \varphi \in C^\infty(\Gamma).$$

(iii) The pseudo-differential operator T is hypoelliptic *with loss of one derivative on Γ. More precisely, we have, for every $s \in \mathbf{R}$,*

$$\varphi \in \mathcal{D}'(\Gamma),\ T\varphi \in H^s(\Gamma) \implies \varphi \in H^{s+1}(\Gamma). \tag{6.27}$$

Moreover, for any $s' < s+1$ there exists a constant $C_{s,s'} > 0$ such that

$$\|\varphi\|^2_{H^{s+1}(\Gamma)} \leq C_{s,s'} \left(\|T\varphi\|^2_{H^s(\Gamma)} + \|\varphi\|^2_{H^{s'}(\Gamma)} \right). \tag{6.28}$$

Proof (i) We decompose the differential operator $Q\left(x', D_{x'}\right)$, given by formula (1.3), as follows:

$$\begin{aligned} Q\left(x', D_{x'}\right)\varphi &:= R\left(x', D_{x'}\right)\varphi + \beta\left(x', D_{x'}\right)\varphi \\ &= \left(\sum_{i,j=1}^{n-1} \alpha^{ij}(x') \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \gamma(x')\varphi \right) + \sum_{i=1}^{n-1} \beta^i(x') \frac{\partial \varphi}{\partial x_i}. \end{aligned}$$

Hence, we have the decomposition formula

$$-T = -\mu(x')\,\Pi - Q\left(x', D_{x'}\right) = \left(-\mu(x')\Pi - R(x', D_{x'})\right) - \beta(x', D_{x'}).$$

Then, by integration by parts it follows that

$$\text{Re}\left(\beta(x', D_{x'})\varphi, \varphi\right)_{L^2(\Gamma)} = -\frac{1}{2}\left(\text{div}\,\beta \cdot \varphi, \varphi\right),$$

so that

$$-\text{Re}\left(\beta(x', D_{x'})\varphi, \varphi\right)_{L^2(\Gamma)} \geq -\frac{1}{2} \max_{x' \in \Gamma} \left|\text{div}\,\beta(x')\right| \cdot \|\varphi\|^2_{L^2(\Gamma)} \quad \text{for all } \varphi \in C^\infty(\Gamma). \tag{6.29}$$

The proof of part (i) is divided into three steps.

Step (I): First, we consider the case of the open subset

$$\Gamma \setminus \Gamma_0 = \left\{x' \in \Gamma : \mu(x') > 0\right\}.$$

In this case, by using local coordinate systems flattening out Γ, together with a partition of unity we can apply Theorems 3.1 and 3.3 (or [25, Theorem 22.3.3]) to the pseudo-differential operator

$$-\mu(x')\Pi - R(x', D_{x'})$$

in the following way:

(1) Apply the Gårding inequality (3.2) to the pseudo-differential operator $A := -\mu(x')\, \Pi$ with $m := 1$ and $s := 0$.
(2) Apply the Fefferman–Phong inequality (3.4) to the differential operator $A := -R\left(x', D_{x'}\right)$.

Case (1): By assertion (6.21b), we remark that the real part $\mu(x')\, p_1\left(x', \xi'\right)$ of the principal symbol of $\mu(x')\Pi$ satisfies the condition: For every compact set $K \subset \Gamma \setminus \Gamma_0$ we can find a constant $\delta_K > 0$ such that

$$-\mu(x')\, p_1(x', \xi') \geq \delta_K \left|\xi'\right| \quad \text{for every } x' \in K.$$

Case (2): On the other hand, it follows from condition (6.21a) that the complete symbol $r(x', \xi')$ of $R\left(x', D_{x'}\right)$ satisfies the condition

$$-r(x', \xi') = \sum_{i,j=1}^{n-1} \alpha^{ij}(x')\, \xi_i\, \xi_j - \gamma(x') \geq 0 \quad \text{on } T^*(\Gamma),$$

since $\gamma(x') \leq 0$ on Γ.

Then we can find constants $c_K > 0$ and $C_K > 0$ such that

$$\begin{aligned} &- \mathrm{Re}\left(\left(\mu(x')\, \Pi + R(x', D_{x'})\right) \varphi, \varphi\right)_{L^2(\Gamma)} \\ &\geq c_K \left\|\varphi\right\|^2_{H^{1/2}(\Gamma)} - C_K \left\|\varphi\right\|^2_{L^2(\Gamma)} \quad \text{for all } \varphi \in C^\infty_K(\Gamma), \end{aligned} \tag{6.30}$$

where

$$C^\infty_K(\Gamma) = \left\{\varphi : \varphi \in C^\infty(\Gamma),\ \mathrm{supp}\, \varphi \subset K\right\}.$$

Indeed, it suffices to note (see [34, Lemma 2.1]) that if $P\left(x', D_{x'}\right)$ is a *first order*, pseudo-differential operator with *real* principal symbol, then we can find some constant $d_0 > 0$ such that

$$\mathrm{Re}\left(\sqrt{-1}\, P\left(x', D_{x'}\right) \varphi, \varphi\right)_{L^2(\Gamma)} \geq -d_0 \left\|\varphi\right\|^2_{L^2(\Gamma)} \quad \text{for all } \varphi \in C^\infty(\Gamma).$$

Therefore, by combining inequalities (6.29) and (6.30) we have the inequality

$$\begin{aligned} &- \mathrm{Re}\left(T\varphi, \varphi\right)_{L^2(\Gamma)} \\ &= -\mathrm{Re}\left(\left(\mu(x')\Pi + R\left(x', D_{x'}\right) + \beta\left(x', D_{x'}\right)\right) \varphi, \varphi\right)_{L^2(\Gamma)} \\ &\geq c_K \left\|\varphi\right\|^2_{H^{1/2}(\Gamma)} - \left(C_K + \frac{1}{2} \max_{x' \in \Gamma} \left|\mathrm{div}\, \beta(x')\right|\right) \left\|\varphi\right\|^2_{L^2(\Gamma)} \text{ for all } \varphi \in C^\infty_K(\Gamma), \end{aligned} \tag{6.31}$$

where K is an arbitrary compact set in $\Gamma \setminus \Gamma_0$.

Step (II): Secondly, we consider the case of an open neighborhood V of the closed subset

$$\Gamma_0 = \left\{x' \in \Gamma : \mu(x') = 0\right\}.$$

In this case we can apply Theorems 3.2 and 2.1 to the pseudo-differential operator

$$-\mu(x')\Pi - R(x', D_{x'})$$

in the following way:

(3) Apply the sharp Gårding inequality (3.3) to the pseudo-differential operator $A := -\mu(x')\, \Pi$ with $m := 1$ and $s := -1/2$.

(4) Apply the Fefferman–Phong subelliptic estimate (2.2) to the differential operator $Q(x', D_{x'}) := R(x', D_{x'})$ with $\varepsilon := 1/2$.

Case (3): By assertion (6.21b), we remark that the real part $\mu(x')\, p_1\left(x', \xi'\right)$ of the principal symbol of $\mu(x')\varPi$ satisfies the condition

$$-\mu(x')\, p_1\left(x', \xi'\right) \geq 0 \quad \text{for every } x' \in V.$$

By applying the sharp Gårding inequality (3.3) to $A := -\mu(x')\,\varPi$ with $m := 1$ and $s := -1/2$, for every compact set $K' \subset V$ we can find constants $d_{K'} > 0$ and $D_{K'} > 0$ such that

$$-\mathrm{Re}\left(\mu(x')\varPi\varphi, \varphi\right)_{L^2(\varGamma)} \geq -d_{K'}\, \|\varphi\|^2_{L^2(\varGamma)} - D_{K'}\, \|\varphi\|^2_{H^{-1/2}(\varGamma)} \quad \text{for all } \varphi \in C^\infty_{K'}(\varGamma).$$

Since the injection $L^2(\varGamma) \hookrightarrow H^{-1/2}(\varGamma)$ is continuous, we can find another constant $C_{K'} > 0$ such that

$$-\mathrm{Re}\left(\mu(x')\varPi\varphi, \varphi\right)_{L^2(\varGamma)} \geq -C_{K'}\, \|\varphi\|^2_{L^2(\varGamma)} \quad \text{for all } \varphi \in C^\infty_{K'}(\varGamma). \tag{6.32}$$

Case (4): On the other hand, since the differential operator $R\left(x', D_{x'}\right)$ satisfies condition (H.2), by applying the Fefferman–Phong subelliptic estimate (2.2) to $R\left(x', D_{x'}\right)$ with $\varepsilon := 1/2$ we can find constants $c'_{K'} > 0$ and $C'_{K'} > 0$ such that

$$-\mathrm{Re}\left(R\left(x', D_{x'}\right)\varphi, \varphi\right)_{L^2(\varGamma)} \geq c'_{K'}\, \|\varphi\|^2_{H^{1/2}(\varGamma)} - C'_{K'}\, \|\varphi\|^2_{L^2(\varGamma)} \quad \text{for all } \varphi \in C^\infty_{K'}(\varGamma). \tag{6.33}$$

Therefore, by combining inequalities (6.29), (6.32) and (6.33) we have the inequality

$$\begin{aligned} -\mathrm{Re}\,(T\varphi, \varphi)_{L^2(\varGamma)} &= -\mathrm{Re}\left(\left(\mu(x')\varPi + R(x', D_{x'}) + \beta(x', D_{x'})\right)\varphi, \varphi\right)_{L^2(\varGamma)} \\ &\geq c'_{K'}\, \|\varphi\|^2_{H^{1/2}(\varGamma)} - \left(C_{K'} + C'_{K'} + \frac{1}{2}\max_{x' \in \varGamma}\left|\mathrm{div}\,\beta(x')\right|\right) \|\varphi\|^2_{L^2(\varGamma)} \\ &\quad \text{for all } \varphi \in C^\infty_{K'}(\varGamma), \end{aligned} \tag{6.34}$$

where K' is an arbitrary compact set in the open neighborhood V of $\varGamma_0$.

Step (III): Now we choose an open subset W of $\varGamma$ such that

$$\begin{cases} \varGamma \subset W \cup V, \\ W \cap \varGamma_0 = \emptyset, \end{cases}$$

and construct two functions $\theta \in C^\infty_0(W)$ and $\omega \in C^\infty_0(V)$ such that

$$\theta(x')^2 + \omega(x')^2 = 1 \quad \text{on } \varGamma.$$

Then any function $\varphi \in C^\infty(\varGamma)$ can be expressed in the form

$$\varphi = \theta\,(\theta\,\varphi) + \omega\,(\omega\,\varphi) \quad \text{on } \varGamma,$$

where

$$\begin{cases} \theta\,\varphi \in C^\infty_K(\varGamma), \quad K = \mathrm{supp}\,\theta \subset \varGamma \setminus \varGamma_0, \\ \omega\,\varphi \in C^\infty_{K'}(\varGamma), \quad K' = \mathrm{supp}\,\omega \subset V. \end{cases}$$

We recall that if $P\left(x', D_{x'}\right)$ is a *first order*, pseudo-differential operator with *real* principal symbol, then we have the inequality

$$\operatorname{Re}\left(\sqrt{-1}\, P\left(x', D_{x'}\right) \varphi, \varphi\right)_{L^2(\Gamma)} \geq -d_0 \,\|\varphi\|^2_{L^2(\Gamma)} \quad \text{for all } \varphi \in C^\infty(\Gamma).$$

Therefore, the desired global inequality (6.26) follows by applying inequality (6.31) to the function $\theta\,\varphi$ and inequality (6.34) to the function $\omega\,\varphi$, respectively. More precisely, the reader might be referred to the proof of Wloka [61, Theorem 19.2].

(ii) By using the generalized Schwarz inequality (see [12, Chapitre II, Théorème 2.8]), we obtain from inequality (6.26) that there exist constants $c_0 > 0$ and $C_0 > 0$ such that

$$\begin{aligned} & c_0 \,\|\varphi\|^2_{H^{1/2}(\Gamma)} - C_0 \,\|\varphi\|^2_{L^2(\Gamma)} \\ & \leq \operatorname{Re}\, (T\varphi, \varphi)_{L^2(\Gamma)} \leq \left|(T\varphi, \varphi)_{L^2(\Gamma)}\right| \leq \|T\varphi\|_{H^{-1/2}(\Gamma)} \,\|\varphi\|_{H^{1/2}(\Gamma)} \\ & \leq \frac{\varepsilon^2}{2} \,\|\varphi\|^2_{H^{1/2}(\Gamma)} + \frac{1}{2\varepsilon^2} \,\|T\varphi\|^2_{H^{-1/2}(\Gamma)} \quad \text{for every } \varepsilon > 0. \end{aligned}$$

Therefore, by taking

$$\varepsilon := \sqrt{c_0},$$

we have the desired inequality (6.28)

$$\|\varphi\|^2_{H^{1/2}(\Gamma)} \leq \frac{1}{c_0^2} \,\|T\varphi\|^2_{H^{-1/2}(\Gamma)} + \frac{2C_0}{c_0} \,\|\varphi\|^2_{L^2(\Gamma)} \leq C_1 \left(\|T\varphi\|^2_{H^{-1/2}(\Gamma)} + \|\varphi\|^2_{L^2(\Gamma)}\right),$$

with

$$C_1 := \max\left\{\frac{1}{c_0^2}, \frac{2C_0}{c_0}\right\}.$$

(iii) By using Proposition 6.2, we find from condition (6.21a) that the principal symbol $t_2(x', \xi')$ of $T = \mu(x')\,\Pi + Q\left(x', D_{x'}\right)$ satisfies the condition (see condition (4.2))

$$-t_2(x', \xi') = \sum_{i,j=1}^{n-1} \alpha^{ij}(x')\,\xi_i\,\xi_j \geq 0 \quad \text{on } T^*(\Gamma).$$

Therefore, part (iii) follows by applying Hörmander [24, Theorem 5.2 and estimate (1.1)] to the pseudo-differential operator $-T$ with

$$m := 2, \quad s_0 := -1/2, \quad s' := 0$$

(see inequality (4.3) and assertion (4.4)).

The proof of Proposition 6.3 is complete.

Similarly, we can prove the following results for the *adjoint* $T^* = \left(\mu(x')\,\Pi + Q\right)^*$ of the pseudo-differential operator $T = \mu(x')\,\Pi + Q$, analogous to Proposition 6.3:

Proposition 6.4 *Assume that condition (H.2) is satisfied. Then we have the following three assertions:*

(i) *There exist constants $c > 0$ and $C > 0$ such that*

$$-\mathrm{Re}\,\left(T^*\psi,\psi\right)_{L^2(\Gamma)} \geq c\,\|\psi\|^2_{H^{1/2}(\Gamma)} - C\,\|\psi\|^2_{L^2(\Gamma)} \quad \text{for all } \psi \in C^\infty(\Gamma). \tag{6.35}$$

Here it should be noticed that we can choose the same constants c and C as in inequality (6.26).

(ii) *There exists a constant $C_2 > 0$ such that*

$$\|\psi\|^2_{H^{1/2}(\Gamma)} \leq C_2 \left(\left\|T^*\psi\right\|^2_{H^{-1/2}(\Gamma)} + \|\psi\|^2_{L^2(\Gamma)}\right) \quad \text{for all } \psi \in C^\infty(\Gamma).$$

(iii) *The pseudo-differential operator T^* is* hypoelliptic *with loss of one derivative on Γ. More precisely, we have, for every $s \in \mathbf{R}$,*

$$\psi \in \mathcal{D}'(\Gamma),\ T^*\psi \in H^s(\Gamma) \implies \psi \in H^{s+1}(\Gamma). \tag{6.36}$$

Moreover, for any $s' < s+1$ there exists a constant $C^*_{s,s'} > 0$ such that

$$\|\psi\|^2_{H^{s+1}(\Gamma)} \leq C^*_{s,s'} \left(\left\|T^*\psi\right\|^2_{H^s(\Gamma)} + \|\psi\|^2_{H^{s'}(\Gamma)}\right). \tag{6.37}$$

Proof Indeed, it suffices to note that

$$\begin{aligned}
&2\,\mathrm{Re}\,\left(T^*\psi,\psi\right)_{L^2(\Gamma)} = \left(T^*\psi,\psi\right)_{L^2(\Gamma)} + \overline{\left(T^*\psi,\psi\right)}_{L^2(\Gamma)} \\
&= \left(T^*\psi,\psi\right)_{L^2(\Gamma)} + \left(\psi,T^*\psi\right)_{L^2(\Gamma)} = \left(T^*\psi,\psi\right)_{L^2(\Gamma)} + (T\psi,\psi)_{L^2(\Gamma)} \\
&= (\psi,T\psi)_{L^2(\Gamma)} + (T\psi,\psi)_{L^2(\Gamma)} = \overline{(T\psi,\psi)}_{L^2(\Gamma)} + (T\psi,\psi)_{L^2(\Gamma)} \\
&= 2\,\mathrm{Re}\,(T\psi,\psi)_{L^2(\Gamma)},
\end{aligned}$$

since we have the formula

$$\left(T^*\right)^* = \left(\left(\mu(x')\,\Pi + Q\right)^*\right)^* = \mu(x')\,\Pi + Q = T$$

for the pseudo-differential operator T (see [12], [25], [28], [58]).

The proof of Proposition 6.4 is complete.

Recall that the operator

$$\mathcal{T}\colon H^{s+3/2}(\Gamma) \longrightarrow H^{s+1/2}(\Gamma)$$

is a densely defined, closed linear operator given by formula (6.24) for $s > -1/2$ (see also Table 6.2).

The *adjoint operator* $\mathcal{T}^*$ of $\mathcal{T}$ is a densely defined, closed linear operator

$$\boxed{\mathcal{T}^*\colon H^{-s-1/2}(\Gamma) \longrightarrow H^{-s-3/2}(\Gamma)}$$

for $s > -1/2$ such that

$${}_{s+1/2}\left(\mathcal{T}\varphi,\psi\right)_{-s-1/2} = {}_{s+3/2}\left(\varphi,\mathcal{T}^*\psi\right)_{-s-3/2} \quad \text{for all } \varphi \in \mathcal{D}(\mathcal{T}) \text{ and } \psi \in \mathcal{D}(\mathcal{T}^*),$$

where ${}_\sigma\left(\cdot,\cdot\right)_{-\sigma}$ denotes the sesquilinear pairing between the Sobolev spaces $H^\sigma(\Gamma)$ and $H^{-\sigma}(\Gamma)$ for each $\sigma \in \mathbf{R}$ (see [62, Chapter VII]). The situation can be visualized in Table 6.3 below.

The next lemma allows us to give a characterization of the adjoint operator $\mathcal{T}^*$ in terms of pseudo-differential operators (see [49, Lemma 8.4.8]):

$$\begin{tikzcd} H^{-s-5/2}(\varGamma) & H^{-s-1/2}(\varGamma) \arrow[l, "T^*"'] \\ H^{-s-3/2}(\varGamma) \arrow[u] & \mathcal{D}(\mathcal{T}^*) \arrow[l, "\mathcal{T}^*"'] \arrow[u] \\ C^\infty(\varGamma) \arrow[u] & C^\infty(\varGamma) \arrow[l, "T^*"] \arrow[u] \end{tikzcd}$$

Table 6.3 The mapping property of the adjoint operator $\mathcal{T}^*$

Lemma 6.2 *Let M be a compact, C^∞ manifold without boundary. If T is a classical, pseudo-differential operator of second order on M, we define a densely defined, closed linear operator*

$$\mathcal{T}\colon H^{s+3/2}(M) \longrightarrow H^{s+1/2}(M) \quad \textit{for some } s \in \mathbf{R}$$

as follows:

(a) The domain $\mathcal{D}(\mathcal{T})$ of $\mathcal{T}$ is the space

$$\mathcal{D}(\mathcal{T}) = \left\{ \varphi \in H^{s+3/2}(M) : T\varphi \in H^{s+1/2}(M) \right\}.$$

(b) $\mathcal{T}\varphi = T\varphi$ for every $\varphi \in \mathcal{D}(\mathcal{T})$.
Here $T\varphi$ is taken in the sense of distributions.

Then the adjoint operator $\mathcal{T}^*$ of $\mathcal{T}$ is characterized as follows:

(c) The domain $\mathcal{D}(\mathcal{T}^*)$ of $\mathcal{T}^*$ is contained in the space

$$\left\{ \psi \in H^{-s-1/2}(M) : T^*\psi \in H^{-s-3/2}(M) \right\},$$

where T^* is the adjoint *of the pseudo-differential operator T.*
(d) $\mathcal{T}^*\psi = T^*\psi$ for every $\psi \in \mathcal{D}(\mathcal{T}^*)$.

Proof Let ψ be an arbitrary element of $\mathcal{D}(\mathcal{T}^*) \subset H^{-s-1/2}(M)$. By using *Friedrichs' mollifiers* ([22, p. 178, Remark], [12, Chapitre IV, Corollaire 10.5]), we can find a sequence $\{\psi_j\}_{j=1}^{\infty}$ of $C^\infty(M)$ such that

$$\begin{cases} \psi_j \longrightarrow \psi & \text{in } H^{-s-1/2}(M), \\ T^*\psi_j \longrightarrow T^*\psi & \text{in } H^{-s-3/2}(M). \end{cases}$$

Then we have, for all $\varphi \in C^\infty(M) \subset \mathcal{D}(\mathcal{T})$,

$$\begin{aligned} {}_{-s-3/2}\left(\mathcal{T}^*\psi, \varphi\right)_{s+3/2} &= {}_{-s-1/2}\left(\psi, \mathcal{T}\varphi\right)_{s+1/2} = {}_{-s-1/2}\left(\psi, T\varphi\right)_{s+1/2} \\ &= \lim_{j\to\infty} \left(\psi_j, T\varphi\right)_{L^2(\varGamma)} = \lim_{j\to\infty} \left(T^*\psi_j, \varphi\right)_{L^2(\varGamma)} \\ &= {}_{-s-3/2}\left(T^*\psi, \varphi\right)_{s+3/2}. \end{aligned}$$

This proves that

$$T^*\psi = \mathcal{T}^*\psi \in H^{-s-3/2}(M).$$

The proof of Lemma 6.2 is complete.

Moreover, the next *regularity* results for the closed operators $\mathcal{T}$ and $\mathcal{T}^*$ follow by combining Propositions 6.3 and 6.4 and Lemma 6.2:

Proposition 6.5 *Assume that condition (H.2) is satisfied. Then we have the following two* regularity *results:*

(i) *If $\varphi \in \mathcal{D}'(\Gamma)$ and $T\varphi \in H^\sigma(\Gamma)$ for $\sigma \in \mathbf{R}$, then it follows from the regularity property* (6.27) *that $\varphi \in H^{\sigma+1}(\Gamma)$. In particular, we have the regularity result for the null space of the closed operator $\mathcal{T}$:*

$$\mathcal{N}(\mathcal{T}) = \left\{\varphi \in H^{s+3/2}(\Gamma) : T\varphi = 0\right\} \subset C^\infty(\Gamma). \tag{6.38}$$

(ii) *If $\psi \in \mathcal{D}'(\Gamma)$ and $T^*\psi \in H^\sigma(\Gamma)$ for $\sigma \in \mathbf{R}$, then it follows from the regularity property* (6.36) *that $\psi \in H^{\sigma+1}(\Gamma)$. In particular, we have the regularity result for the null space of the adjoint operator $\mathcal{T}^*$:*

$$\mathcal{N}(\mathcal{T}^*) = \left\{\psi \in H^{-s-1/2}(\Gamma) : T^*\psi = 0\right\} \subset C^\infty(\Gamma). \tag{6.39}$$

Now we can prove the following index formula for the closed operator $\mathcal{T}$:

Proposition 6.6 *If condition (H.2) is satisfied, then the closed operator $\mathcal{T}$ is a Fredholm operator with index* zero*:*

$$\operatorname{ind} \mathcal{T} = 0 \quad \textit{for every } s > -1/2. \tag{6.40}$$

Proof The proof is divided into three steps.

Step 1: First, by virtue of *Friedrichs' mollifiers* ([22, p. 178, Remark], [12, Chapitre IV, Corollaire 10.5]), for every $\varphi \in \mathcal{D}(\mathcal{T})$ we can find a sequence $\{\varphi_j\}_{j=1}^\infty$ of $C^\infty(\Gamma)$ such that

$$\begin{cases} \varphi_j \longrightarrow \varphi & \text{in } H^{s+3/2}(\Gamma), \\ T\varphi_j \longrightarrow T\varphi & \text{in } H^{s+1/2}(\Gamma). \end{cases}$$

Hence, by passing to the limit in inequality (6.28) with

$$\varphi := \varphi_j, \quad s := s + \frac{1}{2}, \quad s' := s - \frac{1}{2},$$

we have the inequality

$$\|\varphi\|^2_{H^{s+3/2}(\Gamma)} \leq C_3 \left(\|T\varphi\|^2_{H^{s+1/2}(\Gamma)} + \|\varphi\|^2_{H^{s-1/2}(\Gamma)}\right) \text{ for all } \varphi \in \mathcal{D}(\mathcal{T}). \tag{6.41}$$

However, by the *Rellich–Kondrachov theorem* ([1, Theorem 6.3, Parts I and II], [12, p. 95, Proposition 3.4]) it follows that the injection

$$H^{s+3/2}(\Gamma) \longrightarrow H^{s-1/2}(\Gamma)$$

is *compact*.

Therefore, by applying Peetre's lemma (Lemma 6.1) to the closed operator $\mathcal{T}$ we obtain from assertion (6.38) and inequality (6.41) that

$$\dim \mathcal{N}(\mathcal{T}) < \infty. \tag{6.42a}$$

$$\text{The range } \mathcal{R}(\mathcal{T}) \text{ is closed in } H^{s+1/2}(\Gamma). \tag{6.42b}$$

Similarly, it follows from an application of inequality (6.37) with

$$s := -s - 3/2, \quad s' := -s - 5/2$$

that

$$\|\psi\|^2_{H^{-s-1/2}(\Gamma)} \leq C_3^* \left(\|T^*\psi\|^2_{H^{-s-3/2}(\Gamma)} + \|\psi\|^2_{H^{-s-5/2}(\Gamma)} \right) \quad \text{for all } \psi \in \mathcal{D}(\mathcal{T}^*). \tag{6.43}$$

Therefore, by applying Peetre's lemma (Lemma 6.1) to the adjoint operator $\mathcal{T}^*$ we obtain from assertion (6.39) and inequality (6.43) that

$$\dim \mathcal{N}\left(\mathcal{T}^*\right) = \operatorname{codim} \mathcal{R}\left(\mathcal{T}\right) < \infty. \tag{6.44a}$$

$$\text{The range } \mathcal{R}\left(\mathcal{T}^*\right) \text{ is closed in } H^{-s-3/2}(\Gamma), \tag{6.44b}$$

since the injection

$$H^{-s-1/2}(\Gamma) \longrightarrow H^{-s-5/2}(\Gamma)$$

is *compact*.

By combining assertions (6.42a) and (6.44a), we find that

$$\operatorname{ind} \mathcal{T} = \dim \mathcal{N}\left(\mathcal{T}\right) - \dim \mathcal{N}\left(\mathcal{T}^*\right) < \infty. \tag{6.45}$$

Moreover, we remark from assertions (6.38) and (6.39) that $\operatorname{ind} \mathcal{T}$ is *independent* of $s > -1/2$.

Step 2: Now we show that

$$\operatorname{ind}\left(\mathcal{T} - C\right) = 0 \quad \text{for every } s > -1/2, \tag{6.46}$$

where C is the same positive constant as in inequalities (6.26) and (6.35).

First, we show that

$$\mathcal{N}\left(\mathcal{T} - C\right) = \{0\}. \tag{6.47}$$

To do so, we assume that $\varphi \in \mathcal{N}\left(\mathcal{T} - C\right)$:

$$\varphi \in H^{s+3/2}(\Gamma) \text{ and } (T - C)\varphi = 0.$$

Then, by virtue of Friedrichs' mollifiers we can find a sequence $\{\varphi_j\}_{j=1}^{\infty}$ of $C^\infty(\Gamma)$ such that we have, for $s > -1/2$,

$$\begin{cases} \varphi_j \longrightarrow \varphi & \text{in } H^{s+3/2}(\Gamma) \subset L^2(\Gamma), \\ (T - C)\varphi_j \longrightarrow 0 & \text{in } H^{s+1/2}(\Gamma) \subset L^2(\Gamma). \end{cases}$$

However, by using Schwarz's inequality we obtain from inequality (6.26) with $\varphi := \varphi_j$ that

$$\begin{aligned} c \left\|\varphi_j\right\|^2_{H^{1/2}(\Gamma)} &\leq -\operatorname{Re}\left(\left(T - C\right)\varphi_j, \varphi_j\right)_{L^2(\Gamma)} \leq \left| \left(\left(T - C\right)\varphi_j, \varphi_j\right)_{L^2(\Gamma)} \right| \\ &\leq \left\|(T - C)\varphi_j\right\|_{L^2(\Gamma)} \cdot \left\|\varphi_j\right\|_{L^2(\Gamma)}, \end{aligned}$$

so that

$$\varphi_j \longrightarrow 0 \quad \text{in } H^{1/2}(\Gamma) \subset L^2(\Gamma).$$

This proves that $\varphi = 0$.

Similarly, by using inequality (6.35) we can prove that

$$\mathcal{N}\left(\mathcal{T}^{*}-C\right)=\{0\}. \tag{6.48}$$

Therefore, the desired assertion (6.46) follows from assertions (6.47) and (6.48):

$$\operatorname{ind}(\mathcal{T}-C)=\dim\mathcal{N}(\mathcal{T}-C)-\dim\mathcal{N}\left(\mathcal{T}^{*}-C\right)=0.$$

Step 3: Finally, we are in a position to prove the desired assertion

$$\operatorname{ind}\mathcal{T}=0 \quad \text{for every } s>-1/2. \tag{6.40}$$

By the *Rellich–Kondrachov theorem* ([1, Theorem 6.3, Parts I and II], [12, p. 95, Proposition 3.4]), it follows that the constant mapping

$$C\colon H^{s+3/2}(\Gamma)\longrightarrow H^{s+1/2}(\Gamma)$$

is *compact*. However, it is known (see Gohberg–Kreĭn [20]) that the index is *stable* under compact perturbations.

Therefore, the desired assertion (6.40) follows from assertion (6.46):

$$\operatorname{ind}\mathcal{T}=\operatorname{ind}((\mathcal{T}-C)+C)=\operatorname{ind}(\mathcal{T}-C)=0.$$

Now the proof of Proposition 6.6 is complete.

Therefore, by combining formulas (6.25) and (6.40) we obtain the following index formula for the operator $\mathcal{A}=(A,\,B\boldsymbol{\gamma})$:

Theorem 6.3 *If condition (H.2) is satisfied, then we have the assertion*

$$\operatorname{ind}\mathcal{A}=0 \quad \textit{for every } s>-1/2. \tag{6.49}$$

Proof Indeed, by considering $A-\lambda_0$ and $A^{*}-\lambda_0$ for some large number $\lambda_0>0$ we may assume that condition (5.2) is satisfied for the operator $A-\lambda_0$:

$$\mathcal{N}_0\left(A-\lambda_0\right)=\mathcal{N}_0\left(A^{*}-\lambda_0\right)=\{0\}.$$

Then, by formulas (6.16) in Proposition 6.1 with $A:=A-\lambda_0$ we can express the solution $u\in H^{s+2}(\Omega)$ of the non-homogeneous Višik–Ventcel' boundary value problem

$$\begin{cases}(A-\lambda_0)\,u=f & \text{in } \Omega,\\ B\boldsymbol{\gamma}u=\psi & \text{on } \Gamma\end{cases}$$

in the following forms:

$$u=\mathcal{G}_D(\lambda_0)f+\mathcal{P}(\lambda_0)\varphi\in H^{s+2}(\Omega), \tag{6.50a}$$

$$\varphi=u|_{\Gamma}\in H^{s+3/2}(\Gamma). \tag{6.50b}$$

Here we recall that the mapping

$$\mathcal{P}(\lambda_0)\colon H^{s+3/2}(\Gamma)\longrightarrow H^{s+2}(\Omega)$$

is the Poisson operator for the Dirichlet problem (see problem (5.5))

$$\begin{cases}(A-\lambda_0)\, w = 0 & \text{in } \Omega, \\ \gamma_0 w = \varphi & \text{on } \Gamma\end{cases}$$

and that the mapping

$$\mathcal{G}_D(\lambda_0)\colon H^s(\Omega) \longrightarrow H^{s+2}(\Omega)$$

is the Green operator for the Dirichlet problem (see problem (6.4))

$$\begin{cases}(A-\lambda_0)\, v = f & \text{in } \Omega, \\ \gamma_0 v = 0 & \text{on } \Gamma.\end{cases}$$

Moreover, we have the pseudo-differential equation (see equation (6.22))

$$T(\lambda_0)\varphi := B\boldsymbol{\gamma}\left(\mathcal{P}(\lambda_0)\varphi\right) = \psi - \mu(x')\frac{\partial}{\partial \boldsymbol{\nu}}\left(\mathcal{G}_D(\lambda_0)f\right)\Big|_{\Gamma} \in H^{s+1/2}(\Gamma). \tag{6.51}$$

Therefore, by applying the index formulas (6.25) and (6.40) with

$$A := A - \lambda_0, \quad \mathcal{A} := \mathcal{A}(\lambda_0) = (A-\lambda_0, B\boldsymbol{\gamma}), \quad T := T(\lambda_0) = B\boldsymbol{\gamma}\left(\mathcal{P}(\lambda_0)\right),$$

we obtain that

$$\operatorname{ind}\mathcal{A}(\lambda_0) = \operatorname{ind}\mathcal{T}(\lambda_0) = 0 \quad \text{for every } s > -1/2.$$

This gives that

$$\begin{aligned}\operatorname{ind}\mathcal{A} &= \operatorname{ind}(A,\, B\boldsymbol{\gamma}) = \operatorname{ind}\left((A-\lambda_0,\, B\boldsymbol{\gamma}) + (\lambda_0, 0)\right) = \operatorname{ind}\mathcal{A}(\lambda_0) \\ &= 0 \quad \text{for every } s > -1/2,\end{aligned}$$

since the constant mapping

$$(\lambda_0, 0) : H^{s+2}(\Omega) \longrightarrow H^s(\Omega) \oplus H^{s+1/2}(\Gamma)$$

is *compact* for every $s > -1/2$, just as in the proof of assertion (6.40).

The proof of Theorem 6.3 is complete.

7 Regularity theorem for the Višik–Ventcel' boundary problem (1.4)

In this section, by using Propositions 6.1 and 6.5 we prove a regularity theorem for the non-homogeneous Višik–Ventcel' boundary value problem (1.4) under condition (H.2) (Theorem 7.1). Moreover, by applying Sobolev's imbedding theorem we obtain a regularity result for the null space of the mapping

$$\mathcal{A} = (A,\, B\boldsymbol{\gamma}) : H^{s+2}(\Omega) \longrightarrow H^s(\Omega) \oplus H^{s+1/2}(\Gamma) \quad \text{for } s > -1/2$$

(Corollary 7.1). Moreover, we prove the closedness of $\mathfrak{A}$ defined by formula (2.7a) and the regularity property (2.7b) (Proposition 7.1).

(I) First, by using Propositions 6.1 and 6.5 we can prove the following *regularity* property for the Višik–Ventcel' boundary value problem (1.4):

Theorem 7.1 *Assume that condition (H.2) is satisfied. Then we have, for every $s \geq 0$.*

$$\begin{cases} u \in L^2(\Omega), \\ Au = f \in H^s(\Omega), \\ B\boldsymbol{\gamma} u = \psi \in H^{s+1/2}(\Gamma) \end{cases} \implies u \in H^{s+2}(\Omega). \tag{7.1}$$

Proof We recall that all the sufficiently large eigenvalues of the Dirichlet problem for the differential operator A and its formal adjoint A^* lie in the *parabolic* type region, as discussed in [2, pp. 274–277] and [31, Chapter 3]. Hence, by considering $A - \lambda_0$ and $A^* - \lambda_0$ for some large number $\lambda_0 > 0$ we may assume that condition (5.2) is satisfied for the operator $A - \lambda_0$:

$$\mathcal{N}_0\left(A - \lambda_0\right) = \mathcal{N}_0\left(A^* - \lambda_0\right) = \{0\}.$$

Assume that

$$\begin{cases} u \in L^2(\Omega), \\ Au = f \in H^s(\Omega) \text{ and } B\boldsymbol{\gamma} u = \psi \in H^{s+1/2}(\Gamma) \quad \text{for } s \geq 0. \end{cases}$$

Then we have the assertions

$$\begin{cases} u \in L^2(\Omega), \\ (A - \lambda_0)\, u = f - \lambda_0\, u \in L^2(\Omega). \end{cases}$$

By using assertion (6.3) with $A := A - \lambda_0$ and the pseudo-differential equation (6.51), we obtain that

$$\begin{cases} \varphi = u|_{\Gamma} \in H^{-5/2}(\Gamma), \\ T(\lambda_0)\varphi = \psi - \mu(x') \frac{\partial}{\partial \boldsymbol{\nu}} \left(\mathcal{G}_D(\lambda_0)\left(f - \lambda_0\, u\right)\right)\Big|_{\Gamma} \in H^{1/2}(\Gamma). \end{cases}$$

Hence, it follows from the regularity property (6.27) with $T := T(\lambda_0)$ and $s := 1/2$ that

$$\varphi \in H^{3/2}(\Gamma).$$

Moreover, we have, by formula (6.50a) with $f := f - \lambda_0 u$ and $s := 0$,

$$u = \mathcal{G}_D(\lambda_0)\left(f - \lambda_0\, u\right) + \mathcal{P}(\lambda_0)\varphi \in H^2(\Omega).$$

Therefore, by repeating this process (a *bootstrap argument*) we can prove that

$$u \in H^{s+2}(\Omega) \quad \text{for } s \geq 0.$$

The proof of Theorem 7.1 is complete.

(II) Secondly, by applying Sobolev's imbedding theorem (see [1, Theorem 4.12, Part II]) we obtain the following regularity result for the null space of the operator $\mathcal{A} = (A, B\boldsymbol{\gamma})$:

Corollary 7.1 *If condition (H.2) is satisfied, then we have the assertion*

$$\begin{cases} u \in L^2(\Omega), \\ Au = 0 & \text{in } \Omega, \\ B\boldsymbol{\gamma} u = 0 & \text{on } \Gamma \end{cases} \implies u \in \bigcap_{\sigma \geq 0} H^{\sigma}(\Omega) = C^{\infty}(\overline{\Omega}).$$

(III) Finally, we are in a position to prove the following proposition:

Proposition 7.1 *Assume that condition (H.2) is satisfied. Then the operator*

$$\mathfrak{A} \colon L^2(\Omega) \longrightarrow L^2(\Omega)$$

is closed. *Moreover, we have the assertion*

$$\mathcal{D}(\mathfrak{A}) = \{u \in H_A(\Omega) : B\boldsymbol{\gamma} u = 0 \quad \text{on } \Gamma\} \subset H^2(\Omega). \tag{7.2}$$

This implies the desired regularity property (2.7b).

Proof The proof of Proposition 7.1 is divided into two steps.

Step (1): First, we prove the *closedness* of the operator $\mathfrak{A}$. Without loss of generality, we may assume that condition (5.2) is satisfied for the operator $A - \lambda_0$ for some large number $\lambda_0 > 0$, as in the proof of Theorem 7.1.

Let $\{u_j\}_{j=1}^{\infty}$ be an arbitrary sequence in the domain $\mathcal{D}(\mathfrak{A} - \lambda_0 I) = \mathcal{D}(\mathfrak{A})$ such that

$$u_j \longrightarrow u \quad \text{in } L^2(\Omega), \tag{7.3a}$$

$$(A - \lambda_0)\, u_j \longrightarrow v \quad \text{in } L^2(\Omega). \tag{7.3b}$$

Then it follows from assertion (7.3a) that

$$(A - \lambda_0)\, u_j \longrightarrow (A - \lambda_0)\, u \quad \text{in the space } \mathcal{D}'(\Omega) \text{ of distributions,}$$

and further from assertion (7.3b) that

$$(A - \lambda_0)\, u = v \in L^2(\Omega).$$

By definition (6.1) and formula (6.2) with $A := A - \lambda_0$, we find that

$$u \in H_{A-\lambda_0}(\Omega), \tag{7.4a}$$

$$u_j \longrightarrow u \quad \text{in } H_{A-\lambda_0}(\Omega). \tag{7.4b}$$

Moreover, by inequality (6.3) wit $A := A - \lambda_0$ it follows from assertion (7.4b) that

$$B\boldsymbol{\gamma} u = \lim_{j \to \infty} B\boldsymbol{\gamma} u_j = 0 \quad \text{in } H^{-5/2}(\Gamma). \tag{7.5}$$

Therefore, by combining assertions (7.4a) and (7.5) we obtain that

$$\begin{cases} u \in \mathcal{D}(\mathfrak{A} - \lambda_0 I), \\ (\mathfrak{A} - \lambda_0 I)\, u = (A - \lambda_0)\, u = v. \end{cases}$$

This proves the closedness of the operator $\mathfrak{A} - \lambda_0 I$.

However, it is easy to see that the operator $\mathfrak{A}$ is closed if and only if the operator $\mathfrak{A} - \lambda_0 I$ is closed.

Step (2): Secondly, by using the regularity property (7.1) with $s := 0$ we find that

$$\begin{cases} u \in H_A(\Omega), \\ B\boldsymbol{\gamma} u = 0 \text{ on } \Gamma \end{cases} \iff \begin{cases} u \in L^2(\Omega), \\ Au \in L^2(\Omega), \\ B\boldsymbol{\gamma} u = 0 \text{ on } \Gamma \end{cases} \implies u \in H^2(\Omega).$$

This proves the desired assertion (7.2) and hence the regularity property (2.7b).

The proof of Proposition 7.1 is complete.

8 Proof of Theorems 2.2 and 2.3

This section is devoted to the proof of Theorems 2.2 and 2.3. In Subsection 8.1, we prove Theorem 2.2. More precisely, we can prove that if conditions (2.4), (H.1) and (H.2) are satisfied, then the mapping $\mathcal{A}$ is *bijective* for every $s > -1/2$.

The proof of Theorem 2.3 in Subsection 8.2 is essentially the same as that of Theorem 2.2 if we replace formula (1.3) by formula (2.6) and condition (H.2) by condition (H.3), respectively. Indeed, it suffices to note that Propositions 6.3, 6.4, 6.5 and 6.6 and Theorem 6.3 remain valid for the pseudo-differential operator $T = \mu(x')\Pi + Q(x', D_{x'})$ when we replace condition (H.2) by condition (H.3).

8.1 Proof of Theorem 2.2

The proof of Theorem 2.2 is divided into three steps.

Step I: First, we prove the following *uniqueness theorem* for the Višik–Ventcel' boundary value problem (1.4) in the framework of smooth functions:

Theorem 8.1 *Assume that conditions* (2.4) *and (H.1) are satisfied. Then every solution* $u \in C^2(\overline{\Omega})$ *of the Višik–Ventcel' boundary value problem*

$$Au = 0 \quad \text{in } \Omega, \tag{8.1a}$$

$$B\boldsymbol{\gamma}u = 0 \quad \text{on } \Gamma \tag{8.1b}$$

is identically equal to zero *in* Ω*:*

$$u(x) \equiv 0 \quad \text{in } \Omega. \tag{8.2}$$

Proof The proof is divided into two steps.

Step (1): The case where $u(x)$ is constant in Ω. Then we have, by equation (8.1a),

$$0 = Au(x) = c(x)u(x) \quad \text{in } \Omega.$$

This proves the desired assertion (8.2), since condition (2.4) is satisfied.

Step (2): The case where $u(x)$ is not constant in Ω. Our proof is based on a reduction to absurdity.

In this case, by applying the *strong maximum principle* (see [37, p. 64, Theorem 6]) we may assume that there exists a boundary point $x_0' \in \Gamma$ such that (if necessary replacing u by $-u$)

$$u(x_0') = \max_{x \in \overline{\Omega}} u(x) > 0. \tag{8.3}$$

Then, by applying Hopf's *boundary point lemma* (see [37, p. 67, Theorem 8]) we obtain that

$$\frac{\partial u}{\partial \boldsymbol{\nu}}(x_0') < 0, \tag{8.4}$$

and further that

$$\frac{\partial u}{\partial x_i}(x_0') = 0 \quad \text{for } 1 \leq i \leq n-1.$$

Hence, we have, by the boundary condition (8.1b),

$$0 = B\boldsymbol{\gamma}u(x_0') = \mu(x_0')\frac{\partial u}{\partial \boldsymbol{\nu}}(x_0') + Qu(x_0') \tag{8.5}$$

$$
\begin{aligned}
&= \mu(x_0') \frac{\partial u}{\partial \boldsymbol{\nu}}(x_0') + \sum_{i,j=1}^{n-1} \alpha^{ij}(x_0') \frac{\partial^2 u}{\partial x_i \partial x_j}(x_0') + \gamma(x_0') u(x_0') \\
&\leq \mu(x_0') \frac{\partial u}{\partial \boldsymbol{\nu}}(x_0') + \gamma(x_0') u(x_0'),
\end{aligned}
$$

since the matrix $\left(\alpha^{ij}(x_0')\right)_{1 \leq i,j \leq n-1}$ is non-negative definite.

However, in view of condition (H.1) we obtain from assertions (8.3) and (8.4) that

$$
\mu(x_0') \frac{\partial u}{\partial \boldsymbol{\nu}}(x_0') + \gamma(x_0') u(x_0) < 0.
$$

This contradicts inequality (8.5).

The proof of Theorem 8.1 is complete.

Step II: Secondly, by combining Corollary 7.1 and Theorem 8.1 we find that if conditions (2.4), (H.1) and (H.2) are satisfied, then the mapping

$$
\mathcal{A} = (A,\, B\boldsymbol{\gamma}) : H^{s+2}(\Omega) \longrightarrow H^{s}(\Omega) \oplus H^{s+1/2}(\Gamma)
$$

is *injective* for every $s > -1/2$.

Step III: Thirdly, it follows from Theorem 6.3 that if condition (H.2) is satisfied, then we have the assertion

$$
\operatorname{ind} \mathcal{A} = \dim \mathcal{N}(\mathcal{A}) - \operatorname{codim} \mathcal{R}(\mathcal{A}) = 0 \quad \text{for every } s > -1/2. \tag{6.49}
$$

Therefore, we have proved that if conditions (2.4), (H.1) and (H.2) are satisfied, then the mapping

$$
\mathcal{A} = (A,\, B\boldsymbol{\gamma}) : H^{s+2}(\Omega) \longrightarrow H^{s}(\Omega) \oplus H^{s+1/2}(\Gamma) \tag{8.6}
$$

is *bijective* for every $s > -1/2$.

The proof of Theorem 2.2 is complete. □

Remark 8.1 By combining assertion (8.6) and Proposition 6.1, we can prove the following theorem:

Theorem 8.2 *Assume that conditions* (2.4)*, (H.1) and (H.2) are satisfied. Then the closed operator*

$$
\mathcal{T} \colon H^{s+3/2}(\Gamma) \longrightarrow H^{s+1/2}(\Gamma), \tag{8.7}
$$

defined by formula (6.24)*, is* bijective *for every $s \geq 0$ (see Table 6.2).*

Indeed, the above assertion (8.6) implies that the non-homogeneous Višik–Ventcel' boundary value problem (1.4) is uniquely solvable in the framework of Sobolev spaces. Hence, by virtue of Proposition 6.1 we find that the operator $\mathcal{T}$ is bijective for every $s \geq 0$.

8.2 Proof of Theorem 2.3

The proof of Theorem 2.3 is essentially the same as that of Theorem 2.2 if we replace condition (H.2) by condition (H.3). More precisely, Propositions 6.3, 6.4, 6.5 and 6.6 and Theorem 6.3 remain valid for the pseudo-differential operator

$$T = \mu(x')\Pi + Q(x', D_{x'}) = \mu(x')\Pi + \sum_{j=1}^{r} X_j \left(x', D_{x'}\right)^2 + X_0 \left(x', D_{x'}\right) + \gamma(x'),$$

when we replace formula (1.3) by formula (2.6) and condition (H.2) by condition (H.3), respectively.

In fact, if K' is an arbitrary compact set in the open neighborhood V of the boundary portion

$$\Gamma_0 = \left\{x' \in \Gamma : \mu(x') = 0\right\},$$

then, by using the *energy estimate* due to Oleĭnik–Radkevič [33, Theorem 2.5.1] (or [34, Theorem 2.1]) and [33, Theorem 2.2.8] with $s := 0$ and [33, Lemma 2.5.3] with $R(K) := 1$ and $s := -1/2$, we can obtain inequality (6.33) under condition (H.3). Hence, by combining inequalities (6.29), (6.32) and (6.33) we have the fundamental inequality (6.34) under condition (H.3).

In this way, the proof of Theorem 2.3 goes through just as in Section 7 and Subsection 8.1 if conditions (2.4), (H.1) and (H.3) are satisfied. Therefore, assertions (8.6) and (8.7) remain valid if conditions (2.4), (H.1) and (H.3) are satisfied.

The proof of Theorem 2.3 is complete. □

9 Agmon's method

In order to prove an existence and uniqueness theorem for the homogeneous Višik–Ventcel' boundary value problem (1.5) in the framework of Sobolev spaces when $|\lambda| \to \infty$ (Theorem 2.4), we make use of a method essentially due to Agmon ([2], [29]). This is a technique of treating a spectral parameter λ as a second order, elliptic differential operator of an extra variable y on the unit circle S, and relating the old problem to a new one with the additional variable (see [17], [48]).

First, we introduce an auxiliary variable y of the unit circle

$$S = \mathbf{R}/2\pi\mathbf{Z},$$

and replace the complex parameter λ by the second order differential operator

$$-e^{i\theta}\frac{\partial^2}{\partial y^2} \quad \text{for } -\pi < \theta < \pi.$$

More precisely, if we express the complex parameter λ in the form

$$\lambda = r^2 \, e^{i\theta} \quad \text{for } r \geq 0 \text{ and } -\pi < \theta < \pi,$$

then we replace the differential operator

$$A - \lambda = A - r^2 \, e^{i\theta}$$

defined in the original domain Ω by the second order differential operator

$$\widetilde{A}(\theta) := A + e^{i\theta}\,\frac{\partial^2}{\partial y^2} \quad \text{for } -\pi < \theta < \pi, \tag{9.1}$$

defined in the product domain $\Omega \times S$. We remark that the operator $\widetilde{A}(\theta)$ is *strongly uniform elliptic* for $-\pi < \theta < \pi$ in $\Omega \times S$ (see [61, p. 146, Definition 10.6]).

Now we consider instead of the original Višik–Ventcel' boundary value problem with spectral parameter

$$\begin{cases} (A-\lambda)u = f & \text{in } \Omega, \\ B\boldsymbol{\gamma}u = \mu(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}}\Big|_{\Gamma} + Q\left(u|_{\Gamma}\right) = 0 & \text{on } \Gamma. \end{cases} \tag{1.5}$$

the following homogeneous Višik–Ventcel' boundary value problem in the product domain $\Omega \times S$: Given a function $\widetilde{f}(x,y)$ defined in $\Omega \times S$, find a function $\widetilde{u}(x,y)$ in $\Omega \times S$ such that

$$\begin{cases} \widetilde{A}(\theta)\widetilde{u} = \left(A + e^{i\theta}\,\dfrac{\partial^2}{\partial y^2}\right)\widetilde{u} = \widetilde{f} & \text{in } \Omega \times S, \\ B\boldsymbol{\gamma}\widetilde{u} = \mu(x')\dfrac{\partial \widetilde{u}}{\partial \boldsymbol{\nu}}\Big|_{\Gamma\times S} + Q(x', D_{x'})\left(\widetilde{u}|_{\Gamma\times S}\right) = 0 & \text{on } \Gamma \times S. \end{cases} \tag{9.2}$$

In order to prove Theorem 2.4, we associate with the homogeneous Višik–Ventcel' boundary value problem (9.2) a densely defined, closed linear operator

$$\boxed{\widetilde{\mathfrak{A}}(\theta)\colon L^2(\Omega \times S) \longrightarrow L^2(\Omega \times S)}$$

in the Hilbert space $L^2(\Omega \times S)$ as follows (see formulas (2.7a) and (2.7b)):

(a) The domain $\mathcal{D}(\widetilde{\mathfrak{A}}(\theta))$ of definition of $\widetilde{\mathfrak{A}}(\theta)$ is the space

$$\mathcal{D}(\widetilde{\mathfrak{A}}(\theta) = \left\{\widetilde{u} \in L^2(\Omega \times S) : \widetilde{A}(\theta)\widetilde{u} \in L^2(\Omega \times S),\ B\boldsymbol{\gamma}\widetilde{u} = 0 \text{ on } \Gamma \times S\right\} \tag{9.3a}$$

$$= \left\{\widetilde{u} \in H^2(\Omega \times S) :\ B\boldsymbol{\gamma}\widetilde{u} = 0 \text{ on } \Gamma \times S\right\}. \tag{9.3b}$$

(b) $\widetilde{\mathfrak{A}}(\theta)\widetilde{u} = \widetilde{A}(\theta)\widetilde{u}$ for every $\widetilde{u} \in \mathcal{D}(\widetilde{\mathfrak{A}}(\theta))$.

Indeed, since $\widetilde{A}(\theta)\colon L^2(\Omega\times S) \to \mathcal{D}'(\Omega\times S)$ and $B\boldsymbol{\gamma} : H_{\widetilde{A}(\theta)}(\Omega\times S) \to H^{-5/2}(\Gamma\times S)$ are both continuous, it follows that $\widetilde{\mathfrak{A}}(\theta)$ is a closed operator (see the proof of Proposition 7.1). Furthermore, the operator $\widetilde{\mathfrak{A}}(\theta)$ is densely defined, since the domain $\mathcal{D}(\widetilde{\mathfrak{A}}(\theta))$ contains a dense subspace $C_0^\infty(\Omega \times S)$ of $L^2(\Omega \times S)$.

The next theorem asserts that if condition (G) is satisfied, then the operator $\widetilde{\mathfrak{A}}(\theta)$ is a Fredholm operator:

Theorem 9.1 *Let $\theta \in (-\pi, \pi)$. Assume that condition (G) is satisfied. Then the operator $\widetilde{\mathfrak{A}}(\theta)\colon L^2(\Omega \times S) \to L^2(\Omega \times S)$ is a* Fredholm operator. *Moreover, there exists a constant $\widetilde{C}(\theta) > 0$, continuously depending on θ, such that the* a priori *estimate*

$$\|\widetilde{u}\|_{H^2(\Omega\times S)} \leq \widetilde{C}(\theta)\left(\left\|\widetilde{A}(\theta)\widetilde{u}\right\|_{L^2(\Omega\times S)} + \|\widetilde{u}\|_{L^2(\Omega\times S)}\right) \tag{9.4}$$

holds true for all functions $\widetilde{u} \in \mathcal{D}(\widetilde{\mathfrak{A}}(\theta))$.

The proof of Theorem 9.1 will be given in Section 13, due to its length.

10 The Dirichlet problem for Agmon's method

In this section, by using the theory of pseudo-differential operators we consider the following non-homogeneous Dirichlet problem $(\widetilde{D})$ for the second order, *strongly uniform elliptic* differential operator $\widetilde{\Lambda}(\theta)$ for $-\pi < \theta < \pi$ in the framework of Sobolev spaces on the product domain $\Omega \times S$: For given functions $\widetilde{f}$ and $\widetilde{\varphi}$ defined in $\Omega \times S$ and on $\Gamma \times S$, respectively, find a function $\widetilde{u}$ in $\Omega \times S$ such that

$$\begin{cases} \widetilde{\Lambda}(\theta)\widetilde{u} = \widetilde{f} & \text{in } \Omega \times S, \\ \gamma_0 \widetilde{u} = \widetilde{u}|_{\Gamma \times S} = \widetilde{\varphi} & \text{on } \Gamma \times S. \end{cases} \tag{$\widetilde{D}$}$$

Following Seeley [43] and [44], we let (cf. formula (5.1))

$$\mathcal{N}_0\left(\widetilde{\Lambda}(\theta)\right) = \left\{\widetilde{u} \in C^\infty(\overline{\Omega} \times S) : \operatorname{supp} \widetilde{u} \subset \overline{\Omega} \times S,\ \widetilde{\Lambda}(\theta)\widetilde{u} = 0 \text{ in } \Omega \times S\right\}. \tag{10.1}$$

It is known (see [43, Theorem 7]) that $\mathcal{N}_0\left(\widetilde{\Lambda}(\theta)\right)$ is *finite-dimensional*. We remark from formula (10.1) that

$$\mathcal{N}_0\left(\widetilde{\Lambda}(\theta)\right) \subset \mathcal{N}\left(\widetilde{\mathfrak{A}}(\theta)\right).$$

10.1 Symbol of the differential operator $\widetilde{\Lambda}(\theta)$

In this subsection, we calculate explicitly the symbol of the strongly uniform elliptic differential operator $\widetilde{\Lambda}(\theta)$ defined by formula (9.1). However, it is easy to see that there is a *homotopy* in the class of strongly uniform elliptic symbols between the elliptic differential operators

$$\begin{cases} \widetilde{\Lambda}_1(\theta) = \widetilde{\Lambda}(\theta) = A + e^{i\theta}\,\frac{\partial^2}{\partial y^2}, \\ \widetilde{\Lambda}_0(\theta) = \Delta + e^{i\theta}\,\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} + e^{i\theta}\,\frac{\partial^2}{\partial y^2}. \end{cases}$$

For example, we may take

$$\widetilde{\Lambda}_t(\theta) := t\,A + (1-t)\,\Delta + e^{i\theta}\,\frac{\partial^2}{\partial y^2} \quad \text{for } 0 \le t \le 1. \tag{10.2}$$

Therefore, we have only to calculate explicitly the symbol of the strongly uniform elliptic differential operator $\widetilde{\Lambda}_0(\theta)$ for the usual Laplacian $A = \Delta$:

$$\widetilde{\Lambda}_0(\theta) = \Delta + e^{i\theta}\,\frac{\partial^2}{\partial y^2} \quad \text{in } \Omega \times S \text{ for } -\pi < \theta < \pi. \tag{10.3}$$

To do so, let

$$(x, \xi, y, \eta) = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n, y, \eta)$$

be a local coordinate system of the cotangent bundle $T^*(\Omega \times S) = T^*(\Omega) \times T^*(S)$. Then the principal symbol of the differential operator $\widetilde{\Lambda}_0(\theta)$ is equal to the following:

$$-\left(\left(|\xi|^2 + \cos\theta \cdot \eta^2\right) + \sqrt{-1}\,\sin\theta \cdot \eta^2\right).$$

Moreover, we remark that

$$\begin{aligned}&\left(|\xi|^2+\cos\theta\cdot\eta^2\right)+\sqrt{-1}\,\sin\theta\cdot\eta^2\\&=\left(\frac{|\xi|^2+\cos\theta\cdot\eta^2+\sqrt{-1}\,\sin\theta\cdot\eta^2}{|\xi|^2+\eta^2}\right)\left(|\xi|^2+\eta^2\right),\end{aligned}$$

and further that the middle term can be estimated as follows:

$$\left|\frac{|\xi|^2+\cos\theta\cdot\eta^2+\sqrt{-1}\,\sin\theta\cdot\eta^2}{|\xi|^2+\eta^2}\right|\geq\begin{cases}\sqrt{\frac{1+\cos\theta}{2}} & \text{if } |\theta|\leq\pi/2,\\ \frac{1}{\sqrt{2}}\sqrt[4]{\frac{1+\cos\theta}{1-\cos\theta}} & \text{if } \pi/2<|\theta|<\pi.\end{cases}\tag{10.4}$$

By virtue of inequality (10.4), we are reduced to the study of the non-homogeneous Dirichlet problem for the *strongly uniform elliptic* differential operator $\widetilde{\Lambda}_0(\theta)$ defined by formula (10.3).

In this way, we can prove the following *existence and uniqueness theorem* for the non-homogeneous Dirichlet problem $(\widetilde{D})$ in the framework of Sobolev spaces (cf. [3], [12], [19], [29], [44], [61]), analogous to Theorem 5.3:

Theorem 10.1 *The non-homogeneous Dirichlet problem $(\widetilde{D})$ has a unique solution $\widetilde{u}$ in the space $H^{s+2}(\Omega\times S)$ for any $\widetilde{f}\in H^s(\Omega\times S)$ and any $\widetilde{\varphi}\in H^{s+3/2}(\Gamma\times S)$ with $s>-3/2$. Moreover, the unique solution $\widetilde{u}$ of the Dirichlet problem $(\widetilde{D})$,* modulo $\mathcal{N}_0\left(\widetilde{\Lambda}(\theta)\right)\subset C^\infty(\overline{\Omega}\times S)$*, can be expressed in the form (cf. formula* (5.7)*)*

$$\widetilde{u}=\left(\widetilde{\mathcal{F}}(\theta)\widetilde{E}\widetilde{f}\right)\Big|_{\Omega\times S}+\widetilde{\mathcal{P}}(\theta)\left(\widetilde{\varphi}-\left(\widetilde{\mathcal{F}}(\theta)\widetilde{E}\widetilde{f}\right)\Big|_{\Gamma\times S}\right)\quad\text{in } \Omega\times S.\tag{10.5}$$

Here:

(a) $\widetilde{\mathcal{F}}(\theta)\colon H^s(M\times S)\to H^{s+2}(M\times S)$ is the right inverse *to $\widetilde{\Lambda}(\theta)$.*
(b) $\widetilde{\mathcal{P}}(\theta)\colon H^{s-1/2}(\Gamma\times S)\to H^s(\Omega\times S)$ is the Poisson kernel *for $\widetilde{\Lambda}(\theta)$.*
(c) $\widetilde{E}\colon H^s(\Omega\times S)\to H^s(M\times S)$ is the Seeley extension operator *(see [1, Theorems 5.21 and 5.22]).*

By applying Theorem 10.1 with $\widetilde{\varphi}:=0$, we find from formula (10.5) that the homogeneous Dirichlet problem

$$\begin{cases}\widetilde{\Lambda}(\theta)\widetilde{v}=\widetilde{f} & \text{in } \Omega\times S,\\ \gamma_0\widetilde{v}=0 & \text{on } \Gamma\times S\end{cases}\tag{10.6}$$

has a unique solution $\widetilde{v}\in H^{s+2}(\Omega\times S)$ for every $\widetilde{f}\in H^s(\Omega\times S)$. We let (cf. formula (6.5))

$$\widetilde{v}:=\widetilde{\mathcal{G}}_D(\theta)\widetilde{f}=\left(\widetilde{\mathcal{F}}(\theta)\widetilde{E}\widetilde{f}\right)\Big|_{\Omega\times S}-\widetilde{\mathcal{P}}(\theta)\left(\widetilde{\mathcal{F}}(\theta)\widetilde{E}\widetilde{f}\Big|_{\Gamma\times S}\right).\tag{10.7}$$

The operator $\widetilde{\mathcal{G}}_D(\theta)$ is called the *Green operator* for the Dirichlet problem (10.6).

11 A special reduction to the boundary $\Gamma \times S$

In this section, we reduce the homogeneous Višik–Ventcel' boundary value problem (9.2) to the study of a second order, pseudo-differential operator $\widetilde{T}(\theta) := B\boldsymbol{\gamma}(\widetilde{\mathcal{P}}(\theta))$ on the boundary $\Gamma \times S$.

More precisely, we can prove the following result, analogous to Proposition 6.1 with $\psi := 0$ and formulas (6.15) and (6.16):

Proposition 11.1 *For a given function $\widetilde{f} \in H^s(\Omega \times S)$ with $s > -1/2$, there exists a solution $\widetilde{u} \in H^{s+2}(\Omega \times S)$ of the homogeneous Višik–Ventcel' problem* (9.2) *if and only if there exists a solution $\widetilde{\varphi} \in H^{s+3/2}(\Gamma \times S)$ of the equation (cf. formula* (6.15)*)*

$$B\boldsymbol{\gamma}\left(\widetilde{\mathcal{P}}(\theta)\widetilde{\varphi}\right) = -\,\mu(x')\,\frac{\partial}{\partial \boldsymbol{\nu}}\left(\widetilde{\mathcal{G}}_D(\theta)\widetilde{f}\right)\Big|_{\Gamma \times S} \quad \text{on } \Gamma \times S. \tag{11.1}$$

Moreover, the solutions $\widetilde{u}$ and $\widetilde{\varphi}$ are related as follows (cf. formula (6.16)*):*

$$\widetilde{u} = \widetilde{\mathcal{G}}_D(\theta)\widetilde{f} + \widetilde{\mathcal{P}}(\theta)\widetilde{\varphi} \in H^{s+2}(\Omega \times S), \tag{11.2a}$$

$$\widetilde{\varphi} = \widetilde{u}|_{\Gamma \times S} \in H^{s+3/2}(\Gamma \times S). \tag{11.2b}$$

If we introduce a boundary operator $\widetilde{T}(\theta)$ by the formula

$$\begin{aligned} \widetilde{T}(\theta)\colon\ C^\infty(\Gamma \times S) &\longrightarrow C^\infty(\Gamma \times S) \\ \widetilde{\varphi} &\longmapsto B\boldsymbol{\gamma}\left(\widetilde{\mathcal{P}}(\theta)\widetilde{\varphi}\right), \end{aligned}$$

then we have the formula (cf. formula (6.17))

$$\begin{aligned} \widetilde{T}(\theta) = B\boldsymbol{\gamma}\left(\widetilde{\mathcal{P}}(\theta)\right) &= \mu(x')\gamma_1\widetilde{\mathcal{P}}(\theta) + Q\left(x', D_{x'}\right)\left(\gamma_0\widetilde{\mathcal{P}}(\theta)\right) \\ &= \mu(x')\widetilde{\Pi}(\theta) + Q\left(x', D_{x'}\right), \end{aligned} \tag{11.3}$$

where $\gamma_0\widetilde{\mathcal{P}}(\theta) = I$ and $\widetilde{\Pi}(\theta) = \gamma_1\widetilde{\mathcal{P}}(\theta)$ is called the *Dirichlet-to-Neumann operator* defined as follows:

$$\widetilde{\Pi}(\theta)\widetilde{\varphi} := \frac{\partial}{\partial \boldsymbol{\nu}}\left(\widetilde{\mathcal{P}}(\theta)\widetilde{\varphi}\right)\Big|_{\Gamma \times S} \quad \text{for all } \widetilde{\varphi} \in C^\infty(\Gamma \times S). \tag{11.4}$$

By combining formulas (11.1) and (11.2), we have proved that the homogeneous Višik–Ventcel' boundary value problem (9.2) can be reduced to the study of the pseudo-differential equation on $\Gamma \times S$ (cf. formula (6.22) with $\psi := 0$)

$$\widetilde{T}(\theta)\widetilde{\varphi} = \mu(x')\widetilde{\Pi}(\theta)\widetilde{\varphi} + Q\left(x', D_{x'}\right)\widetilde{\varphi} = -\,\mu(x')\frac{\partial}{\partial \boldsymbol{\nu}}\left(\widetilde{\mathcal{G}}_D(\theta)\widetilde{f}\right)\Big|_{\Gamma \times S}. \tag{11.5}$$

12 Symbolic calculus

The purpose of this section is to prove that if condition (G) is satisfied, then the closed realization

$$\widetilde{\mathcal{T}}(\theta) \colon H^{3/2}(\Gamma \times S) \longrightarrow H^{1/2}(\Gamma \times S),$$

defined by formula (12.7), is a Fredholm operator for every $-\pi < \theta < \pi$ (Proposition 12.3).

First, we show that the pseudo-differential operator

$$\widetilde{T}(\theta) = \mu(x')\widetilde{\Pi}(\theta) + Q\left(x', D_{x'}\right)$$

associated with the differential operator

$$\widetilde{\Lambda}_1(\theta) = \widetilde{\Lambda}(\theta) = A + e^{i\theta}\,\frac{\partial^2}{\partial y^2} \quad \text{for } -\pi < \theta < \pi \tag{9.1}$$

is *hypoelliptic* with loss of one derivative if condition (G) is satisfied (Proposition 12.2). By using the homotopy (10.2) in the class of strongly uniform elliptic symbols, we have only to calculate the complete symbol

$$\widetilde{t}_2(x', \xi', y, \eta; \theta) + \widetilde{t}_1(x', \xi', y, \eta; \theta) + \dots$$

of the pseudo-differential operator $\widetilde{T}(\theta)$ for the differential operator

$$\widetilde{\Lambda}_0(\theta) = \Delta + e^{i\theta}\,\frac{\partial^2}{\partial y^2} \quad \text{for } -\pi < \theta < \pi, \tag{10.3}$$

just as in Subsection 10.1.

12.1 Symbols of $\widetilde{T}(\theta)$ for the differential operator $\widetilde{\Lambda}_0(\theta)$

In this subsection, we calculate explicitly the principal symbols of the pseudo-differential operators $\widetilde{\Pi}(\theta)$ and $\widetilde{T}(\theta)$ associated with the differential operator $\widetilde{\Lambda}_0(\theta)$ defined by formula (10.3).

Step 1: First, we calculate the symbol of the *Dirichlet-to-Neumann operator* $\widetilde{\Pi}(\theta)$ defined by formula (11.4). To do this, we let

$$(x', \xi', y, \eta) = (x_1, \dots, x_{n-1}, \xi_1, \dots, \xi_{n-1}, y, \eta)$$

be a local coordinate system of the cotangent bundle $T^*(\Gamma \times S) = T^*(\Gamma) \times T^*(S)$. Then it is known (see [50, Section 10.2]) that the complete symbol of $\widetilde{\Pi}(\theta)$ is given by the following formula:

$$\begin{aligned} &\Big(\widetilde{p}_1(x', \xi', y, \eta; \theta) + \sqrt{-1}\,\widetilde{q}_1(x', \xi', y, \eta; \theta)\Big) \\ &+ \Big(\widetilde{p}_0(x', \xi', y, \eta; \theta) + \sqrt{-1}\,\widetilde{q}_0(x', \xi', y, \eta; \theta)\Big) + \text{terms of order} \le -1, \end{aligned}$$

where $-\widetilde{p}_1(x', \xi', y, \eta; \theta) > 0$ on the bundle $T^*(\Gamma \times S) \setminus \{0\}$ of non-zero cotangent vectors, for $-\pi < \theta < \pi$. More precisely, we have the formulas

- $\widetilde{p}_1(x', \xi', y, \eta; \theta)$

$$= -\frac{1}{\sqrt{2}} \left[\left[\left(|\xi'|^2 + \cos\theta \cdot \eta^2 \right)^2 + \sin^2\theta \cdot \eta^4 \right]^{1/2} + \left(|\xi'|^2 + \cos\theta \cdot \eta^2 \right) \right]^{1/2}.$$

- $\widetilde{q}_1(x', \xi', y, \eta; \theta)$

$$= -\frac{1}{\sqrt{2}} \left[\left[\left(|\xi'|^2 + \cos\theta \cdot \eta^2 \right)^2 + \sin^2\theta \cdot \eta^4 \right]^{1/2} - \left(|\xi'|^2 + \cos\theta \cdot \eta^2 \right) \right]^{1/2}.$$

Hence, we have the formula

$$\widetilde{p}_1(x', \xi', y, \eta; \theta)^2 + \widetilde{q}_1(x', \xi', y, \eta; \theta)^2 = \left[\left(|\xi'|^2 + \cos\theta \cdot \eta^2 \right)^2 + \sin^2\theta \cdot \eta^4 \right]^{1/2}.$$

Moreover, it is easy to see that

$$-\widetilde{p}_1(x', \xi', y, \eta; \theta) \geq \begin{cases} \sqrt{\frac{1+\cos\theta}{2}} \left(|\xi'|^2 + \eta^2 \right)^{1/2} & \text{if } |\theta| \leq \pi/2, \\ \frac{1}{\sqrt{2}} \sqrt[4]{\frac{1+\cos\theta}{1-\cos\theta}} \left(|\xi'|^2 + \eta^2 \right)^{1/2} & \text{if } \pi/2 < |\theta| < \pi, \end{cases} \tag{12.1}$$

and further that

$$\widetilde{p}_1(x', \xi', y, \eta; \theta)^2 + \widetilde{q}_1(x', \xi', y, \eta; \theta)^2 \geq \frac{1+\cos\theta}{2} \left(|\xi'|^2 + \eta^2 \right), \ -\pi < \theta < \pi. \tag{12.2}$$

Therefore, we obtain that the operator $\widetilde{T}(\theta) = \mu(x')\widetilde{\Pi}(\theta) + Q(x', D_{x'})$ is a classical, pseudo-differential operator of second order on the boundary $\Gamma \times S$ and further that its complete symbol is given by the following formula:

$$\begin{aligned} &\widetilde{t}_2(x', \xi', y, \eta; \theta) + \widetilde{t}_1(x', \xi', y, \eta; \theta) + \text{terms of order} \leq 0 \qquad (12.3) \\ &:= -\left[\sum_{j,k=1}^{n-1} \alpha^{jk}(x')\xi_j\xi_k \right] + \mu(x')\, \widetilde{p}_1(x', \xi', y, \eta; \theta) \\ &\quad + \sqrt{-1} \left(\mu(x')\, \widetilde{q}_1(x', \xi', y, \eta; \theta) + \sum_{k=1}^{n-1} \beta^k(x')\xi_k \right) + \text{terms of order} \leq 0. \end{aligned}$$

Step 2: Summing up, we have the following proposition for the pseudo-differential operator $\widetilde{T}(\theta)$ associated with the differential operator $\widetilde{\Lambda}_0(\theta)$ in the case where $A = \Delta$, analogous to Proposition 6.2:

Proposition 12.1 *The first two symbols*

$$\widetilde{t}_2(x', \xi', y, \eta; \theta) + \widetilde{t}_1(x', \xi', y, \eta; \theta)$$

of the pseudo-differential operator

$$\widetilde{T}(\theta) = \mu(x')\widetilde{\Pi}(\theta) + Q(x', D_{x'}) \quad \text{for } -\pi < \theta < \pi \tag{11.3}$$

are given respectively by the following formulas (see formula (12.3)*) :*

$$\widetilde{t}_2(x', \xi', y, \eta; \theta) = -\sum_{j,k=1}^{n-1} \alpha^{jk}(x')\xi_j\xi_k, \tag{12.4a}$$

$$\widetilde{t}_1(x', \xi', y, \eta; \theta) = \mu(x')\, \widetilde{p}_1(x', \xi', y, \eta; \theta) \tag{12.4b}$$

$$+\sqrt{-1}\left(\mu(x')\,\widetilde{q}_1(x',\xi',y,\eta;\theta)+\sum_{k=1}^{n-1}\beta^k(x')\xi_k\right),$$

where the symbols $\widetilde{p}_1(x',\xi',y,\eta;\theta)$ *and* $\widetilde{q}_1(x',\xi',y,\eta;\theta)$ *satisfy inequalities* (12.1) *and* (12.2).

12.2 Hypoellipticity of $\widetilde{T}(\theta)$ for the differential operator $\widetilde{\Lambda}(\theta)$

In light of formulas (12.3) and (12.4), by using the *homotopy* (10.2) we can prove the following proposition for the pseudo-differential operator $\widetilde{T}(\theta)$ associated with the original differential operator $\widetilde{\Lambda}_1(\theta)=\widetilde{\Lambda}(\theta)$ defined by formula (9.1), analogous to Proposition 6.3:

Proposition 12.2 *Let* $-\pi<\theta<\pi$. *If condition (G) is satisfied, then we have the following three assertions:*

(i) There exist constants $\widetilde{c}_0(\theta)>0$ *and* $\widetilde{C}_0(\theta)>0$, *depending continuously on* θ, *such that*

$$-\mathrm{Re}\left(\widetilde{T}(\theta)\widetilde{\varphi},\widetilde{\varphi}\right)_{L^2(\Gamma\times S)}\geq\widetilde{c}_0(\theta)\,\|\widetilde{\varphi}\|^2_{H^{1/2}(\Gamma\times S)}-\widetilde{C}_0(\theta)\,\|\widetilde{\varphi}\|^2_{L^2(\Gamma\times S)}\quad\text{for all }\widetilde{\varphi}\in C^\infty(\Gamma\times S).\tag{12.5}$$

(ii) There exists a constant $\widetilde{C}_1(\theta)>0$, *depending continuously on* θ, *such that*

$$\|\widetilde{\varphi}\|^2_{H^{1/2}(\Gamma\times S)}\leq\widetilde{C}_1(\theta)\left(\left\|\widetilde{T}(\theta)\widetilde{\varphi}\right\|^2_{H^{-1/2}(\Gamma\times S)}+\|\widetilde{\varphi}\|^2_{L^2(\Gamma\times S)}\right)\quad\text{for all }\widetilde{\varphi}\in C^\infty(\Gamma\times S).$$

(iii) The pseudo-differential operator $\widetilde{T}(\theta)$ *is* hypoelliptic *with loss of one derivative on* $\Gamma\times S$. *More precisely, we have, for every* $s\in\mathbf{R}$,

$$\widetilde{\varphi}\in\mathcal{D}'(\Gamma\times S),\ \widetilde{T}(\theta)\widetilde{\varphi}\in H^s(\Gamma\times S)\implies\widetilde{\varphi}\in H^{s+1}(\Gamma\times S).\tag{12.6}$$

Moreover, for any $t<s+1$ *there exists a constant* $\widetilde{C}_{s,t}(\theta)>0$, *depending continuously on* θ, *such that*

$$\|\widetilde{\varphi}\|^2_{H^{s+1}(\Gamma\times S)}\leq\widetilde{C}_{s,t}(\theta)\left(\left\|\widetilde{T}(\theta)\widetilde{\varphi}\right\|^2_{H^s(\Gamma\times S)}+\|\widetilde{\varphi}\|^2_{H^t(\Gamma\times S)}\right).$$

Here $(\cdot,\cdot)_{L^2(\Gamma\times S)}$ *is the inner product of the Hilbert space* $L^2(\Gamma\times S)=H^0(\Gamma\times S)$ *and* $\|\cdot\|_{H^s(\Gamma\times S)}$ *is the norm of the Sobolev space* $H^s(\Gamma\times S)$, *respectively.*

Indeed, by formula (12.4b), inequality (12.1) and condition (G) it suffices to note that we have, for $-\pi<\theta<\pi$,

$$-\mathrm{Re}\,\widetilde{t}_1(x',\xi',y,\eta;\theta)=\mu(x')\left(-\widetilde{p}_1(x',\xi',y,\eta;\theta)\right)>0\quad\text{on the bundle }T^*(\Gamma\times S)\setminus\{0\}\text{ of non-zero cotangent vectors.}$$

Hence, the desired inequality (12.5) follows by applying Hörmander [25, Theorem 22.3.3] (with $m:=1$) to the pseudo-differential operator $-\widetilde{T}(\theta)=-\mu(x')\widetilde{\Pi}(\theta)-Q(x',D_{x'})$.

Remark 12.1 Proposition 12.2 remains valid for the adjoint $\widetilde{T}(\theta)^*$ of the pseudo-differential operator $\widetilde{T}(\theta)$, analogous to Proposition 6.4.

Now we can associate with the Višik–Ventcel' boundary value problem (9.2) a densely defined, closed linear operator

$$\boxed{\widetilde{\mathcal{T}}(\theta)\colon H^{3/2}(\Gamma \times S) \longrightarrow H^{1/2}(\Gamma \times S)}$$

as follows (cf. formula (6.24) and Table 6.2):

(1) The domain $\mathcal{D}(\widetilde{\mathcal{T}}(\theta))$ of definition of $\widetilde{\mathcal{T}}(\theta)$ is the space

$$\mathcal{D}(\widetilde{\mathcal{T}}(\theta)) = \left\{\widetilde{\varphi} \in H^{3/2}(\Gamma \times S) : \widetilde{T}(\theta)\widetilde{\varphi} \in H^{1/2}(\Gamma \times S)\right\}. \tag{12.7}$$

(2) $\widetilde{\mathcal{T}}(\theta)\widetilde{\varphi} = \widetilde{T}(\theta)\widetilde{u}$ for every $\widetilde{\varphi} \in \mathcal{D}(\widetilde{\mathcal{T}}(\theta))$.

Then, by using Proposition 12.2 with $s := 1/2$ and $t := -1/2$ we can prove the following fundamental results for the closed operator $\widetilde{\mathcal{T}}(\theta)$ associated with the original differential operator $\widetilde{A}_1(\theta) = \widetilde{A}(\theta)$, analogous to assertions (6.45) and (6.41):

Proposition 12.3 *Let $-\pi < \theta < \pi$. If condition (G) is satisfied, then we have the following two assertions:*

(i) The operator $\widetilde{\mathcal{T}}(\theta)$ defined by formula (12.7) *is a* Fredholm operator.
(ii) There exists a constant $\widetilde{C}_2(\theta) > 0$, depending continuously on θ, such that we have, for all $\widetilde{\varphi} \in \mathcal{D}(\widetilde{\mathcal{T}}(\theta))$,

$$\|\widetilde{\varphi}\|_{H^{3/2}(\Gamma\times S)} \leq \widetilde{C}_2(\theta) \left(\left\|\widetilde{\mathcal{T}}(\theta)\widetilde{\varphi}\right\|_{H^{1/2}(\Gamma\times S)} + \|\widetilde{\varphi}\|_{H^{-1/2}(\Gamma\times S)} \right). \tag{12.8}$$

13 Proof of Theorem 9.1

This section is devoted to the proof of Theorem 9.1. More precisely, we show how Theorem 9.1 follows from Propositions 11.1, 12.2 and 12.3 if condition (G) is satisfied. The proof of Theorem 9.1 is divided into three steps.

Step 1: By virtue of the pseudo-differential equation (11.5) and the regularity property (12.6), we find from Proposition 11.1 that if $\widetilde{f} \in H^s(\Omega \times S)$, then every solution $\widetilde{u} \in L^2(\Omega\times S)$ of the homogeneous Višik–Ventcel' boundary value problem

$$\begin{cases} \widetilde{A}(\theta)\widetilde{u} = \widetilde{f} & \text{in } \Omega \times S, \\ B\gamma\widetilde{u} = 0 & \text{on } \Gamma \times S, \end{cases} \tag{9.2}$$

belongs to the Sobolev space $H^{s+2}(\Omega\times S)$ for every $s \geq 0$, and it can be expressed, unique *modulo* $\mathcal{N}_0\left(\widetilde{A}(\theta)\right) \subset C^\infty(\overline{\Omega} \times S)$, in the form

$$\widetilde{u} = \widetilde{\mathcal{G}}_D(\theta)\widetilde{f} + \widetilde{\mathcal{P}}(\theta)\widetilde{\varphi} \in H^{s+2}(\Omega \times S), \tag{11.2a}$$

$$\widetilde{\varphi} = \widetilde{u}|_{\Gamma\times S} \in H^{s+3/2}(\Gamma \times S). \tag{11.2b}$$

Therefore, we obtain the following *regularity* result for the homogeneous Višik–Ventcel' boundary value problem (9.2) in the framework of Sobolev spaces, analogous to Theorem 7.1:

Theorem 13.1 *Let $\theta \in (-\pi, \pi)$. Assume that condition (G) is satisfied. If $\widetilde{f}$ is a function in $H^s(\Omega \times S)$, then every solution $\widetilde{u} \in L^2(\Omega \times S)$ of the* homogeneous *Višik–Ventcel' boundary value problem* (9.2) *belongs to $H^{s+2}(\Omega \times S)$ for every $s \geq 0$.*

Rephrased, Theorem 13.1 asserts that every solution $\widetilde{u}$ of the homogeneous Višik–Ventcel' boundary value problem (9.2) has the *elliptic gain* of 2 derivatives from $\widetilde{f}$ in the framework of Sobolev spaces.

Step 2: Furthermore, we have the following two assertions:

(i) The regularity property (9.3b) holds true for the operator $\widetilde{\mathfrak{A}}(\theta)$ defined by formula (9.3a), if we take $s := 0$ in Theorem 13.1.

(ii) It follows from Propositions 11.1 and 12.3, Peetre's lemma (Lemma 6.1) and Theorem 13.1 that the null space

$$\mathcal{N}\left(\widetilde{\mathfrak{A}}(\theta)\right) = \left\{\widetilde{u} \in H^2(\Omega \times S) : \widetilde{\Lambda}(\theta)\widetilde{u} = 0 \text{ in } \Omega \times S, \ B\boldsymbol{\gamma}\widetilde{u} = 0 \text{ on } \Gamma \times S\right\}$$

is a *finite-dimensional*, subspace of $C^\infty(\overline{\Omega} \times S)$. Indeed, it suffices to note that

$$\widetilde{u} \in \mathcal{N}\left(\widetilde{\mathfrak{A}}(\theta)\right) \iff \begin{cases} \widetilde{u} = \widetilde{\mathcal{P}}(\theta)\widetilde{\varphi}, & \widetilde{\varphi} \in H^{3/2}(\Gamma \times S), \\ \widetilde{\mathcal{T}}(\theta)\widetilde{\varphi} = 0. \end{cases}$$

In order to prove the desired *a priori* estimate (9.4), we may assume that (see formula (10.1))

$$\mathcal{N}_0\left(\widetilde{\Lambda}(\theta)\right) = \mathcal{N}\left(\widetilde{\mathfrak{A}}(\theta)\right) = \{0\},$$

since all norms on the finite-dimensional space $\mathcal{N}\left(\widetilde{\mathfrak{A}}(\theta)\right)$ are *equivalent*. More precisely, the reader might be referred to Taira [52, pp. 1314–1315, Proof of Theorem 7.1].

Step 3: Now it follows from an application of Theorem 10.1 and formula (10.7) with $s := 0$ that the Green operator

$$\widetilde{\mathcal{G}}_D(\theta) \colon L^2(\Omega \times S) \longrightarrow H^2(\Omega \times S) \tag{13.1}$$

and the Poisson kernel

$$\widetilde{\mathcal{P}}(\theta) \colon H^{3/2}(\Gamma \times S) \longrightarrow H^2(\Omega \times S) \tag{13.2}$$

are both continuous.

Therefore, by using assertions (13.1) and (13.2) and the *a priori* estimate (12.8) we obtain from formulas (11.2) and (11.5) that if a function $\widetilde{u} \in H^2(\Omega \times S)$ is a solution of the homogeneous Višik–Ventcel' boundary value problem (9.2), then we have the inequality

$$\begin{aligned} &\|\widetilde{u}\|_{H^2(\Omega\times S)} \qquad\qquad (13.3)\\ &\leq \left\|\widetilde{\mathcal{G}}_D(\theta)\widetilde{f}\right\|_{H^2(\Omega\times S)} + \left\|\widetilde{\mathcal{P}}(\theta)\widetilde{\varphi}\right\|_{H^2(\Omega\times S)} \leq \widetilde{C}_1(\theta)\left(\left\|\widetilde{f}\right\|_{L^2(\Omega\times S)} + \|\widetilde{\varphi}\|_{H^{3/2}(\Gamma\times S)}\right) \\ &\leq \widetilde{C}_1(\theta)\left(\left\|\widetilde{f}\right\|_{L^2(\Omega\times S)} + \widetilde{C}_2(\theta)\left(\left\|\widetilde{\mathcal{T}}(\theta)\widetilde{\varphi}\right\|_{H^{1/2}(\Gamma\times S)} + \|\widetilde{\varphi}\|_{H^{-1/2}(\Gamma\times S)}\right)\right) \\ &= \widetilde{C}_1(\theta)\left(\left\|\widetilde{f}\right\|_{L^2(\Omega\times S)} + \widetilde{C}_2(\theta)\left\|\mu(x')\frac{\partial}{\partial\boldsymbol{\nu}}\left(\widetilde{\mathcal{G}}_D(\theta)\widetilde{f}\right)\Big|_{\Gamma\times S}\right\|_{H^{1/2}(\Gamma\times S)}\right. \end{aligned}$$

$$
\begin{aligned}
&+ \widetilde{C}_2(\theta) \left\| \left(\widetilde{u} - \widetilde{\mathcal{G}}_D(\theta)\widetilde{f} \right)\Big|_{\Gamma \times S} \right\|_{H^{-1/2}(\Gamma \times S)} \Bigg) \\
&= \widetilde{C}_1(\theta) \Bigg(\left\| \widetilde{f} \right\|_{L^2(\Omega \times S)} + \widetilde{C}_2(\theta) \left\| \mu(x')\gamma_1 \left(\widetilde{\mathcal{G}}_D(\theta)\widetilde{f} \right) \right\|_{H^{1/2}(\Gamma \times S)} \\
&+ \widetilde{C}_2(\theta) \left\| \gamma_0 \left(\widetilde{u} - \widetilde{\mathcal{G}}_D(\theta)\widetilde{f} \right) \right\|_{H^{-1/2}(\Gamma \times S)} \Bigg) \\
&\leq \widetilde{C}_3(\theta) \left(\left\| \widetilde{f} \right\|_{L^2(\Omega \times S)} + \left\| \widetilde{\mathcal{G}}_D(\theta)\widetilde{f} \right\|_{H^2(\Omega \times S)} + \left\| \widetilde{u} - \widetilde{\mathcal{G}}_D(\theta)\widetilde{f} \right\|_{L^2(\Omega \times S)} \right) \\
&\leq \widetilde{C}_3(\theta) \Big(\left\| \widetilde{f} \right\|_{L^2(\Omega \times S)} + \left\| \widetilde{\mathcal{G}}_D(\theta)\widetilde{f} \right\|_{H^2(\Omega \times S)} + \|\widetilde{u}\|_{L^2(\Omega \times S)} + \left\| \widetilde{\mathcal{G}}_D(\theta)\widetilde{f} \right\|_{L^2(\Omega \times S)} \Big) \\
&\leq \widetilde{C}_4(\theta) \left(\left\| \widetilde{f} \right\|_{L^2(\Omega \times S)} + \left\| \widetilde{\mathcal{G}}_D(\theta)\widetilde{f} \right\|_{H^2(\Omega \times S)} + \|\widetilde{u}\|_{L^2(\Omega \times S)} \right) \\
&\leq \widetilde{C}_5(\theta) \left(\left\| \widetilde{f} \right\|_{L^2(\Omega \times S)} + \|\widetilde{u}\|_{L^2(\Omega \times S)} \right) \\
&= \widetilde{C}_5(\theta) \left(\left\| \widetilde{A}(\theta)\widetilde{u} \right\|_{L^2(\Omega \times S)} + \|\widetilde{u}\|_{L^2(\Omega \times S)} \right),
\end{aligned}
$$

since the trace operators

$$
\begin{cases}
\gamma_0 \colon \mathcal{N}\left(\widetilde{A}(\theta), 0\right) \longrightarrow H^{-1/2}(\Gamma \times S), \\
\gamma_1 \colon H^2(\Omega \times S) \longrightarrow H^{1/2}(\Gamma \times S)
\end{cases}
$$

are both continuous (see assertion (5.4a) with $A := \widetilde{A}(\theta)$ and $s := 0$). Here the letter $\widetilde{C}_j(\theta)$ denotes a generic positive constant.

In this way, we find from inequality (13.3) that the desired *a priori* estimate (9.4) holds true for all $\widetilde{u} \in \mathcal{D}(\widetilde{\mathfrak{A}}(\theta))$.

Now the proof of Theorem 9.1 is complete. □

14 Proof of Theorem 2.4

This section is devoted to the proof of Theorem 2.4. We recall that conditions (H.1), (H.2) and (H.3) are trivially satisfied under condition (G). The proof is divided into four steps.

Step I: We associate with the Višik–Ventcel' boundary value problem (1.5) a densely defined, closed linear operator

$$
\boxed{\mathfrak{A} \colon L^2(\Omega) \longrightarrow L^2(\Omega)}
$$

in the Hilbert space $L^2(\Omega)$ as follows (see Proposition 7.1):

(a) The domain $\mathcal{D}(\mathfrak{A})$ of definition of $\mathfrak{A}$ is the space

$$
\mathcal{D}(\mathfrak{A}) = \left\{ u \in H^2(\Omega) : B\gamma u = 0 \text{ on } \Gamma \right\}. \tag{2.7b}
$$

(b) $\mathfrak{A}u = Au$ for every $u \in \mathcal{D}(\mathfrak{A})$.

Step II: By applying Theorem 2.2 with $A := A - \lambda$ and $\psi := 0$, we can obtain that if condition (G) is satisfied, then the homogeneous Višik–Ventcel' boundary value problem (1.5) has a unique solution $u \in H^2(\Omega)$ for any $f \in L^2(\Omega)$, provided that $\lambda > 0$. Indeed, it suffices to note that condition (2.4) is satisfied:

$$c(x) - \lambda \leq \lambda < 0 \quad \text{in } \Omega.$$

In particular, we have the assertion for $\lambda = 1$

$$\operatorname{ind}(\mathfrak{A} - I) = 0. \tag{14.1}$$

However, by the *Rellich–Kondrachov theorem* ([1, Theorem 6.3, Parts I and II], [12, p. 95, Proposition 3.4]) it follows that the constant mapping

$$(1 - \lambda)\, I \colon H^2(\Omega) \longrightarrow L^2(\Omega)$$

is *compact* for all complex number $\lambda \in \mathbf{C}$.

Hence, we obtain from assertion (14.1) that the index of the operator $\mathfrak{A} - \lambda I$ is equal to *zero* for all complex number $\lambda \in \mathbf{C}$:

$$\operatorname{ind}(\mathfrak{A} - \lambda\, I) = \operatorname{ind}((\mathfrak{A} - I) + (1 - \lambda) I) = \operatorname{ind}(\mathfrak{A} - I) = 0,$$

since the index is *stable* under compact perturbations (see Gohberg–Kreĭn [20]).

In this way, we have proved the following fundamental theorem (cf. Theorem 6.3) :

Theorem 14.1 *If condition (G) is satisfied, then we have the assertion*

$$\operatorname{ind}(\mathfrak{A} - \lambda\, I) = 0 \quad \textit{for all number } \lambda \in \mathbf{C}. \tag{14.2}$$

Step III: The next theorem plays an essential role in the proof of the resolvent estimates (2.8) and (2.9) due to Taira [49, Corollary 8.4.2] based on Agmon [2, p. 272, Theorem 15.4]:

Theorem 14.2 *Let $\theta \in (-\pi, \pi)$. Assume that the* a priori *estimate*

$$\|\widetilde{u}\|_{H^2(\Omega \times S)} \leq \widetilde{C}(\theta) \left(\left\| \widetilde{A}(\theta) \widetilde{u} \right\|_{L^2(\Omega \times S)} + \|\widetilde{u}\|_{L^2(\Omega \times S)} \right) \tag{9.4}$$

holds true for all functions $\widetilde{u} \in H^2(\Omega \times S)$ satisfying the boundary condition $B\boldsymbol{\gamma}\widetilde{u} = 0$ on $\Gamma \times S$. Then, for every $-\pi < \theta < \pi$ there exists a constant $R(\theta) > 0$, continuously depending on θ, such that if $\lambda = r^2\, e^{i\theta}$ and $|\lambda| = r^2 \geq R(\theta)$, we have, for all functions $u \in H^2(\Omega)$ satisfying the boundary condition $B\boldsymbol{\gamma} u = 0$ on Γ (that is, $u \in D(\mathfrak{A})$),

$$|u|_2 + |\lambda|^{1/2} \cdot |u|_1 + |\lambda| \cdot \|u\|_{L^2(\Omega)} \leq C(\theta)\, \|(A - \lambda)u\|_{L^2(\Omega)}\,, \tag{14.3}$$

with a constant $C(\theta) > 0$ continuously depending on θ. Here $|\cdot|_j$ is the seminorm on the Sobolev space $H^2(\Omega)$ defined by the formula

$$|u|_j = \left(\int_\Omega \sum_{|\beta| = j} \left| D^\beta u(x) \right|^2 dx \right)^{1/2} \quad \textit{for } j = 1,\, 2.$$

Proof Now let $u(x)$ be an arbitrary function in the domain $\mathcal{D}(\mathfrak{A})$:

$$u \in H^2(\Omega) \text{ and } B\boldsymbol{\gamma}u = 0 \text{ on } \Gamma.$$

We choose a function $\zeta(y)$ in $C^\infty(S)$ such that

$$\begin{cases} 0 \le \zeta(y) \le 1 \quad \text{on } S, \\ \operatorname{supp}\zeta \subset \left[\frac{\pi}{3}, \frac{5\pi}{3}\right], \\ \zeta(y) = 1 \quad \text{for} \quad \dfrac{\pi}{2} \le y \le \dfrac{3\pi}{2}, \end{cases}$$

and let

$$\widetilde{v}_\eta(x,y) := u(x) \otimes \zeta(y)e^{i\eta y} \quad \text{for all } x \in \Omega,\ y \in S \text{ and } \eta \ge 0.$$

Then we have the assertions

- $\widetilde{v}_\eta \in H^2(\Omega \times S) \quad \text{for all } \eta \ge 0,$
- $\widetilde{A}(\theta)\widetilde{v}_\eta = \left(A + e^{i\theta}\dfrac{\partial^2}{\partial y^2}\right)\widetilde{v}_\eta$

$$= \left(A - \eta^2 e^{i\theta}\right) u \otimes \zeta(y)e^{i\eta y}$$
$$+ 2(i\eta)e^{i\theta}u \otimes \zeta'(y)e^{i\eta y} + e^{i\theta}u \otimes \zeta''(y)e^{i\eta y} \in L^2(\Omega \times S) \quad \text{for all } \eta \ge 0,$$

and also the boundary condition

- $B\boldsymbol{\gamma}\left(\widetilde{v}_\eta(x,y)\right) = \left(B\boldsymbol{\gamma}u(x)\right) \otimes \zeta(y)e^{i\eta y} = 0 \quad \text{for all } \eta \ge 0.$

Thus, by applying the *a priori* estimate (9.4) to the functions

$$\widetilde{v}_\eta(x,y) = u(x) \otimes \zeta(y)e^{i\eta y} \in \mathcal{D}(\widetilde{\mathfrak{A}}(\theta)) \quad \text{for all } \eta \ge 0,$$

we obtain that

$$\begin{aligned} &\left\| u \otimes \zeta e^{i\eta y} \right\|_{H^2(\Omega \times S)} \\ &\le \widetilde{C}(\theta)\left(\left\|\widetilde{A}(\theta)(u \otimes \zeta e^{i\eta y})\right\|_{L^2(\Omega\times S)} + \|u \otimes \zeta e^{i\eta y}\|_{L^2(\Omega\times S)}\right) \quad \text{for all } \eta \ge 0. \end{aligned} \tag{14.4}$$

We can estimate each term of inequality (14.4) as follows:

- $$\begin{aligned} \left\| u \otimes \zeta e^{i\eta y} \right\|_{L^2(\Omega\times S)} &= \left(\int_{\Omega\times S} |u(x)|^2\, |\zeta(y)|^2\, dx\, dy\right)^{1/p} \\ &= \|\zeta\|_{L^2(S)} \cdot \|u\|_{L^2(\Omega)}. \end{aligned} \tag{14.5}$$

- $$\begin{aligned} &\left\|\widetilde{A}(\theta)(u \otimes \zeta e^{i\eta y})\right\|_{L^2(\Omega\times S)} \\ &\le \left\|(A - \eta^2 e^{i\theta})u \otimes \zeta e^{i\eta y}\right\|_{L^2(\Omega\times S)} + 2\eta\|u \otimes \zeta' e^{i\eta y}\|_{L^2(\Omega\times S)} + \|u \otimes \zeta'' e^{i\eta y}\|_{L^2(\Omega\times S)} \\ &\le \|\zeta\|_{L^2(S)} \cdot \left\|(A - \eta^2 e^{i\theta})u\right\|_{L^2(\Omega)} + \left(2\eta\left\|\zeta'\right\|_{L^2(S)} + \left\|\zeta''\right\|_{L^2(S)}\right)\|u\|_{L^2(\Omega)}. \end{aligned} \tag{14.6}$$

- $$\left\| u \otimes \zeta e^{i\eta y} \right\|^2_{H^2(\Omega\times S)} \tag{14.7}$$

$$
\begin{aligned}
&= \sum_{|\alpha|\leq 2} \int_{\Omega\times S} \left| D^{\alpha}_{x,y}(u(x)\otimes \zeta(y) e^{i\eta y})\right|^2 dx\, dy \\
&\geq \sum_{|\alpha|\leq 2} \int_{\Omega} \int_{\pi/2}^{3\pi/2} \left| D^{\alpha}_{x,y}(u(x)\otimes e^{i\eta y})\right|^2 dx\, dy \\
&= \sum_{k+|\beta|\leq 2} \int_{\Omega} \int_{\pi/2}^{3\pi/2} \left| \eta^k D^{\beta} u(x)\right|^2 dx\, dy \\
&\geq \pi \left(\sum_{|\beta|=2} \int_{\Omega} \left| D^{\beta} u(x)\right|^2 dx + \eta^2 \sum_{|\beta|=1} \int_{\Omega} |D^{\beta} u(x)|^2 dx + \eta^4 \int_{\Omega} |u(x)|^2 dx \right) \\
&= \pi \left(|u|_2^2 + \eta^2 |u|_1^2 + \eta^4 \|u\|^2_{L^2(\Omega)} \right).
\end{aligned}
$$

Therefore, by carrying these three inequalities (14.5), (14.6) and (14.7) into inequality (14.4) we obtain that

$$
\begin{aligned}
&|u|_2 + \eta\, |u|_1 + \eta^2 \, \|u\|_{L^2(\Omega)} \\
&\leq \widetilde{C}'(\theta) \left(\left\| (A - \eta^2 e^{i\theta}) u \right\|_{L^2(\Omega)} + \eta \, \|u\|_{L^2(\Omega)} \right) \quad \text{for all } \eta \geq 0,
\end{aligned}
$$

with another constant $\widetilde{C}'(\theta) > 0$ continuously depending θ. If η is so large that

$$\eta \geq 2\widetilde{C}'(\theta),$$

then we can eliminate the last term on the right-hand side to obtain that

$$|u|_2 + \eta\, |u|_1 + \eta^2 \, \|u\|_{L^2(\Omega)} \leq 2\widetilde{C}'(\theta) \left\| (A - \eta^2 e^{i\theta}) u \right\|_{L^2(\Omega)} \quad \text{for all } \eta \geq 2\widetilde{C}'(\theta).$$

This proves the desired *a priori* estimate (14.3) if we take

$$\lambda := \eta^2 e^{i\theta}, \quad R(\theta) := 4\, \widetilde{C}'(\theta)^2, \quad C(\theta) := 2\, \widetilde{C}'(\theta). \tag{14.8}$$

The proof of Theorem 14.2 is now complete.

By combining Theorems 9.1 and 14.2, we can obtain the desired resolvent estimates (2.8) and (2.9) for the operator $\mathfrak{A} - \lambda I$. More precisely, we prove the following corollary:

Corollary 14.1 *Assume that condition (G) is satisfied. Then, for every* $0 < \varepsilon < \pi/2$ *there exist constants* $r(\varepsilon) > 0$ *and* $c(\varepsilon) > 0$ *such that we have, for all* $\lambda = r^2\, e^{i\theta}$ *satisfying the conditions* $r \geq r(\varepsilon)$ *and* $-\pi + \varepsilon \leq \theta \leq \pi - \varepsilon$,

$$|u|_2 + |\lambda|^{1/2} \cdot |u|_1 + |\lambda| \cdot \|u\|_{L^2(\Omega)} \leq c(\varepsilon)\, \|(\mathfrak{A} - \lambda I)u\|_{L^2(\Omega)} \quad \textit{for all } u \in \mathcal{D}(\mathfrak{A}). \tag{14.9}$$

Proof By the *a priori* estimate (14.3), we have, for all $\lambda = r^2\, e^{i\theta}$, $-\pi < \theta < \pi$ and $|\lambda| = r^2 \geq R(\theta)$,

$$|u|_2 + |\lambda|^{1/2} \cdot |u|_1 + |\lambda| \cdot \|u\|_{L^2(\Omega)} \leq C(\theta)\, \|(\mathfrak{A} - \lambda I)u\|_{L^2(\Omega)} \quad \text{for all } u \in \mathcal{D}(\mathfrak{A}).$$

However, we find from formulas (14.8) that the constants $R(\theta)$ and $C(\theta)$ depend *continuously* on $\theta \in (-\pi, \pi)$, so that they may be chosen uniformly in $\theta \in [-\pi + \varepsilon, \pi - \varepsilon]$, for every $\varepsilon > 0$. This proves the existence of the constants $r(\varepsilon)$ and $c(\varepsilon)$. Namely, the desired *a priori* estimate (14.9) holds true for all $\lambda = r^2\, e^{i\theta}$ satisfying the conditions $r \geq r(\varepsilon)$ and $\theta \in [-\pi + \varepsilon, \pi - \varepsilon]$.

The proof of Corollary 14.1 is complete.

Step IV: The *a priori* estimate (14.9) asserts that the operator $\mathfrak{A}-\lambda I$ is *injective* if λ belongs to the set

$$\Sigma(\varepsilon) = \left\{ \lambda = r^2\, e^{i\theta} : r \geq r(\varepsilon), -\pi + \varepsilon \leq \theta \leq \pi - \varepsilon \right\}.$$

Hence, it follows from assertion (14.2) that $\mathfrak{A} - \lambda I$ is *bijective* for all $\lambda \in \Sigma(\varepsilon)$.

Summing up, we have proved that the resolvent set of $\mathfrak{A}$ contains the set $\Sigma(\varepsilon)$ and further that the resolvent $(\mathfrak{A} - \lambda I)^{-1}$ satisfies the estimate

$$\left\| (\mathfrak{A} - \lambda I)^{-1} \right\| \leq \frac{c(\varepsilon)}{|\lambda|} \quad \text{for all } \lambda \in \Sigma(\varepsilon). \tag{2.8}$$

Finally, we remark that the resolvent estimate (2.9) is a special case of the resolvent estimate (2.8).

Now the proof of Theorem 2.4 is complete. □

15 The Višik–Ventcel' boundary value problem via the Boutet de Monvel calculus

This Section 15 and the next Section 16 are devoted to the proof of Theorem 2.5. Our proof of Theorem 2.5 is based on Agmon [2, Theorems 14.4 and 15.1] which are summarized in [55, Section 4].

In Section 15, for some large number $\lambda_0 > 0$ (see condition (5.2) with $A := A - \lambda_0$) we study the homogeneous Višik–Ventcel' boundary value problem

$$\begin{cases} (A - \lambda_0)\, u = f & \text{in } \Omega, \\ B\boldsymbol{\gamma} u = 0 & \text{on } \Gamma \end{cases} \tag{15.1}$$

in the framework of Sobolev spaces, if condition (G) is satisfied. However, in the hypoelliptic (degenerate) case, we cannot use Green's formula to characterize the adjoint operator $\mathfrak{A}^* - \lambda_0\, I$ of the boundary value problem (15.1) in the framework of Sobolev spaces. Therefore, we shift our attention to the Green operator (the resolvent) $(\mathfrak{A} - \lambda_0\, I)^{-1}$ and its adjoint operator $(\mathfrak{A}^* - \lambda_0\, I)^{-1}$, just as in Taira [53], [55]. In Sections 15 and 16, we make use of the *Boutet de Monvel calculus* ([8], [39], [41], [50, Appendix B]) in order to study the mapping properties of $(\mathfrak{A} - \lambda_0\, I)^{-1}$ and $(\mathfrak{A}^* - \lambda_0\, I)^{-1}$ (Theorems 15.2 and 16.1). In this way, we can verify all the conditions of [55, Theorem 4.1] and [55, Remark 4.1].

In order to study the homogeneous Višik–Ventcel' boundary value problem (15.1), we consider a *homotopy* in the class of second order, uniformly elliptic symbols between the elliptic differential operators $A - \lambda_0$ and $\Delta - 1$, by taking

$$A_t := (1 - t)\,(A - \lambda_0) + t\,(\Delta - 1) \quad \text{for } 0 \leq t \leq 1. \tag{15.2}$$

Therefore, we are reduced to the study of the differential operator

$$A_1 = \Delta - 1 = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} - 1, \tag{15.3}$$

just as in Section 10.

First, we construct a right inverse $\mathcal{Q}_1$ to the differential operator $A_1 = \Delta - 1$ adapted to the Boutet de Monvel calculus. Following Rempel–Schulze [39], we denote by f_0 the extension of f to the whole Euclidean space $\mathbf{R}^n$ with $f_0 = 0$ outside Ω:

$$f_0(x) = \begin{cases} f(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \in \mathbf{R}^n \setminus \Omega. \end{cases}$$

Let $G_2(x)$ be the *Bessel potential* of order 2 (see [45, Chapter V, Section 3]), that is,

$$G_2(x) = \frac{1}{(4\pi)^{n/2}} \int_0^\infty e^{-t-\frac{|x|^2}{4t}}\, t^{\frac{2-n}{2}}\, \frac{dt}{t}, \tag{15.4a}$$

$$\widehat{G}_2(\xi) = \int_{\mathbf{R}^n} e^{-ix\xi} G_2(x)\, dx = \frac{1}{1+|\xi|^2}. \tag{15.4b}$$

If we let

$$\mathcal{Q}_1 f(x) = -G_2 * f_0(x) = -\int_{\boldsymbol{\Omega}} G_2(x-y) f(y)\, dy \quad \text{for all } x \in \Omega, \tag{15.5}$$

then we obtain from the *transmission property* of the Bessel potential $G_2(x)$ (see Boutet de Monvel [8], Rempel–Schulze [39, p. 161, Theorem 2]) that the operator

$$\mathcal{Q}_1 \colon H^s(\Omega) \longrightarrow H^{s+2}(\Omega)$$

is continuous for all $s > -1/2$, and further from [39, p. 100, Lemma 5] that (cf. formula (5.6))

$$A_1 \mathcal{Q}_1 f = (\Delta - 1)\, \mathcal{Q}_1 f = f \quad \text{in } \Omega. \tag{15.6}$$

This proves that $\mathcal{Q}_1$ is a *right inverse* to A_1.

The main purpose of Section 15 is to characterize the mapping property of the Green operator (the resolvent) $\mathcal{G}$ of the homogeneous Višik–Ventcel' boundary value problem

$$\begin{cases} A_1 u = (\Delta - 1)\, u = f & \text{in } \Omega, \\ B\boldsymbol{\gamma} u = \mu(x')\gamma_1 u + Q\left(x', D_{x'}\right)(\gamma_0 u) = 0 & \text{on } \Gamma \end{cases} \tag{15.7}$$

in the framework of Sobolev spaces if condition (G) is satisfied (Theorem 15.2).

15.1 The Green operator for the Dirichlet problem

First, we study the *Green operator* $\mathcal{G}_D$ for the homogeneous Dirichlet problem from the viewpoint of the Boutet de Monvel calculus. For every function $f \in H^s(\Omega)$ with $s > -1/2$, the function $v = \mathcal{G}_D f \in H^{s+2}(\Omega)$ is the unique solution of the homogeneous Dirichlet problem for the elliptic differential operator $A_1 = \Delta - 1$ defined by formula (15.3):

$$\begin{cases} A_1 v = (\Delta - 1)\, v = f & \text{in } \Omega, \\ \gamma_0 v = v|_\Gamma = 0 & \text{on } \Gamma. \end{cases} \tag{15.8}$$

By using formula (15.6), we obtain that (cf. formula (6.5))

$$v = \mathcal{G}_D f = \mathcal{Q}_1 f - \mathcal{P}_1\left(\gamma_0\left(\mathcal{Q}_1 f\right)\right) \quad \text{for } f \in H^s(\Omega). \tag{15.9}$$

Here $\mathcal{P}_1$ is the *Poisson kernel* for the differential operator $A_1 = \Delta - 1$.

First, the next theorem characterizes the Green operator $\mathcal{G}_D$ defined by formula (15.9) in terms of the Boutet de Monvel calculus (cf. [55, Theorem 8.1]):

Theorem 15.1 *The Green operator $\mathcal{G}_D$ given by formula* (15.9) *can be expressed in the matrix form (see Table 15.1 below)*

$$\mathcal{D} = \begin{pmatrix} \mathcal{Q}_1 & -\mathcal{P}_1 \\ \gamma_0 \mathcal{Q}_1 & I \end{pmatrix} \tag{15.10}$$

with the principal symbol

$$\sigma(\mathcal{D}) = \begin{pmatrix} -\frac{1}{2\langle\xi'\rangle}\left(\frac{1}{\langle\xi'\rangle+i\nu} + \frac{1}{\langle\xi'\rangle-i\nu}\right) & -\frac{1}{\langle\xi'\rangle+i\nu} \\ -\frac{1}{2\langle\xi'\rangle}\,\frac{1}{\langle\xi'\rangle-i\tau} & 1 \end{pmatrix}. \tag{15.11}$$

Here and in the following we use the notation

$$\begin{aligned}
&\xi = (\xi', \nu) = (\xi_1, \xi_2, \ldots, \xi_{n-1}, \nu) \in \mathbf{R}^n, \\
&\langle\xi'\rangle = \sqrt{1 + |\xi'|^2}, \\
&\xi = (\xi', \nu) = (\xi_1, \xi_2, \ldots, \xi_{n-1}, \nu) \in \mathbf{R}^n \quad \textit{for potential operators}, \\
&\xi = (\xi', \tau) = (\xi_1, \xi_2, \ldots, \xi_{n-1}, \tau) \in \mathbf{R}^n \quad \textit{for trace operators.}
\end{aligned}$$

Proof (1) We remark from formulas (15.4) and (15.5) that the *principal symbol* of $\mathcal{Q}_1$ of $-G_2(x)$ is given by the formula

$$\sigma(\mathcal{Q}_1) = -\frac{1}{\langle\xi'\rangle^2 + \nu^2} = -\frac{1}{2\langle\xi'\rangle}\left(\frac{1}{\langle\xi'\rangle + i\nu} + \frac{1}{\langle\xi'\rangle - i\nu}\right). \tag{15.12}$$

(2) By using Rempel–Schulze [39, p. 100, Lemma 4], we obtain from formula (15.12) that the operator $\gamma_0 \mathcal{Q}_1$ is a *trace operator* of order -2 with the principal symbol

$$\sigma(\gamma_0 \mathcal{Q}_1) = -\frac{1}{2\langle\xi'\rangle}\,\frac{1}{\langle\xi'\rangle - i\tau}. \tag{15.13}$$

(3) On the other hand, we find from formula (15.12) ([39, p. 102, Proposition 6]) that the Poisson kernel $\mathcal{P}_1$ is a *potential operator* of order -1 with the principal symbol

$$\sigma(\mathcal{P}_1) = \frac{1}{\langle\xi'\rangle + i\nu}. \tag{15.14}$$

Therefore, the desired assertions (15.10) and (15.11) follow by combining formulas (15.12), (15.13) and (15.14).

Finally, the mapping property of the Green operator $\mathcal{G}_D$ follows from an application of Rempel–Schulze [39, p. 176, Theorem 1], as is shown in Table 15.1 below.

The proof of Theorem 15.1 is complete.

$$\begin{tikzcd} H^s(\Omega) \arrow[r, "\gamma_0 \mathcal{Q}_1"] \arrow[d, "\mathcal{Q}_1"'] & H^{s+3/2}(\Gamma) \arrow[d, "I"] \\ H^{s+2}(\Omega) & H^{s+3/2}(\Gamma) \arrow[l, "-\mathcal{P}_1"] \end{tikzcd}$$

Table 15.1 The mapping property of the Green operator $\mathcal{G}_D$ for $s \geq 0$

15.2 The Green operator for the Višik–Ventcel' boundary problem (15.7)

In this subsection, by using formulas (6.5), (6.6), (6.16), (6.17) and (6.22) with

$$A := A_1 = \Delta - 1, \quad \mathcal{P} := \mathcal{P}_1, \quad T := T_1 = B\boldsymbol{\gamma}\mathcal{P}_1 = \mu(x')\, \Pi_1 + Q(x', D_{x'}),$$

we prove the mapping property of the Green operator (resolvent) $\mathcal{G} = (\mathfrak{A} - I)^{-1}$ of the Višik–Ventcel' boundary value problem (15.7) under condition (G) in the framework of Sobolev spaces:

Theorem 15.2 *Assume that condition (G) is satisfied. Then the Green operator* $\mathcal{G} = (\mathfrak{A} - I)^{-1}$*, given by the formula*

$$\mathcal{G}f = \mathcal{G}_D f - \mathcal{P}_1 \left(\mathcal{T}_1^{-1} \left(\mu(x')\, \gamma_1 \left(\mathcal{G}_D f \right) \right) \right) \quad \text{for } f \in L^2(\Omega), \tag{15.15}$$

maps $H^s(\Omega)$ *continuously into* $H^{s+2}(\Omega)$ *for every* $s \geq 0$ *(see Table 15.3 below).*

Proof By applying Theorem 8.2 with $A := A_1 = \Delta - 1$ under condition (G), we find from assertion (8.7) that the closed operator

$$\mathcal{T}_1 \colon H^{s+3/2}(\Gamma) \longrightarrow H^{s+1/2}(\Gamma) \tag{15.16}$$

is *bijective* for every $s \geq 0$ in formula (15.15) (see definition (6.24) with $T := T_1$). The situation can be visualized in Table 15.2 below (see Table 6.2 with $T := T_1$).

$$\begin{tikzcd} H^{s+3/2}(\Gamma) \arrow[r, "T_1"] & H^{s-1/2}(\Gamma) \\ \mathcal{D}(\mathcal{T}_1) \arrow[u] \arrow[r, "\mathcal{T}_1"] & H^{s+1/2}(\Gamma) \arrow[u] \\ C^\infty(\Gamma) \arrow[u] \arrow[r, "T_1"'] & C^\infty(\Gamma) \arrow[u] \end{tikzcd}$$

Table 15.2 The mapping property of the operator $\mathcal{T}_1$ for $s \geq 0$

Therefore, we find from formulas (6.19) and (6.20) that the principal term $s(x', \xi')$ of the symbol of the pseudo-differential operator T_1^{-1} is "formally" given by the formula

$$s(x', \xi') = -\frac{1}{\sum_{j,k=1}^{n-1} \alpha^{jk}(x')\xi_j \xi_k + \mu(x') \langle \xi' \rangle} \tag{15.17}$$

in terms of the Boutet de Monvel calculus, where

$$\mu(x') > 0 \quad \text{on } \Gamma.$$

The proof of Theorem 15.2 is divided into four steps.

Step (1): First, we obtain from formulas (15.10) and (15.11) that the Green operator

$$\mathcal{G}_D = \mathcal{Q}_1 - \mathcal{P}_1 \left(\gamma_0 \mathcal{Q}_1 \right), \tag{15.9}$$

has the principal symbol

$$\sigma\left(\mathcal{G}_D\right) = -\frac{1}{2\langle \xi' \rangle} \left(\frac{1}{\langle \xi' \rangle + i\nu} + \frac{1}{\langle \xi' \rangle - i\nu} \right) + \frac{1}{2\langle \xi' \rangle} \frac{1}{\langle \xi' \rangle + i\nu} \frac{1}{\langle \xi' \rangle - i\tau}. \tag{15.18}$$

Step (2): Secondly, it follows from formula (6.19) that the principal symbol of the Dirichlet-to-Neumann operator $\Pi_1 = \gamma_1 \mathcal{P}_1$ is given by the formula

$$\sigma\left(\Pi_1\right) = -\langle \xi' \rangle. \tag{15.19}$$

Moreover, by using Rempel–Schulze [39, p. 100, Lemma 4] we obtain from formula (15.12) that the operator $\gamma_1 \mathcal{Q}_1$ is a *trace operator* of order -1 with the principal symbol

$$\sigma\left(\gamma_1 \mathcal{Q}_1\right) = -\frac{1}{2} \frac{1}{\langle \xi' \rangle - i\tau}. \tag{15.20}$$

Therefore, by combining formulas (15.20), (15.19) and (15.13) we find that the operator

$$\gamma_1 \mathcal{G}_D = \gamma_1 \mathcal{Q}_1 - \Pi_1 \left(\gamma_0 \mathcal{Q}_1 \right)$$

is a *trace operator* of order -1 with the principal symbol

$$\sigma\left(\gamma_1 \mathcal{G}_D\right) = -\frac{1}{\langle \xi' \rangle - i\tau}. \tag{15.21}$$

Step (3): Thirdly, since $\gamma_0 \mathcal{G}_D = 0$ we have the formula

$$B\boldsymbol{\gamma}\mathcal{G}_D = \mu(x')\, \gamma_1 \mathcal{G}_D + Q(x', D_{x'})\left(\gamma_0 \mathcal{G}_D\right) = \mu(x')\, \gamma_1 \mathcal{G}_D.$$

Hence, it follows from formula (15.21) that the operator $B\boldsymbol{\gamma}\mathcal{G}_D = \mu(x')\, \gamma_1 \mathcal{G}_D$ is a *trace operator* of order -1 with the principal symbol

$$\sigma\left(B\boldsymbol{\gamma}\mathcal{G}_D\right) = \mu(x')\, \sigma\left(\gamma_1 \mathcal{G}_D\right) = -\frac{\mu(x')}{\langle \xi' \rangle - i\tau}. \tag{15.22}$$

Step (4): In terms of the Boutet de Monvel calculus, we can express each operator in the representation formula (15.15) in the matrix form (see Table 15.3 below)

$$\mathcal{R} = \begin{pmatrix} \mathcal{G}_D & -\mathcal{P}_1 \\ \mu(x')\, \gamma_1 \mathcal{G}_D & \mathcal{T}_1^{-1} \end{pmatrix} \tag{15.23}$$

and further from formulas (15.18), (15.14), (15.22) and (15.17) that the principal symbol of $\mathcal{R}$ is “formally” given by the formula

$$\sigma(\mathcal{R}) = \begin{pmatrix} -\frac{1}{\langle\xi'\rangle^2+\nu^2} + \frac{1}{2\langle\xi'\rangle}\frac{1}{\langle\xi'\rangle+i\nu}\frac{1}{\langle\xi'\rangle-i\tau} & -\frac{1}{\langle\xi'\rangle+i\nu} \\ -\frac{\mu(x')}{\langle\xi'\rangle-i\tau} & s(x',\xi') \end{pmatrix}. \tag{15.24}$$

Therefore, we find from Rempel–Schulze [39, p. 176, Theorem 1] that the resolvent $\mathcal{G} = (\mathfrak{A} - I)^{-1}$ maps $H^s(\Omega)$ continuously into $H^{s+2}(\Omega)$ for every $s \geq 0$, as is shown in Table 15.3 below.

The proof of Theorem 15.2 is complete.

$$\begin{tikzcd} H^s(\Omega) \arrow[r, "\mu(x')\,\gamma_1\mathcal{G}_D"] \arrow[d, "\mathcal{G}_D"'] & H^{s+1/2}(\Gamma) \arrow[d, "\mathcal{T}_1^{-1}"'] \\ H^{s+2}(\Omega) & H^{s+3/2}(\Gamma) \arrow[l, "-\mathcal{P}_1"] \end{tikzcd}$$

Table 15.3 The mapping property of the resolvent $\mathcal{G}$ for $s \geq 0$

16 Proof of Theorem 2.5 via the Boutet de Monvel calculus

This section is devoted to the proof of Theorem 2.5. By virtue of the *homotopy* (15.2), we are reduced to the study of the homogeneous Višik–Ventcel' boundary value problem (15.7). The proof is divided into two steps.

Step 1: First, by using Theorems 15.1 and 15.2 we can characterize explicitly the mapping property of the adjoint operator $\mathcal{G}^*$ of the Green operator $\mathcal{G}$ (defined by formula (15.15)) as follows:

Theorem 16.1 *Assume that condition (G) is satisfied. Then the* adjoint operator $\mathcal{G}^*$ *of* $\mathcal{G}$ *is given by the formula*

$$\mathcal{G}^* g = \mathcal{G}_D{}^* g - \left(\mu(x')\,\gamma_1\mathcal{G}_D\right)^* \left(\mathcal{T}_1{}^*\right)^{-1} \left(\mathcal{P}_1{}^* g\right) \quad \text{for } g \in L^2(\Omega), \tag{16.1}$$

and it maps $H^s(\Omega)$ *continuously into* $H^{s+2}(\Omega)$ *for every* $s \geq 0$ *(see Table 16.2 below).*

Proof First, we find from formula (15.15) and assertion (15.16) that

$$T_1{}^* = (B\boldsymbol{\gamma}\mathcal{P}_1)^* = \left(\mu(x')\,\Pi_1\right)^* + Q(x', D_{x'})^*$$

and the adjoint operator

$$\mathcal{T}_1{}^* \colon H^{s+3/2}(\Gamma) \longrightarrow H^{s+1/2}(\Gamma)$$

is *bijective* for every $s \geq 0$ in formula (16.1). The situation can be visualized in Table 16.1 below (see Table 6.3 with $T := T_1$ and $s := -s - 2$).

$$\begin{tikzcd}
H^{s-1/2}(\Gamma) & H^{s+3/2}(\Gamma) \arrow[l, "\mathcal{T}_1{}^*"'] \\
H^{s+1/2}(\Gamma) \arrow[u] & \mathcal{D}(\mathcal{T}_1{}^*) \arrow[l, "\mathcal{T}_1{}^*"'] \arrow[u] \\
C^\infty(\Gamma) \arrow[u] & C^\infty(\Gamma) \arrow[l, "\mathcal{T}_1{}^*"] \arrow[u]
\end{tikzcd}$$

Table 16.1 The mapping properties of the adjoint operator $\mathcal{T}_1{}^*$ for $s \geq 0$

Secondly, by virtue of formulas (15.23) and (15.24) we obtain that the *adjoint operator* $\mathcal{R}^*$ of $\mathcal{R}$ in the Boutet de Monvel calculus can be expressed in the matrix form (see Table 16.2 below)

$$\mathcal{R}^* = \begin{pmatrix} \mathcal{G}_D{}^* & \left(\mu(x')\,\gamma_1 \mathcal{G}_D\right)^* \\ -\mathcal{P}_1{}^* & \left(\mathcal{T}_1^*\right)^{-1} \end{pmatrix} \tag{16.2}$$

with principal symbol

$$\sigma\left(\mathcal{R}^*\right) = \begin{pmatrix} -\frac{1}{\langle\xi'\rangle^2+\nu^2} - \frac{1}{2\langle\xi'\rangle}\frac{1}{\langle\xi'\rangle+i\nu}\frac{1}{\langle\xi'\rangle-i\tau} & -\frac{\mu(x')}{\langle\xi'\rangle+i\nu} \\ -\frac{1}{\langle\xi'\rangle-i\tau} & s(x',\xi') \end{pmatrix}. \tag{16.3}$$

Indeed, by Rempel–Schulze [39, p. 151, Corollary 11] it follows from formula (15.22) that the adjoint operator $(B\boldsymbol{\gamma}\mathcal{G}_D)^* = \left(\mu(x')\,\gamma_1\mathcal{G}_D\right)^*$ is a *potential operator* of order -1 with the principal symbol

$$-\frac{\mu(x')}{\langle\xi'\rangle + i\nu}. \tag{16.4}$$

Therefore, we obtain from formula (15.15) that the adjoint operator

$$\mathcal{G}^* = \left((\mathfrak{A} - I)^{-1}\right)^* = \left(\mathfrak{A}^* - I\right)^{-1}$$

is given by the formula

$$\mathcal{G}^* = \mathcal{G}_D{}^* - \left(\mu(x')\,\gamma_1\mathcal{G}_D\right)^* \left(\mathcal{T}_1{}^*\right)^{-1} \mathcal{P}_1{}^*.$$

Moreover, we obtain from formulas (16.2), (16.3) and (16.4) and [39, p. 176, Theorem 1] that the adjoint operator

$$\mathcal{G}^* : H^s(\Omega) \longrightarrow H^{s+2}(\Omega)$$

is continuous for every $s \geq 0$, as is shown in Table 16.2 below.

The proof of Theorem 16.1 is complete.

Step 2: By virtue of part (i) of Theorem 2.4, Theorem 15.2 and Theorem 16.1, we can apply [55, Theorem 4.1] and [55, condition (4.4)] with $\mathcal{A} := \mathfrak{A} - I$ to obtain Theorem 2.5.

Now the proof of Theorem 2.5 is complete. □

$$\begin{tikzcd}
H^s(\Omega) \arrow[r, "-\mathcal{P}_1{}^*"] \arrow[d, "\mathcal{G}_D{}^*"'] & H^{s+1/2}(\Gamma) \arrow[d, "(\mathcal{T}_1{}^*)^{-1}"'] \\
H^{s+2}(\Omega) & H^{s+3/2}(\Gamma) \arrow[l, "(\mu(x')\,\gamma_1\mathcal{G}_D)^*"]
\end{tikzcd}$$

Table 16.2 The mapping property of the adjoint operator $\mathcal{G}^*$ for $s \geq 0$

17 Concluding remarks

In this last section, we state a brief history of the *stochastic analysis* methods for Višik–Ventcel' boundary value problems. More precisely, we remark that the Višik–Ventcel' boundary value problem (1.5) was studied by Anderson [5], [6], Cattiaux [11] and Takanobu–Watanabe [56] from the viewpoint of stochastic analysis (see also Ikeda–Watanabe [26, Chapter IV, Section 7]).

(I) Anderson [5] and [6] studies the non-degenerate case under low regularity in the framework of the submartingale problem and shows the existence and uniqueness of solutions to the considered submartingale problem.

(II) Takanobu–Watanabe [56] study certain cases of both degenerate interior and boundary operators under minimal assumptions of regularity based on the theory of stochastic differential equations, and they show the existence and uniqueness of solutions. Such existence and uniqueness results on the diffusion processes corresponding to the boundary value problems imply the existence and uniqueness of the associated Feller semigroups on the space of continuous functions.

(III) Cattiaux [11] studies the hypoellipticity for diffusions with Višik–Ventcel' boundary conditions. By making use of a variant of the Malliavin calculus under Hömander's type conditions, he proves that some laws and conditional laws of such diffusions have a smooth density with respect to the Lebesgue measure.

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