Stochastic stability under logit choice in coalitional bargaining problems*

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Abstract This study examines a dynamic process of n-person coalitional bargaining problems. We investigate the evolution of social conventions by embedding a coalitional bargaining setting in a dynamic process. In each period, a group of players may make some coalitional move, that is, forming a new team or negotiating the division of a surplus. Players revise their coalitions and surplus divisions over time in the presence of stochastic noise, which leads players to make a suboptimal decision. Under a logit specification of choice probabilities, we find that the stability of a core allocation decreases in the wealth of the richest player. Furthermore, stochastically stable allocations are core allocations that minimize the wealth of the richest player.

Keywords: Stochastic stability; Coalitional bargaining; Coalition; Logit-response dynamics; Bargaining.

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1 Introduction

We consider a dynamic model of coalitional bargaining, where subsets of n players meet randomly and may form coalitions over time. The potential gains from cooperation are stylized by a characteristic function that maps each coalition to the surplus it generates. Feldman (1974) and Green (1974) show that such dynamic recontracting processes converge in core allocations. Agastya (1997) and Rozen (2013) model the cooperative setting as non-cooperative games with individual strategies. Both studies show that, in such games, myopic best-response dynamics converge in core allocations. Arnold and Schwalbe (2002) and Nax (2018) consider similar coalitional bargaining games for perturbed best-response dynamics and aspiration-based dynamics, respectively, and both obtain convergence to the core.

Agastya (1999) applies a stochastic stability analysis (Foster and Young 1990, Kandori et al. 1993 and Young 1993a) to the non-cooperative game of Agastya (1997).[1] Newton (2012b) unites the approach adopted by Feldman (1974) and Green (1974) with a stochastic stability analysis by incorporating joint deviations into a non-cooperative adaptive process. Following the spirit of Newton (2012b), we study the stochastically stable states of an adaptive process, where joint deviations are allowed. In each period, a group of players come to the bargaining table with a proposal for the division of a surplus. If the players in the group all agree, they form a coalition and distribute the surplus according to the proposal. Any existing coalition is dissolved if it includes any member of the newly formed coalition. Our approach differs from that of Newton (2012b) in two respects: (i) joint deviations are restricted to groups of players who are in a coalition or are forming a coalition; and (ii) players adopt the logit choice rule with one-period memory. We review the related literature in more detail in Section 2.

We find that stochastically stable allocations are core allocations whenever the set of interior points in the core, called the *strict core*, is nonempty. Moreover, we find the following two properties: (i) the stability of an allocation decreases in the wealth of the richest player; and (ii) stochastically stable allocations are core allocations that minimize the wealth of the richest player among all core allocations. These results are interesting because they show that equity considerations play an important role, even with myopic payoff-maximizing players.

This study makes two contributions to the literature. First, we characterize stochastically stable allocations under behavioral and institutional assumptions that differ from those of extant studies. We consider a situation where a group of players meet, discuss forming a coalition, and make joint decisions. We model this situation as a dynamic process, where groups of players stochastically receive opportunities to revise coalitions over time. Players are assumed to follow the logit choice rule. This is the main difference from extant works, which impose different assumptions, such as players submitting requests to an authority and making decisions according to the "best response

[1]Examples of studies that apply stochastic stability analyses to bargaining settings include those of Young (1993b), Sáez-Martí and Weibull (1999), Naidu et al. (2010), and Hwang et al. (2018). Jackson and Watts (2002), Klaus et al. (2010), Newton (2012a), Sawa (2014), Nax and Pradelski (2015, 2016), Newton and Sawa (2015), Klaus and Newton (2016), and Boncinelli and Pin (2018), among many others, apply the same analysis technique to cooperative settings. See Newton (2018) for a review of recent developments in this area.

with mutations" rule in the style of Young (1993a). We provide a more detailed comparison in Section 2.

Second, we relax the strong assumptions often imposed on the characteristic function. For example, Chatterjee et al. (1993) assume strict super-additivity; that is, the production of $S \cup T$ has to be strictly greater than the sum of the productions of S and T for any two disjoint coalitions S, T. Here, we simply assume the existence of the strict core. However, we do briefly consider settings with an empty core at the end of Section 5.2. Some studies on coalition formation (e.g., Gomes and Jehiel (2005)) consider more general settings, without assuming any particular characteristic function. However, we do assume a characteristic function in our model because this helps us to characterize allocations that arise in the long run.

With regard to the second contribution described above, we employ a somewhat relaxed definition of the core. The traditional definition of the core requires the formation of a grand coalition (e.g., Definition 4.2 of Moulin (1988)). This assumption has been relaxed by several studies on coalition formation, such as Konishi and Ray (2003). In a similar way, we allow multiple coalitions to coexist in a core allocation. The selection results depend on whether a setting has a core with a grand coalition or with multiple coalitions. Accordingly, we define the former as a *core* and the latter as a *dispersed core*. Then, we conduct separate stochastic stability analyses for each in Sections 5.1 and 5.2, respectively.

The paper is organized as follows. Section 2 provides a literature review. Section 3 introduces the coalitional bargaining model and defines the core and the dispersed core. In Section 4, we describe the dynamic of the bargaining model, after which we characterize stochastically stable allocations in Section 5. Section 6 extends the model to incorporate heterogeneous utility functions. Lastly, Section 7 concludes the paper.

2 Related literature

Table 1 summarizes related studies on coalitional bargaining with myopic or rational players. Feldman (1974) and Green (1974) conducted pioneering work on the dynamic recontracting process. Agastya (1997) and Rozen (2013) consider dynamic non-cooperative processes of coalitional bargaining, where players submit a demand or a pair comprising a demand and a list of acceptable partners. The model of Arnold and Schwalbe (2002) is similar to that of Agastya (1997), except that players choose a demand and a coalition to join (or to form a singleton coalition). Arnold and Schwalbe (2002) relax the super-additivity of the characteristic function, and consider the perturbed dynamics in which players make uniform errors outside of the core. Nax (2018) considers aspiration-based dynamics. Here, rather than providing a best response, players revise their individual strategies according to their aspiration levels. Such dynamics have fewer information requirements than those of best-response dynamics; that is, the dynamics are uncoupled (Hart and Mas-Colell, 2003). These studies all assume myopic players and show that the dynamic processes converge to some core allocation. Our (unperturbed) process is similar to that of Feldman

Table 1: Coalitional bargaining models and selection results

Model	Agency[a]	Foresight	Noise[b]	Selection result
Feldman (1974)	C	myopic	No	Some core allocation
Green (1974)	C	myopic	No	Some core allocation
Agastya (1997)	I	myopic	No	Some core allocation
Agastya (1999)	I	myopic	Yes (U)	Minimizing payoff for the richest
AS (2002) [c]	I	myopic	Yes (U)	Some core allocation
Newton (2012b)	C	myopic	Yes (U)	Maximizing payoff for the poorest
Rozen (2013)	I	myopic	No	Some (strict) core allocation
Nax (2018)	C	myopic	No	Some core allocation
Okada (1996)	C	perfect	No	Maximizing per capita ($\max_{J \subset N} v(J)/\lvert J\rvert$)
CJ (2010)[d]	C	perfect	No	Maximizing product of payoffs
This study	C	myopic	Yes (L)	Minimizing payoff for the richest player

[a] "Agency" indicates whether the study employs a unilateral (I) or a collective (C) decision-making process.

[b] "Noise" indicates whether the study examines robustness against stochastic noise. "(U)" and "(L)" indicate that the study employs a uniform error model and a payoff-dependent error model (logit), respectively.

[c] Arnold and Schwalbe (2002).

[d] Compte and Jehiel (2010).

(1974) in that we assume that forming a new coalition dissolves all existing coalitions.

The selection results are more refined in the works of Agastya (1999), Newton (2012b), Okada (1996), and Compte and Jehiel (2010), as well as in our study. Here (as in Agastya (1999) and Newton (2012b)), we assume myopic players and examine the robustness of allocations against stochastic noise by perturbing a dynamic process. These models all select a subset of the core that is independent from the initial state. The works of Okada (1996), and Compte and Jehiel (2010) assume rational players who are perfectly farsighted and make no mistakes, thus obtaining finer selection results. The refined selection results of these studies favor the egalitarian outcome: if it is in the core (or in the strict core), then it will be selected under certain conditions.[2] Even if the egalitarian outcome is not in the core, equity considerations seem to play a role. For example, surplus is equally distributed among coalition members in Okada (1996), and the wealth of the richest player is minimized in Agastya (1999) and in our study.

The institutional assumptions, particularity those related to the unit of agency, vary among the studies. This difference superficially appears as the state space of the dynamic process. Agastya (1997, 1999); Arnold and Schwalbe (2002); Rozen (2013) assume individual agency; that is, players make unilateral decisions. In their processes, players submit requests (e.g., a demand and a list of acceptable partners), and then the process chooses which coalitions to form, which are typically the coarsest coalition structures. The state of a typical process is the current profile of these indi-

[2] The model of Okada (1996) requires one additional condition for the egalitarian outcome:

$$\frac{v(S)}{|S|} \leq \frac{v(T)}{|T|} \quad \text{for all } S, T \subseteq N \text{ with } S \subseteq T.$$

vidual strategies. Other studies allow players to jointly choose strategies (i.e., collective agency). The latter studies, other than that of Newton (2012b), assume that players considering forming a coalition make joint decisions. In a typical process, a state is given by a claim profile and a set of existing coalitions. A proposal for a coalition and the division of a surplus is made in each period, and the proposed coalition is formed if all members jointly accept the proposal. The assumption is relaxed in Newton (2012b), who makes a distinction between groups of players who jointly choose their strategies and the coalitions they would like to form. Together with the difference in the behavioral assumption described below, Newton (2012b) predicts the set of allocations that maximizes a Rawlsian social welfare function for a class of utility functions, which is a finer prediction than ours. Despite their differences in the coalition-formation process, these studies all show that players choose an allocation in the core.

In addition to the institutional assumptions, there are two other differences between our study and those of Agastya (1999) and Newton (2012b). The first is related to behavioral assumptions. Agastya (1999) and Newton (2012b) employ a behavioral model similar to that of Young (1993b). That is, players possess more than one-period memory and follow the "best response with mutations" choice rule. In our model, players possess one-period memory and follow the logit choice rule. The logit choice is a prominent discrete-choice model that assumes that error probabilities depend on payoffs. In recent experimental studies, Mäs and Nax (2016), Lim and Neary (2016), and Hwang et al. (2018) found evidence of payoff-sensitivity in errors. These differing behavioral assumptions may lead to different predictions. Since the difference in predictions would be clearer for heterogeneous preferences, this will be further discussed in Section 6.

The second difference is related to restrictions on the characteristic function v. Agastya (1999) assumes convexity: $v(J \cup J') \geq v(J) + v(J') - v(J \cap J')$, for any pair of coalitions J, J'. Newton (2012b) assumes super-additivity: $v(J \cup J') \geq v(J) + v(J')$ if $J \cap J' = \varnothing$. Our model does not impose any restriction on v, except the existence of strict core allocations, which the other two studies assume as well. Example 2 in Section 5.1 considers a setting with a non-convex characteristic function and illustrates the difference in the institutional assumptions.

Finally, we offer a brief comparison with studies that assume rational players. Examples include the works of Okada (1996) and Compte and Jehiel (2010), which resemble ours in that the proposer is chosen randomly in each period. In these studies, discount factors are assumed to be close to one, and a proposer chooses her proposal in a rational manner. Consequently, the selection results of the rational models are sensitive to the chances of being a proposer; the probability of a player being a proposer becomes her bargaining power. In our model, players are assumed to be myopic and the probability of a subgroup of players being chosen plays little role. Our results hold as long as each subgroup is chosen with positive probability.

3 Model

We consider a non-cooperative model of multi-player coalitional-bargaining situations, as examined in several studies (e.g., Chatterjee et al. (1993)). The model is given by a tuple $(N, v, \{u_i\}_{i \in N})$, where $N = \{1, \ldots, n\}$ denotes a set of players, v is a characteristic function, and u_i is player i's utility function. Let $\mathcal{R}$ be the class of all nonempty subsets of N. Any $J \in \mathcal{R}$ may form a team, and the (monetary) surplus that a team generates is determined by the characteristic function $v \colon \mathcal{R} \to \mathbb{R}_+$, with $v(\varnothing) = 0$. The team of size n is called the grand coalition. We fix small $\Delta > 0$ and let $S_i = \{0, \Delta, 2\Delta, \ldots, \bar{S}\}$ denote the set of player i's claims s_i, where $\bar{S} \geq \max_{J \in \mathcal{R}} v(J)$.[3] We call $s \in \prod_{i \in N} S_i$ a *claim profile*. Thus, $s_J \in \prod_{i \in J} S_i$, for $J \in \mathcal{R}$, is the *claim profile of team* J. A claim profile of a team $J \in \mathcal{R}$ is feasible if it satisfies the following feasibility constraint:

$$\sum_{i \in J} s_i \leq v(J). \tag{1}$$

We impose the following assumption on $v(\cdot)$ and Δ.

Assumption 3.1. *$v(J)/\Delta \in \mathbb{Z}$ for all $J \in \mathcal{R}$, and $v(N)/(n\Delta) \in \mathbb{Z}$.*

The first condition of Assumption 3.1 ensures the existence of claim profiles for which the feasibility constraint is binding for all formed teams. The second condition ensures that the feasibility constraint is binding for the egalitarian allocation, where each receives $v(N)/n$.

The utility of player i who receives s_i is represented by a function $u_i : S_i \to \mathbb{R}$, which is strictly increasing and strictly concave.[4] We assume that the utility function is common across players (i.e., $u_i(\cdot) = u(\cdot)$) throughout this paper, except in Section 6, where we consider settings with heterogeneous utility functions. The set of feasible claim profiles of team J is denoted by

$$S^J = \left\{ s_J \in \prod_{i \in J} S_i : s_J \text{ satisfies (1).} \right\}.$$

We interpret $v(\{i\})$ as the reservation surplus for player i, and assume that player i receives $v(\{i\})$ if she forms a singleton team (i.e., $S^{\{i\}} = \{v(\{i\})\}$, for all $i \in N$).[5]

Each player can participate in exactly one team. A player forms a singleton team if she does not form a team with others. The set of existing teams is expressed as a partition of N.[6] Let $\mathrm{part}(N)$ denote the set of partitions of N. We refer to a pair comprising a claim profile s and a set of existing teams as an *allocation*. The set of allocations for $\mathcal{M} \in \mathrm{part}(N)$ is defined as

$$\Omega_{\mathcal{M}} = \left\{ (s, \mathcal{M}) \colon s_J \in S^J \ \forall J \in \mathcal{M} \right\}.$$

[3]We assume that the set S_i is finite, for mathematical convenience. Our results hold for all sufficiently small Δ.

[4]We implicitly assume that a surplus $v(\cdot)$ is transferable, but that utility is not. A similar strictly concave utility function is assumed in Agastya (1999).

[5]This assumption is made for the sake of simplicity. Our results still hold for an alternative constraint where player i receives at most $v(\{i\})$: $S^{\{i\}} = \{s_i \leq v(\{i\})\}$.

[6]$\mathcal{M}$ is a partition of N if $N = \cup_{M \in \mathcal{M}} M$, and $M \cap M' = \varnothing \ \forall M, M' \in \mathcal{M}, M \neq M'$.

The allocation space, denoted by Ω, is given by the union of the sets of allocations for all partitions of N. That is,

$$\Omega = \bigcup_{\mathcal{M} \in \text{part}(N)} \Omega_{\mathcal{M}}.$$

We say that coalition J *blocks* allocation $(s, \mathcal{M}) \in \Omega$ if there exists $s' \in S^J$, such that

$$s'_i > s_i \ \forall i \in J.$$

Similarly, coalition $J \notin \mathcal{M}$ *weakly blocks* allocation $(s, \mathcal{M}) \in \Omega$ if there exists $s' \in S^J$, such that

$$s'_i \geq s_i \ \forall i \in J.$$

For $J \in \mathcal{M}$, "weakly block" is defined as follows: $J \in \mathcal{M}$ *weakly blocks* $((s_J, s_{-J}), \mathcal{M})$ if there exists $s' \in S^J$ with $s' \neq s_J$, such that $s'_i \geq s_i$ for all $i \in J$. If $(s, \mathcal{M})$ cannot be weakly blocked by J, then for any claim profile $s' \in S^J$, there is at least one player in J who would be strictly worse off than she is in s. We define the *core* and its extended notion, the *dispersed core*, as follows.

Definition 3.2. *The* core *consists of the claim profiles with $\mathcal{M} = \{N\}$ that cannot be blocked by any coalition $J \in \mathcal{R}$. The* strict core *consists of those claim profiles that cannot be weakly blocked by any $J \in \mathcal{R}$.*

Definition 3.3. *The* dispersed core *consists of the claim profiles with $\mathcal{M} \in \text{part}(N)$ that cannot be blocked by any $J \in \mathcal{R}$. The* strict dispersed core *consists of those claim profiles that cannot be weakly blocked by any $J \in \mathcal{R}$.*

In other words, $(s, \mathcal{M})$ is a core allocation if, for any $J \in \mathcal{R}$, there is no coalitional deviation that strictly improves the payoffs of all players in J. It is strict if there is no deviation that weakly improves the payoffs of all players in J. The dispersed core is a weaker definition in that it does not require that a grand coalition be formed. A similar definition is used in studies on coalition formation, such as those of Arnold and Schwalbe (2002) and Konishi and Ray (2003).

Observe that the grand coalition weakly blocks any allocation $(s, \mathcal{M})$ with $\mathcal{M} \neq \{N\}$ if the strict core is nonempty. Thus, when the strict core exists, it coincides with the strict dispersed core. A strict dispersed-core allocation is not weakly blocked, even by any existing team, which implies that the surplus is distributed without loss in any strict-core allocation. That is, $\sum_{i \in J} s_i = v(J)$ for all $J \in \mathcal{M}$ if $(s, \mathcal{M})$ is in the strict dispersed core. Note too that a strict dispersed-core allocation $(s, \mathcal{M})$ satisfies the set of inequalities:

$$\sum_{i \in J} s_i \geq v(J) + \Delta \quad \forall J \in \mathcal{R} \setminus \mathcal{M}, \tag{2}$$

where $\mathcal{M} = \{N\}$ if the strict core is nonempty.

In what follows, we assume that the strict dispersed core is nonempty. Let Ω^*_Δ denote the set of strict dispersed-core allocations, given Δ, and $\mathscr{C}_\Delta$ denote the set of claim profiles of the strict dispersed-core allocations (i.e., $\mathscr{C}_\Delta = \{s : (s, \mathcal{M}) \in \Omega^*_\Delta\}$).

Let $s_{(i)}$ denote the i-th largest claim for a claim profile s. Define

$$s_{\min} = \min_{s' \in \mathscr{C}_\Delta} s'_{(1)},$$
$$_{\min}\Omega^*_\Delta = \left\{(s, \mathcal{M}) \in \Omega^*_\Delta : s_{(1)} = s_{\min}\right\}.$$

In other words, $s_{\min}$ is the lowest claim of the richest player among all strict dispersed-core allocations. Then, $_{\min}\Omega^*_\Delta$ is the set of allocations that minimize the wealth of the richest player over all strict dispersed-core allocations.

We say that $(s, \mathcal{M})$ is the egalitarian allocation if $s_i = v(N)/n$, for all $i \in N$ and $\mathcal{M} = \{N\}$, and is denoted as $(s^E, \{N\})$. Assumption 3.1 guarantees that the egalitarian allocation is in the allocation space (i.e., $(s^E, \{N\}) \in \Omega$). The following lemma shows that the wealth difference is at least 2Δ if an allocation is not egalitarian.

Lemma 3.4. *Suppose that the strict core exists. For all $s \in \mathscr{C}_\Delta$, such that $s \neq s^E$, $s_{(1)} - s_{(n)} \geq 2\Delta$.*

Proof. The lemma is proved by way of contradiction. Suppose that $s \in \mathscr{C}_\Delta$ is such that $s_{(1)} - s_{(n)} = \Delta$. That is, $s \in \mathscr{C}_\Delta$ implies that s must be efficient (i.e., $\sum_i s_i = v(\{N\})$). If $s_{(1)} \leq v(\{N\})/n$, then $\sum_i s_i < v(\{N\})$. If $s_{(n)} \geq v(\{N\})/n$, then $\sum_i s_i > v(\{N\})$. Therefore, it must be that $s_{(1)} = s_{(n)}$, which is a contradiction. $\square$

4 Dynamics

4.1 The Coalitional Logit Dynamic

We consider a dynamic process of coalitional bargaining with the state space Ω, and examine the robustness of allocations to stochastic noise. The dynamic process of coalitional bargaining proceeds as follows. The state of the process in period t is denoted by $\omega^t = (s^t, \mathcal{M}^t) \in \Omega$, where s^t denotes the claim profile in period t, and $\mathcal{M}^t$ is the set of existing teams in t. At the beginning of period t, $J \in \mathcal{R}$ is chosen with probability q_J. We assume that $q_J > 0$ for every $J \in \mathcal{R}$. Then a payment proposal $s = \{s_i\}_{i \in N} \in S(J, s^t, \mathcal{M}^t)$ is chosen with probability $q_s(J, s^t, \mathcal{M}^t)$, where the set $S(J, s^t, \mathcal{M}^t)$ is specified soon. Each player in J decides whether to accept or reject proposal s. If at least one player in J rejects s, the state remains unchanged. If all players in J accept the proposal, team J is formed and any existing team containing some $i \in J$ is dissolved. Then, members other than i of these previous teams form singleton teams. If some $J \in \mathcal{M}^t$ is chosen to be revised, this is interpreted to mean that the members of J negotiate its surplus division. In such cases, they may change the claim profile, but do not change $\mathcal{M}^t$.

Given s^t, $\mathcal{M}^t$ and J, a proposal s is feasible if it satisfies the following conditions:

$$\begin{aligned} & s_J \in S^J, \\ & s_i = v(\{i\}) && \forall i \in M \setminus J,\ \forall M \in \mathcal{M}^t \text{ such that } M \cap J \neq \varnothing, \end{aligned}$$

$$s_i = s_i^t \qquad\qquad \forall i \in M,\ \forall M \in \mathcal{M}^t \text{ such that } M \cap J = \varnothing.$$

The first condition is the feasibility constraint for J. The second condition implies that players in teams being dissolved form singleton teams and earn their reservation surplus. The third condition implies that forming J should not affect any team that includes no member of J. Here, $S(J, s, \mathcal{M})$ denotes the set of feasible proposals for team J, where the current allocation is $(s, \mathcal{M}) \in \Omega$. We assume that $q_{s'}(J, s, \mathcal{M}) > 0$ for all $s' \in S(J, s, \mathcal{M})$ and all $(s, \mathcal{M}) \in \Omega$.

We assume that players' utilities are temporarily affected by stochastic shocks. Following Blume (1993), these shocks are assumed to follow a type-I extreme-value distribution, which makes players follow the logit choice rule. To describe the logit choice rule, suppose that the current claim profile is s, a randomly chosen coalition is J, and s' is proposed as the surplus distribution. The probability that player i in coalition J agrees with s' is given by

$$\Psi_i^{\eta}(s, s') = \frac{\exp\left[\eta^{-1} u(s_i')\right]}{\exp\left[\eta^{-1} u(s_i')\right] + \exp\left[\eta^{-1} u(s_i)\right]}, \tag{3}$$

where $\eta \in (0, \infty)$ denotes the noise level of the logit choice rule. Note that player i takes into account other players' new claims (i.e., s_J') in Equation (3). The probability that all members in J agree with s' is given by $\prod_{i \in J} \Psi_i^{\eta}(s, s')$.

The dynamic process described above forms a Markov chain with a state space of Ω.[7] A transition from $(s, \mathcal{M})$ occurs when coalition $J \notin \mathcal{M}$ forms a new team, or when $J \in \mathcal{M}$ redistributes its surplus. We express the transition from $(s, \mathcal{M})$ to $(s', \mathcal{M}')$ as $((s, \mathcal{M}), (s', \mathcal{M}'))$. A transition $((s, \mathcal{M}), (s', \mathcal{M}'))$ is said to be *possible* if the following conditions are satisfied.

(i) For $\mathcal{M} \neq \mathcal{M}'$, there exists $J \in \mathcal{M}'$, such that

$$\begin{aligned} &\{i\} \in \mathcal{M}' \quad \forall i \in M \setminus J, && \forall M \in \mathcal{M} \text{ such that } J \cap M \neq \varnothing, \\ &M \in \mathcal{M}', \text{ and } \ s_i = s_i' \quad \forall i \in M, && \forall M \in \mathcal{M} \text{ such that } J \cap M = \varnothing. \end{aligned}$$

(ii) For $\mathcal{M} = \mathcal{M}'$, there exists $J \in \mathcal{M}'$, such that

$$s_i = s_i' \qquad\qquad \forall i \in M',\ \forall M' \in \mathcal{M}' \setminus \{J\}.$$

Condition (i) states that when a new team J is formed and there is at least one player leaving an existing team M to join J, M is dissolved. The other existing teams remain intact and their distributions are unaffected. Condition (ii) applies to cases of a surplus redistribution ($\mathcal{M} = \mathcal{M}'$). It states that at most one team can redistribute its surplus in a transition. Condition (ii) also applies to cases in which a proposal is rejected. Then, $\mathcal{M} = \mathcal{M}'$ holds, and the claim profile s remains

[7] Transitions depend not only on the current claim profile, but also on the set of existing teams. If a coalition forms a new team, it will affect other players in terms of teams being dissolved. Which players are affected depends on the existing teams.

unchanged. This implies that transition (ω, ω) is possible for any $\omega \in \Omega$.

Define

$$R_{(s,\mathcal{M}),(s',\mathcal{M}')} = \begin{cases} \{J \in \mathcal{R} : \ J \text{ satisfies (i).}\} & \text{if } \mathcal{M} \neq \mathcal{M}', \\ \{J \in \mathcal{R} : \ J \text{ satisfies (ii).}\} & \text{if } \mathcal{M} = \mathcal{M}'. \end{cases}$$

Here, $R_{(s,\mathcal{M}),(s',\mathcal{M}')}$ is the set of coalitions potentially leading from $(s, \mathcal{M})$ to $(s', \mathcal{M}')$. A transition from $(s, \mathcal{M})$ to $(s', \mathcal{M}')$ is impossible if $R_{(s,\mathcal{M}),(s',\mathcal{M}')} = \varnothing$.

Recall that q_J is the probability that coalition J is chosen for revision, and $q_{s'}(J, s, \mathcal{M})$ is the probability that s' is chosen, given J, s, and $\mathcal{M}$. For the perturbed process, the transition probability from $(s, \mathcal{M})$ to $(s', \mathcal{M}')$ is given by

$$P^{\eta}_{(s,\mathcal{M}),(s',\mathcal{M}')} = \sum_{J \in R_{(s,\mathcal{M}),(s',\mathcal{M}')}} q_J \, q_{s'}(J, s, \mathcal{M}) \prod_{i \in J} \Psi^{\eta}_i(s, s') \qquad \forall (s', \mathcal{M}') \neq (s, \mathcal{M}). \tag{4}$$

The probability of staying in the same state ω is given by $1 - \sum_{\hat{\omega} \neq \omega} P^{\eta}_{\omega,\hat{\omega}}$.

For the unperturbed process ($\eta = 0$), the transition probability $P^{0}_{(s,\mathcal{M}),(s',\mathcal{M}')}$ is given by replacing $\Psi^{\eta}_i(s, s')$ in Equation (4) with $\Psi^{0}_i(s, s')$, as follows:

$$\Psi^{0}_i(s, s') = \begin{cases} 0 & u(s_i) > u(s'_i) \\ \frac{1}{2} & u(s_i) = u(s'_i) \\ 1 & u(s_i) < u(s'_i). \end{cases}$$

4.2 Limiting Stationary Distributions and Stochastic Stability

The Markov chain induced by $P^{\eta}_{\cdot,\cdot}$ is irreducible for $\eta > 0$, and thus admits a unique stationary distribution. Let $\pi^{\eta}(\omega)$ denote the mass that the stationary distribution places on state $\omega \in \Omega$. The players' long-term behavior is summarized by $\pi^{\eta}(\cdot)$. For example, as the Markov chain is also aperiodic, $\pi^{\eta}(\omega)$ is the probability that state ω is observed at any given time t, provided that t is sufficiently large. $\pi^{\eta}(\omega)$ also represents the fraction of time in which ω is observed over a long time horizon. We say that ω is stochastically stable if the limiting stationary distribution imposes a positive probability on ω.[8]

Definition 4.1. *State ω is* stochastically stable *if* $\lim_{\eta \to 0} \pi^{\eta}(\omega) > 0$.

Next, we introduce several definitions that we use to compute stochastically stable states. Given a state ω, define an ω-tree, denoted by $T(\omega)$, as a directed graph on Ω that has a unique path from any state $\omega' \in \Omega$ to the root state ω. An edge of an ω-tree, denoted by $(\omega', \omega'') \in T(\omega)$, represents a transition from ω' to ω'' in the dynamic.

[8]This definition follows the $\mathcal{R}$-stochastic stability of Sawa (2014). A state is $\mathcal{R}$-stochastically stable if it is stochastically stable on the limiting perturbed process in which any coalition $J \in \mathcal{R}$ may make joint deviations.

Let $\omega = (s, \mathcal{M})$ and $\omega' = (s', \mathcal{M}')$. We define the cost of a transition (ω, ω') as follows:

$$c_{\omega,\omega'} = \begin{cases} \min\limits_{J \in R_{\omega,\omega'}} \left[\sum_{i \in J} \max\{u(s_i) - u(s'_i), 0\}\right] & \text{if } R_{\omega,\omega'} \neq \varnothing, \\ \infty & \text{if } R_{\omega,\omega'} = \varnothing. \end{cases} \tag{5}$$

In other words, the cost of a transition is the sum of the utility losses of players involved in a revision in the transition. Equation (5) shows a difference between our model and that of Alós-Ferrer and Netzer (2010), who consider settings where players make unilateral deviations simultaneously. The cost of (ω, ω') in our model evaluates the payoff disadvantages of a coalitional deviation s'_J, for $J \in R_{\omega,\omega'}$, rather than those of unilateral deviations.

The next lemma shows that the cost $c_{\omega,\omega'}$ is equal to the exponential rate of decay of the corresponding transition probability, $P^{\eta}_{\omega,\omega'}$.[9]

Lemma 4.2. *If $R_{\omega,\omega'} \neq \varnothing$, then*

$$-\lim_{\eta \to 0} \eta \log P^{\eta}_{\omega,\omega'} = c_{\omega,\omega'}.$$

Proof. See Sawa (2014). □

Lemma 4.2 implies that the utility loss of a transition determines the likelihood of the transition. Let $\mathcal{T}(\omega)$ denote the set of ω-trees. The cost of a tree $T \in \mathcal{T}(\omega)$ is defined as

$$W(T) = \sum_{(\omega',\omega'') \in T} c_{\omega',\omega''}. \tag{6}$$

The cost of a tree is the sum of the payoff losses along the tree. The stochastic potential of state ω is defined as

$$W(\omega) = \min_{T \in \mathcal{T}(\omega)} W(T).$$

A tree $T(\omega)$ is called a minimum-cost tree of ω if $W(\omega) = W(T(\omega))$.

As η approaches zero, the stationary distribution converges to a unique limiting stationary distribution. Identifying stochastically stable states is simplified by Young (1993a), who proposes an equivalence theorem with states that minimize the stochastic potential (see also Freidlin and Wentzell (1998)). This version, which postulates coalitional deviations under the logit choice rule, follows the work of Sawa (2014). Our main results in Section 5 are built on Theorem 4.3.

Theorem 4.3. *A state is stochastically stable if and only if it minimizes $W(\omega)$ among all states.*

Proof. See Young (1993a) and Sawa (2014). □

[9] See Chapter 12 of Sandholm (2010) for a discussion on defining transition costs in this way.

5 Characterization of Stochastically Stable Allocations

5.1 Core Allocations with a Grand Coalition

In this section, we address coalitional bargaining problems with a nonempty strict core. We examine problems with a strict dispersed core in Section 5.2. We first consider the unperturbed process (i.e., there is no stochastic shock). The next lemma shows that an unperturbed process starting from any state can reach a strict-core allocation. Note that the proofs in this section are relegated to the Appendix.

Lemma 5.1. *Starting from $(s, \mathcal{M}) \in \Omega$, with $s \notin \mathscr{C}_\Delta$, the unperturbed process induced by P^0 reaches some $(s^*, \{N\}) \in \Omega$ with $s^* \in \mathscr{C}_\Delta$ with positive probability.*

Lemma 5.1 resembles Theorem 2 of Feldman (1974) and implies that an unperturbed process reaches some strict-core allocation within finite time and with probability one. Similarly to Feldman (1974), we assume that any existing team is dissolved if it includes any member of a newly formed coalition. The assumption plays a key role in proving Lemma 5.1, as well as in the results of Lemmas 5.2 and 5.7. For example, if a single player deviates from the grand coalition, then the grand coalition is dissolved and every player forms a singleton team. It is easy to see that the process can reach a strict-core allocation from such a state. Using a similar assumption, we prove that the process can reach a strict-core allocation from any state. In contrast to Theorem 2 of Feldman (1974), we drop the assumption that the number of teams of multiple players is at most one. The process still converges to the (strict) core under our relaxed assumptions.

Let $\mathcal{R}_i = \{J \in \mathcal{R} : i \in J\}$, which is the set of coalitions that include player i. Let $I_\$(s) = \{i : s_i = s_{(1)}\}$ denote the set of the richest players in $s \in \mathscr{C}_\Delta$. Now we can define the following condition, Condition (7).

$$\sum_{i \in J} s_i \geq v(J) + 2\Delta \qquad \forall J \in \mathcal{R}_{i_\$} \setminus \{N\},\ \forall i_\$ \in I_\$(s). \tag{7}$$

Any allocation satisfying Condition (7) is a strict-core allocation. Furthermore, even if a player transfers Δ of her surplus to another in an allocation satisfying (7), the resulting allocation satisfies Inequality (2); that is, it is still a strict-core allocation.

For $\omega = (s, \{N\})$, with $s \in \mathscr{C}_\Delta$, let $R(\omega)$ denote the *lowest cost of escaping* from ω, that is, the minimum cost for the process to escape from ω to some other $\omega' = (s', \{N\})$, with $s' \in \mathscr{C}_\Delta$. See Equation (10) in the Appendix for the formal definition of $R(\cdot)$. A sequence of transitions from ω to ω', with $s' \in \mathscr{C}_\Delta$, is called a *least-cost escape from ω* if the cost of the sequence is $R(\omega)$. We say that a least-cost escape from ω *leads the process to ω'* if there is a least-cost escape from ω that ends at ω'. Condition (7) is a key part of identifying which allocation is most likely to be reached after any departure from a strict-core allocation. This is shown in the following lemma.

Lemma 5.2. *For $s \in \mathscr{C}_\Delta$, s has three properties:*

(i) *The least cost of escaping from state* $\omega = (s, \{N\})$ *is given by*

$$R(\omega) = u(s_{(1)}) - u(s_{(1)} - \Delta). \tag{8}$$

(ii) *Suppose that allocation* s *satisfies Condition (7). If* $s \neq s^E$*, there exists a least-cost escape from* s *to* $s' \in \mathscr{C}_\Delta$*, where* s' *is either an allocation with the richest player claiming* $s_{(1)} - \Delta$ *or that in which the number of players claiming* $s_{(1)}$ *is strictly fewer than that of* s*. If* $s = s^E$*, any least-cost escape from* s *leads the process to some* $s' \in \mathscr{C}_\Delta$*, where* $s'_{(1)} = s_{\min} + \Delta$*.*

(iii) *If allocation* s *violates Condition (7), then for any* $s' \in \mathscr{C}_\Delta$*, there exists a least-cost escape from* s *that leads the process to* s'*.*

Lemma 5.2 shows that the stability of a core allocation depends on the richest player. Moreover, the concavity of $u(\cdot)$ implies that $u(x) - u(x - \Delta) < u(y) - u(y - \Delta)$, for all $x > y$. Therefore, this lemma and the concavity suggest that the stability of a core allocation decreases in the wealth of the richest player. Our main result in this section is described in the following theorem. The stochastically stable allocations are core allocations that minimize the wealth of the richest player.

Theorem 5.3. *State* $(s, \mathcal{M})$ *is stochastically stable if and only if* $(s, \mathcal{M}) \in {}_{\min}\Omega^*_\Delta$*.*

Remark 5.4. The implication of Theorem 5.3 is identical to that of Theorem 1 of Agastya (1999) for homogeneous preferences. This is because of the following observation for the strictly increasing and strictly concave utility function $u(\cdot)$:

$$\operatorname*{argmax}_{s \in \mathscr{C}_\Delta} \min_{i \in N} u(s_i) - u(s_i - \Delta) \quad = \quad \operatorname*{argmin}_{s \in \mathscr{C}_\Delta} \max_{i \in N} s_i \quad = \quad \operatorname*{argmax}_{s \in \mathscr{C}_\Delta} \min_{i \in N} \frac{u(s_i) - u(s_i - \Delta)}{u(s_i)}.$$

The LHS is the set of allocations that minimize the marginal utility of the richest player, which is the set of stochastically stable states in our model. The second set is the set of allocations that minimize the wealth of the richest player. The RHS is the set of allocations that minimize the growth rate of marginal utility of the richest player, which is the set of stochastically stable states in Agastya (1999). The three sets coincide for homogeneous preferences. Predictions based on the two models may differ for heterogeneous preferences. We discuss this further in Section 6.

For the "only if" part of the proof, suppose a strict-core allocation $\omega \notin {}_{\min}\Omega^*_\Delta$. Let $T(\omega)$ denote a minimum-cost tree of ω. The lowest cost of escaping from ω is given by $R(\omega) = u(s_{(1)}) - u(s_{(1)} - \Delta)$. Lemma 5.2 implies that a sequence of least-cost escapes can lead the process from ω to another strict-core allocation $\hat{\omega}$, in which the wealth of the richest players is strictly smaller than $s_{(1)}$. Add the sequence to $T(\omega)$ of ω and delete the edges exiting from the states in the sequence. Because $R(\omega) < R(\hat{\omega})$, the cost of the resulting tree is strictly smaller than the cost of $T(\omega)$. Thus, stochastically stable allocations must be in ${}_{\min}\Omega^*_\Delta$.

Next, we provide a corollary and two examples of the theorem. For the egalitarian allocation s^E, the corresponding state must be the unique element of ${}_{\min}\Omega^*_\Delta$ if $s^E \in \mathscr{C}_\Delta$. Corollary 5.5 follows immediately from Theorem 5.3. Example 1 is an application to a cost-sharing problem, and Example 2 illustrates the difference in institutional assumptions between our model and that of Agastya (1999). Note that our selection result may disagree with that of Agastya (1999) for settings with a non-convex characteristic function.

Corollary 5.5. *If $s^E \in \mathscr{C}_\Delta$, then $(s^E, \{N\})$ is uniquely stochastically stable.*

Example 1 (Cost-sharing problem)**.** This is based on Example 4.2 of Moulin (1988). A public utility (a water system) serves four consumers, with the following symmetric cost structure:

cost of serving:	one consumer, alone	40
	two consumers	60
	three consumers	70
	all four consumers	80

The monetary benefits to the consumers from using the facility are

$$b_1 = 41, \qquad b_2 = 24, \qquad b_3 = 22, \qquad b_4 = 13.$$

Consumer i agrees to purchase this facility if she is charged no more than b_i. Given this setting, the characteristic function is given by $v(J) = \max\{\sum_{i \in J} b_i - c(J),\ 0\}$, where $c(J)$ is the corresponding cost structure. Observe that $v(J) = 0$ if a coalition J does not include consumer 1. Then, $v(J)$ is computed as follows for $J \subset N$, with $1 \in J$:

$$\begin{aligned} &v(1) = 1, && v(12) = 5, && v(13) = 3, && v(14) = 0, \\ &v(123) = 17, && v(124) = 8, && v(134) = 6, && v(1234) = 20. \end{aligned}$$

Let $\Delta = 1$. Any coalition J with $1 \notin J$ has no incentive to deviate from an allocation if every member receives at least Δ. The strict core of this game is a relatively large subset of the set of allocations and, thus, provides only loose guidelines for allocations. For example, $(17, 1, 1, 1)$ is a strict-core allocation, as is $(2, 8, 8, 2)$.

This problem has a unique stochastically stable allocation, namely, $(6, 6, 6, 2)$. To see this, observe that $s_1 + s_2 + s_3 \geq v(123) + \Delta = 18$ must hold in any strict-core allocation. Thus, $(6, 6, 6)$ is the allocation for the three players that minimizes the maximum wealth. The cost share corresponding to the allocation is $(35, 18, 16, 11)$.

Example 2 (Example 1 of Agastya (1999))**.** Let $N = \{1, 2, 3\}$ and $u(x) = \sqrt{x}$. The characteristic function is such that $v(\{1, 2\}) = v(\{1, 3\}) = 300$, $v(\{N\}) = 302$, and $v(\cdot) = 0$ for any other coalition. Suppose that $\Delta = 1$, for which $(s, \mathcal{M}) = ((300, 1, 1), \{N\})$ is the unique allocation in the strict core. Assume that the probabilistic rule in Agastya (1999) is such that the only coalitions that are formed are the largest that satisfy claims, and that each forms with equal probability. Our model predicts a strict-core allocation that differs from that of the model of Agastya (1999).

Consider that every player claims 150 (i.e., a claim profile $(150, 150, 150)$). The claim profile is not feasible in our model, whereas players can submit such a profile in Agastya (1999). The probabilistic rule accommodates their claims and chooses $\{1,2\}$ and $\{1,3\}$ with equal likelihood in each period. The players evaluate the expected payoffs of the claim profile and have no incentive to deviate. For example, if player 2 deviates by claiming 151, no allocation will be feasible except $\{1,3\}$. Then, coalition $\{1,3\}$ forms with probability one, and player 2 obtains zero. The cost of escaping from $(150, 150, 150)$ is positive and proportional to $u(150) - u(150 - \Delta)$. Furthermore, the cost is greater than the cost of escaping from the core allocation, which is proportional to $u(300) - u(300 - \Delta)$. Thus, the core allocation is not stochastically stable in Agastya (1999).

5.2 Core Allocations with Non-Grand Coalitions

Thus far, we have restricted our attention to settings with a nonempty strict core. In this section, we consider settings with a nonempty strict dispersed core. That is, we switch the solution concept from Definition 3.2 to 3.3.[10] Recall that Ω^*_Δ is the set of strict dispersed-core allocations. In addition, let $V_{\max} = \max_{\mathcal{M} \in \text{part}(N)} \sum_{J \in \mathcal{M}} v(J)$, and $\mathcal{M}_{\max} = \{\mathcal{M} \in \text{part}(N) : \sum_{J \in \mathcal{M}} v(J) = V_{\max}\}$. Here, $V_{\max}$ is the maximum attainable sum of the surpluses generated among all profiles of teams $\mathcal{M} \in \text{part}(N)$. $\mathcal{M}_{\max}$ is the set of profiles of teams that can attain $V_{\max}$.

The next lemma identifies the properties of strict dispersed-core allocations. The lemma shows that if there is some strict-core allocation with $\mathcal{M}^* \neq \{N\}$, then $\mathcal{M}^*$ is unique and maximizes the sum of the generated surpluses (i.e., $\mathcal{M}_{\max} = \{\mathcal{M}^*\}$).

Lemma 5.6. *Suppose that $\mathscr{C}_\Delta$ is nonempty. Then, we have that*

$$\text{(i)} \quad \mathcal{M}^* \in \mathcal{M}_{\max} \quad \forall (s, \mathcal{M}^*) \in \Omega^*_\Delta,$$
$$\text{(ii)} \quad |\mathcal{M}_{\max}| = 1.$$

We briefly describe the proof. For (i), if $\mathcal{M}^* \notin \mathcal{M}_{\max}$, then there exists some coalition $J \in \mathcal{M} \in \mathcal{M}_{\max}$ such that $v(J) > \sum_{i \in J} s_i$.[11] Then, there is some allocation for coalition J under which its members are weakly better off. This contradicts that $(s, \mathcal{M}^*)$ is in the strict dispersed core. For (ii), if $\mathcal{M}', \mathcal{M}'' \in \mathcal{M}_{\max}$, then for any allocation $(s', \mathcal{M}')$, there exists $J'' \in \mathcal{M}''$, such that $v(J'') \geq \sum_{i \in J''} s'_i$.[12] This implies that there is some allocation under which the members of J'' are weakly better off. Therefore, no allocation can be in the strict dispersed core, which contradicts that the core is nonempty.

The next lemma guarantees that the unperturbed dynamic will converge to the set of strict dispersed-core allocations, Ω^*_Δ. The proof is more involved than that of Lemma 5.1. For the settings described in Section 5.1, there is a unique optimal team, called the grand coalition, that can

[10] A core allocation with a grand coalition is a dispersed-core allocation. All results in this section can be applied to strict-core allocations with a grand coalition.

[11] If such a team does not exist, then $\sum_{i \in J} s_i \geq v(J)$, for all $J \in \mathcal{M}$. This contradicts that $\mathcal{M}^* \notin \mathcal{M}_{\max}$ and $\mathcal{M} \in \mathcal{M}_{\max}$.

[12] Otherwise, $v(J'') < \sum_{i \in J''} s'_i$ for all $J'' \in \mathcal{M}''$, which contradicts that $\mathcal{M}'' \in \mathcal{M}_{\max}$.

weakly block any allocation with non-grand coalitions. For the settings described in this section, there exists a optimal profile of teams, that is, $\mathcal{M}^* \in \mathcal{M}_{\max}$. These teams cannot form at the same time because at most one team is formed in each period. Moreover, not every team $J \in \mathcal{M}^*$ can weakly block the current allocation. For some $J \in \mathcal{M}^*$ to form, it will need to take on at least some players from existing teams, and may or may not be able to compensate them better than their current allocation does; that is, $v(J)$ does not necessarily exceed the sum of the current shares of all $i \in J$.[13]

Nevertheless, we find that, for any state, there is a particular order of teams in the profile $\mathcal{M}^*$. In the dynamics, there is a non-vanishing probability that each team of $\mathcal{M}^*$ will form sequentially according to this order. Using this observation, Lemma 5.7 shows that the process will eventually enter the strict dispersed core.

Lemma 5.7. *Starting from $(s, \mathcal{M}) \in \Omega$, with $s \notin \mathscr{C}_\Delta$, the unperturbed dynamic induced by P^0 reaches some $(s^*, \mathcal{M}^*) \in \Omega^*_\Delta$ with positive probability.*

Similarly to Lemma 5.1, Lemma 5.7 guarantees that the unperturbed process of our decentralized coalition formation will result in some allocation in the strict dispersed core. Recall that

$$_{\min}\Omega^*_\Delta = \left\{ (s, \mathcal{M}) \in \Omega^*_\Delta : s_{(1)} = s_{\min} \right\},$$

which is the set of strict-core allocations that minimizes the claims of the richest players. Based on this lemma and the above definition, we can now state our second main theorem. All stochastically stable allocations are in the strict dispersed core and, moreover, are in the set $_{\min}\Omega^*_\Delta$. Note that this characterization is slightly weaker than that of Theorem 5.3. For settings with the strict core, we have shown that all states in $_{\min}\Omega^*_\Delta$ are stochastically stable. For the strict dispersed core, stochastically stable states must be in $_{\min}\Omega^*_\Delta$, but the converse is not necessarily true.

Theorem 5.8. $\lim_{\eta \to 0} \pi^\eta(_{\min}\Omega^*_\Delta) = 1$.[14]

The next two examples illustrate the implications of Theorem 5.8. For settings with a dispersed core, not all allocations in $_{\min}\Omega^*_\Delta$ are necessarily stochastically stable. The first example highlights the difference between Theorems 5.3 and 5.8. The second example shows that there exist settings with a dispersed core in which all allocations in $_{\min}\Omega^*_\Delta$ are stochastically stable.

Example 3. Suppose that $N = \{1, 2, 3, 4\}$ and $\Delta = 1$. Let $J_{ij} = \{i, j\}$, which denotes the team of two players, i and j. The characteristic function is given by

$$v(J_{12}) = 4, \qquad v(J_{34}) = 10, \qquad v(N) = 12,$$

and $v(J) = 0$ for any coalition that is not listed above. Because $v(J_{12}) + v(J_{34}) > v(N)$, players 1 and 2 and players 3 and 4 form teams in any strict dispersed-core allocation. More precisely,

[13] The grand coalition includes all members of all teams. Thus, if the strict core exists with a grand coalition, $v(N)$ exceeds the sum of their current shares.

[14] Let $\pi^\eta(X)$ denote the sum of the probabilities of the states of X (i.e., $\sum_{\omega \in X} \pi^\eta(\omega)$).

the strict dispersed core includes all allocations in which J_{12} and J_{34} are formed and every player receives at least one unit of surplus (i.e., $s_1 + s_2 = 4$, $s_3 + s_4 = 10$, $s_i \geq \Delta$, for all $i \in N$).

Observe that

$$_{\min}\Omega^*_\Delta = \{(s, \mathcal{M}^*) : s \in \{(1,3,5,5), (2,2,5,5), (3,1,5,5)\}\}, \;\; \mathcal{M}^* = \{J_{12}, J_{34}\}.$$

Our theorem tells us that $\lim_{\eta \to 0} \pi^\eta({}_{\min}\Omega^*_\Delta) = 1$.

For the setting in Example 3, a careful inspection shows that the allocation $(2,2,5,5)$ is uniquely stochastically stable. It is the most stable because each coalition distributes its surplus equally. Intuitively, the formation of teams J_{12} and J_{34} should be examined separately because there is no incentive for players 1 and 2 to form a team with 3 or 4. The allocation for team J_{12} that minimizes the wealth of the richest player, $(2,2)$, is more stable than other allocations, and is also uniquely stochastically stable. It seems that considering equity works in all coalitions for settings with a dispersed core.

However, it does not always work for all coalitions. Consider the next example, where we slightly modify the setting by adding a fifth player.

Example 4. Suppose that $N = \{1,2,3,4,5\}$, $\Delta = 1$, $J_{ij} = \{i,j\}$, and the characteristic function is given by

$$v(J_{12}) = 4, \qquad v(J_{34}) = 10, \qquad v(N) = 12,$$
$$v(\{1,2,5\}) = 7, \qquad v(\{5\}) = 4.$$

Here, $v(J) = 0$ for any coalition that is not listed above. It is easy to see that

$$_{\min}\Omega^*_\Delta = \{(s, \mathcal{M}^*) : s \in \{(1,3,5,5,4), (2,2,5,5,4), (3,1,5,5,4)\}\},$$

where $\mathcal{M}^* = \{J_{12}, J_{34}, \{5\}\}$. For this modified setting, all allocations in ${}_{\min}\Omega^*_\Delta$ are stochastically stable. For any pair of $\omega, \omega' \in {}_{\min}\Omega^*_\Delta$, the cost of escaping from ω to ω' is $u(4) - u(3)$. For $\omega = (s, \mathcal{M}^*)$, the cost of players 1,2, and 5 forming a team and accepting the allocation $\hat{s}_{1,2,5} = (s_1, s_2, 3)$ is $u(4) - u(3)$, because the utility of player 5 decreases by that amount. Then, there is non-vanishing probability that the process reaches $\omega' = (s', \mathcal{M}^*)$. That is, player 5 leaves team $\{1,2,5\}$ and forms a singleton team. Then, players 1 and 2 form a team with allocation $\hat{\hat{s}}_{1,2} = (s'_1, s'_2)$, and the process reaches ω'. Because the transition cost between any two states in ${}_{\min}\Omega^*_\Delta$ is the same, all states in the set are stochastically stable. In this case, players 1 and 2 will end up in one of $(1,3)$, $(2,2)$ and $(3,1)$.

To conclude this section, we relax the assumption that $\mathscr{C}_\Delta \neq \varnothing$ and consider predictions for settings with an empty core. We first define individually rational allocations and unbeatable teams.

Definition 5.9. (1) *An allocation* $(s, \mathcal{M}) \in \Omega$ *is* individually rational *if* $s_i \geq v(\{i\})$ *for all* $i \in N$.

Let Ω_{IR} be the set of individually rational allocations, and $S^J_{IR} \subseteq S^J$ be the set of individually rational claim profiles for team J (i.e., $s_i \geq v(\{i\})$ for all $i \in J$ if $s_J \in S^J_{IR}$).

(2) *Team J is* unbeatable under claim profile s_J *if*

$$\sum_{i \in J} s_i = v(J), \text{ and} \qquad \sum_{i \in J \cap \hat{J}} s_i > \sum_{i \in J \cap \hat{J}} \hat{s}_i \quad \forall \hat{s}_J \in S^{\hat{J}}_{IR} \text{ and } \hat{J} \neq J.$$

Here, s_J is an unbeatable claim profile for J *if J is unbeatable under s_J.*

Individually rational allocations are those in which each player i receives at least their reservation surplus, $v(\{i\})$. Unbeatable teams have claim profiles that cannot be weakly blocked by any coalition. This implies that if some players in an unbeatable team with an unbeatable claim profile deviate, then at least one of the deviating players must be worse off.

Let $\mathcal{R}_{ub}$ be a set of teams, each of which is unbeatable under some claim profile. The definition implies that any two unbeatable teams are disjoint, or $J \cap J' = \varnothing$, for all $J, J' \in \mathcal{R}_{ub}$. Let $\Omega_{ub} = \{(s, \mathcal{M}) \in \Omega_{IR} : \mathcal{R}_{ub} \subset \mathcal{M}\}$ be the set of states under which all unbeatable teams are formed. Furthermore, let $S^J_{ub} = \{s_J \in S^J : J \text{ is unbeatable under } s_J.\}$, which is the set of claim profiles under which J is unbeatable. Define

$$s^{ub}_{\min} = \max_{J \in \mathcal{R}_{ub}} \underbrace{\min_{s^J \in S^J_{ub}} \max_{i \in J} s^J_i}_{\substack{\text{Minimum wealth} \\ \text{of the richest in } J}}, \qquad {}_{\min}\Omega_{ub} = \left\{(s, \mathcal{M}) \in \Omega_{ub} : \max_{J \in \mathcal{R}_{ub}} \max_{i \in J} s_i \leq s^{ub}_{\min}\right\}.$$

In other words, $s^{ub}_{\min}$ is the minimum claim of the richest player in all unbeatable teams for all unbeatable claim profiles. Thus, ${}_{\min}\Omega_{ub}$ is the set of states where all unbeatable teams are formed and the wealth of the richest in the unbeatable teams is at most $s^{ub}_{\min}$. Corollary 5.10 characterizes stochastically stable allocations for settings with an empty core.

Corollary 5.10. (1) $\lim_{\eta \to 0} \pi^\eta(\Omega_{IR}) = 1$.

(2) *If $\mathcal{R}_{ub} \neq \varnothing$, then* $\lim_{\eta \to 0} \pi^\eta({}_{\min}\Omega_{ub}) = 1$.

The first claim is straightforward. Any state that is not individually rational is not stochastically stable. For the second claim, note that the definition of an unbeatable team J implies that if the state is not in Ω_{ub}, then there is some claim profile that makes every member of J weakly better off. Therefore, J can be formed with a zero transition cost, and the process will place probability one on all states where unbeatable teams are formed. Similarly to our main result, Corollary 5.10 provides an upper bound $s^{ub}_{\min}$ on the wealth of the richest player in the unbeatable teams.

6 Heterogeneity in Utility Functions with a Strict Core

Here, we relax the assumption of homogeneous preferences and examine stochastically stable allocations with heterogeneous utility functions for settings with a nonempty strict core. Let player i's utility function be $u_i : \mathbb{R} \to \mathbb{R}$, which is concave and strictly increasing. These functions may vary between players; that is, it might be that $u_i(x) \neq u_j(x)$, for $x \in \mathbb{R}$ and $i \neq j$. All other aspects of the setup are the same as that in Section 3. We omit most of the proofs in this section because the analysis does not differ significantly from those in Sections 4 and 5.

The discussion on the stochastic potential in Section 4 still applies to the settings here; that is, the stochastically stable states are those minimizing the stochastic potential. However, the characterization differs from that in Section 5.1 because transition costs depend on the players' utility functions. Formally, we have the following lemma, which is analogous to Lemma 5.2. The minimum cost of escaping from an allocation is approximately given by the minimum marginal utility of the players.

Lemma 6.1. *For $s \in \mathscr{C}_\Delta$, s has three properties:*

(i) *The lowest cost of escaping from s is given by*

$$R(s, \{N\}) = \min_{i \in N} u_i(s_i) - u_i(s_i - \Delta). \tag{9}$$

(ii) *If allocation s satisfies Condition (7) and $s \notin \operatorname{argmin}_{s \in \mathscr{C}_\Delta} \min_{i \in N} u_i(s_i) - u_i(s_i - \Delta)$, then the least-cost escape from s leads the process to $s' \in \mathscr{C}_\Delta$, where s' has one of the following two properties:*

$$\begin{aligned}
& \min_{i \in N} u_i(s'_i) - u_i(s'_i - \Delta) < \min_{i \in N} u_i(s_i) - u_i(s_i - \Delta), \\
\textit{or} \qquad & |\operatorname*{argmin}_{i \in N} u_i(s'_i) - u_i(s'_i - \Delta)| = |\operatorname*{argmin}_{i \in N} u_i(s_i) - u_i(s_i - \Delta)| - 1, \\
& \textit{where} \ \ \min_{i \in N} u_i(s'_i) - u_i(s'_i - \Delta) = \min_{i \in N} u_i(s_i) - u_i(s_i - \Delta).
\end{aligned}$$

(iii) *If allocation s violates Condition (7), then the least-cost escape from s leads the process to any $s' \in \mathscr{C}_\Delta$.*

Recall that Ω^*_Δ is the set of strict-core allocations. Define

$${}_{\min}\Omega^{**}_\Delta = \left\{ (s, \{N\}) \in \Omega^*_\Delta : s \in \operatorname*{argmax}_{s \in \mathscr{C}_\Delta} \min_{i \in N} u_i(s_i) - u_i(s_i - \Delta) \right\}.$$

We say that an allocation in the strict core, $(s^{sw}, \{N\}) \in \Omega^*_\Delta$, *maximizes the social welfare* if

$$\sum_{i \in N} u_i(s^{sw}_i) = \max_{s \in \mathscr{C}_\Delta} \sum_{i \in N} u_i(s_i).$$

This is an allocation that maximizes the sum of the players' utilities among all strict-core allocations. The next lemma shows that such allocations must be in ${}_{\min}\Omega^{**}_\Delta$ for bilateral and trilateral bargaining. The proof is given in the Appendix.

Lemma 6.2. *Suppose that $n \leq 3$. If $(s, \{N\})$ maximizes the social welfare, then $(s, \{N\}) \in {}_{\min}\Omega^{**}_\Delta$.*

We have the following result, corresponding to Theorem 5.3. Despite the heterogeneity in utility functions, the expression for stochastically stable states is similar; that is, the minimum marginal utility is approximately maximized. The theorem (along with Lemma 6.2) also states that the utilitarian social welfare is maximized in a stochastically stable allocation for $n \leq 3$. However, this does not necessarily imply that the maximum wealth is minimized. The following example applies the theorem and highlights the difference between this case and the settings of homogeneous utility functions.

Theorem 6.3. $\lim_{\eta \to 0} \pi^{\eta}({}_{\min}\Omega_{\Delta}^{**}) = 1$.

Example 5 (Heterogeneity in the degree of sensitivity)**.** Suppose that $N = \{1, 2, 3\}$, $\Delta = 1$, and that the characteristic function is given by $v(N) = 12$ and $v(J) = 0$, for all $J \neq N$. The strict core is given by the set of the profiles (s_1, s_2, s_3), which satisfy that $\sum_i s_i = 12$, for $s_i > 0$ and all $i \in N$. The egalitarian allocation $(4, 4, 4)$ is in the strict core and, thus, is uniquely stochastically stable if the utility functions are homogeneous.

Suppose that player i's utility function is given by $u_i(x) = \sqrt{\beta_i x}$, where $\beta_i = i$ for $i \in N$. Then, β_i represents the degree of sensitivity to monetary payoffs. Players with a higher index are more sensitive.

Observe that ${}_{\min}\Omega_{\Delta}^{**} = \{((2, 4, 7), \{N\})\}$, because

$$\min\{\sqrt{2} - 1, \sqrt{8} - \sqrt{6}, \sqrt{21} - \sqrt{18}\} > \max\{\sqrt{3} - \sqrt{2}, \sqrt{10} - \sqrt{8}, \sqrt{24} - \sqrt{21}\}.$$

The LHS gives the cost of escaping from (2,4,7), and the RHS gives the upper bound of the cost of escaping for other strict-core allocations. Theorem 6.3 implies that the claim profile (2,4,7) is uniquely stochastically stable and, thus, maximizes the social welfare among allocations in Ω_{Δ}^{*} (Lemma 6.2). Those who are more sensitive to monetary payoffs (i.e., players with higher β_i) obtain a larger share.

To conclude this section, we briefly compare our results with those of Young (1993b) and Agastya (1999). Suppose a two-person bargaining game, given by $N = \{1, 2\}$, $v(\{1\}) = v(\{2\}) = 0$, and $v(N) = V > 0$. Young (1993b) shows that if the players have the same size memory, then the stochastically stable allocation maximizes the Nash product, $u_1(s_1)u_2(V - s_1)$. The same prediction is obtained by Agastya (1999) for two-person bargaining games. In their models, the escape cost for player i is approximated by $u_i'(s_i)/u_i(s_i)$ for sufficiently small Δ, where $u_i'(\cdot)$ denotes player i's marginal utility. The minimum escape cost of two players is maximized when the state is such that $u_1'(s_1)/u_1(s_1) = u_2'(V - s_1)/u_2(V - s_1)$ (i.e., the Nash-bargaining solution).

In contrast, the stochastically stable allocation in our model maximizes the utilitarian welfare, $u_1(s_1) + u_2(V - s_1)$. As in Lemma 6.1, the escape cost is approximated by the minimum marginal utility of the players, $\min_{i \in \{1,2\}} u_i'(s_i)$. The escape cost is maximized when the players have the same marginal utility, which is the state that maximizes the sum of their utilities. For homogeneous preferences, the sets of stochastically stable states of the two models coincide. However, the predictions differ when preferences are heterogeneous. For two-person bargaining games, Young

(1993b) and Agastya (1999) support the Nash-bargaining solution, whereas our result supports the utilitarian solution.

7 Concluding remarks

We have characterized stochastically stable allocations in coalitional bargaining games. For games with a nonempty core, allocations that minimize the maximum payoff over all players have been shown to be stochastically stable. Then, we extended this result to games with an empty core, for which the maximum wealth of unbeatable teams is minimized. In future work, we plan to further investigate the dependence of stochastically stable allocations on behavioral models and institutional settings. A comparison of the findings of Agastya (1999), Newton (2012b), and this study suggests that combinations of behavioral models and institutional settings influence the stability of allocations. Recent experimental studies have revealed behavioral biases in people (e.g. Hwang et al. 2018; Lim and Neary 2016; Mäs and Nax 2016). For example, Mäs and Nax (2016) found an increase in errors when subjects had changed their action in the previous period; that is, errors might be dependent not only on the strategy distribution, but also on some other variables. Thus, merging these behavioral biases in a stochastic stability analysis could be fruitful.

A Appendix

Proofs for Section 5

Proof of Lemma 5.1. By definition, it is obvious that any strict-core allocation is an absorbing state in the unperturbed dynamic. Let $(s^*, \{N\})$ denote an arbitrary strict-core allocation. We show that, for any $(s, \mathcal{M}) \in \Omega$ with $s \notin \mathscr{C}_\Delta$, the unperturbed dynamic starting from $(s, \mathcal{M})$ reaches $(s^*, \{N\})$ with a positive probability.

First, suppose that the process is in $(s, \mathcal{M})$, in which allocation s is in the core, but not in the strict core. Then, there exists a coalition $J \subset N$ that weakly blocks s. Let s' be such that

$$s'_i \geq s_i \quad \forall i \in J \quad \text{with} \sum_{i \in J} s'_i \leq v(J), \qquad\qquad s'_i = v(\{i\}) \quad \forall i \notin J.$$

Because J weakly blocks s, such s' is feasible. Let players form team J and accept s'.[15] Let s'' be such that $s''_i = s'_i$ for all $i \in N$. Such s'' is feasible for the grand coalition because the existence of the strict core implies that $v(J) + \sum_{i \notin J} v(\{i\}) \leq v(N)$. Let players form the grand coalition and accept s''. Then, let player $i^* \notin J$, who receives $v(\{i^*\})$ in s'', form a singleton team and accept $s'''_{i^*} = v(\{i^*\})$. Because i^* leaves the grand coalition, it is dissolved and the state becomes $(s''', \cdot)$, with $s'''_i = v(\{i\})$ for all $i \neq i^*$. Let players form the grand coalition again and accept $s^* \in \mathscr{C}_\Delta$.

[15]Here, "let players form J and accept s" means there is a positive probability that coalition J and allocation s are chosen, and that the players of J accept this allocation. In addition, we assume this event is realized in the dynamic.

Second, suppose that the process is in $(s, \mathcal{M})$, in which allocation s is not in the core. If no multi-player team is formed in $(s, \mathcal{M})$, let players form the grand coalition and accept s^*. Next, suppose that at least one multi-player team is in $(s, \mathcal{M})$. Let $\hat{s}$ be such that $\hat{s}_i = s_i$ for all $i \in N$. Owing to the existence of the core, such $\hat{s}$ must be feasible for a grand coalition (i.e., $\sum_i \hat{s}_i \leq v(N)$). Let players form a grand coalition and accept $\hat{s}$. If $\hat{s}$ is a core allocation, then there is positive probability that the process reaches some $s^* \in \mathscr{C}_\Delta$, as shown above. If $\hat{s}$ is not a core allocation, then there exist J and s' such that J blocks $\hat{s}$. Let players form J and accept s'. Following similar reasoning to that in the previous paragraph, we can show that there is a positive probability that the process reaches $(s^*, \mathcal{M})$. □

For the proofs of Lemma 5.2 and Theorem 5.3, we first define the minimum deviation cost from a state. A directed graph $d(\omega_1, \omega_k)$ on Ω is a *path* if $d(\omega_1, \omega_k)$ is a finite, repetition-free sequence of transitions $\{(\omega_1, \omega_2), (\omega_2, \omega_3), \ldots, (\omega_{k-1}, \omega_k)\}$, such that $\omega_i \in \Omega$ for all $i = 1, \ldots, k$. A path $d(\omega_1, \omega_k)$ is *possible* if $R_{\omega_i \omega_{i+1}}$ is nonempty for all $i = 1, \ldots, k-1$. Let $\mathcal{D}(\omega, \omega')$ be the set of all paths with initial point ω and terminal point ω'. Let the cost $W(d(\omega, \omega'))$ be the sum of the transition costs of $d(\omega, \omega')$ (i.e., $W(d(\omega, \omega')) = \sum_{(\omega_i, \omega_{i+1}) \in d(\omega, \omega')} c_{\omega_i, \omega_{i+1}}$).

The *basin of attraction* of state ω, $B(\omega) \subseteq \Omega$, is the set of all states ω' such that there exists a revision path $d \in \mathcal{D}(\omega', \omega)$, with $W(d) = 0$. We define the *lowest cost of escaping* from state ω as

$$R(\omega) = \min_{\omega' \notin B(\omega)} \left\{ W(d) \,\middle|\, d \in \mathcal{D}(\omega, \omega') \right\}, \tag{10}$$

where $R(\omega)$ is the minimum cost for the process to move away from the basin attraction of ω.

Recall that $s_{\min}$ denotes the lowest claim of the richest player among all strict-core allocations, and let $s_{\max}$ be the highest of these:

$$s_{\min} = \min_{s' \in \mathscr{C}_\Delta} s'_{(1)}, \qquad s_{\max} = \max_{s' \in \mathscr{C}_\Delta} s'_{(1)}.$$

Proof of Lemma 5.2. Observe that the RHS of Equation (8) gives the minimum cost of a mistake over all mistakes in allocation s. Because s is a core allocation, some player's share must decrease as a result of a transition from s. Owing to the concavity of $u(\cdot)$, the lowest cost of such a transition is a Δ transfer from the richest player's share to another player's share. We prove that this least-cost mistake is enough for the process to move to another strict-core allocation.

First, suppose that the current allocation s satisfies Condition (7) and that $s \neq s^E$. Recall that $I_\$(s) = \{i \in N : s_i = s_{(1)}\}$ (i.e., the set of the richest players). Let $i_\$ \in I_\$(s)$. Choose h such that $s_h \leq s_{(1)} - 2\Delta$. Lemma 3.4 guarantees that such h exists. We show that a transfer of Δ from $i_\$$ to h results in a new strict-core allocation. Let s' be such that $s'_{i_\$} = s_{(1)} - \Delta$, $s'_h = s_h + \Delta$, and $s'_i = s_i$ for $i \notin \{i_\$, h\}$. The escape cost from s to s' is given by (8). Observe that s' satisfies Inequality (2) and is a strict-core allocation. In addition, s' is either an allocation in which the richest player claims $s_{(1)} - \Delta$ or that in which the next richest players claim $s_{(1)}$. If $s = s^E$, then any transfer will increase some player's share by Δ to $s_{\min} + \Delta$. Then, the claim is immediate. This proves claim (ii).

Next, suppose Condition (7) does not hold for allocation s. Then, there exists at least one richest player $i_\$ \in I_\$(s)$ and one coalition $J \in \mathcal{R}_{i_\$}$, such that $\sum_{i \in J} s_i = v(J) + \Delta$. Consider allocation s', such that $s'_{i_\$} = s_{i_\$} - \Delta$, $s'_h = s_h + \Delta$ for some $h \notin J$, and $s'_i = s_i$ for $i \notin \{i_\$, h\}$. Note that the cost of switching from $(s, \{N\})$ to $(s', \{N\})$ is given by (8) (i.e., $u(s_{(1)}) - u(s_{(1)} - \Delta)$). Suppose that the process starts with $(s, \{N\})$ and that the following events occur sequentially:

(i) Players form the grand coalition and accept s'. The state becomes $(s', \{N\})$.

(ii) Let $s''_J \in S^J$ be such that $s''_j = s'_j$ for all $j \in J$, and $s'' = (s''_J, s''_{-J})$ be such that $s''_i = v(\{i\})$ for $i \notin J$. Let the players of J form a new team and accept s''. Note that the grand coalition is dissolved.

(iii) Let $\hat{s}$ be such that $\hat{s}_i = s''_i$ for all $i \in J$, and $\hat{s}_i = v(\{i\})$ otherwise. The existence of the strict core guarantees that $\hat{s}$ is feasible. Let the players form the grand coalition and accept $\hat{s}$. This moves the process to $(\hat{s}, \{N\})$.

(iv) Let $\tilde{i} \notin J$ and $\tilde{s}_i = v(\{i\})$ for all $i \in N$. Player i forms a singleton team $\{\tilde{i}\}$ and switches from $\hat{s}$ to $\tilde{s}$. Note that the grand coalition is dissolved when player $\tilde{i}$ quits.

(v) Let $s^* \in \mathscr{C}_\Delta$. The players form the grand coalition and accept s^*.

Observe that (ii)–(v) occur with non-vanishing probability; that is, the cost of these transitions is zero. Once the process reaches $(s', \{N\})$, it can reach any strict-core allocation s^* without incurring a cost. Thus, the escape cost is given by (8). This proves claim (iii). We have also proved claim (i) because the cost of escaping is given by (8) for both cases, regardless of whether or not (7) is satisfied. $\square$

Proof of Theorem 5.3. First, we prove the "only if" part by way of contradiction. Suppose that there exists $\omega_0 = (s_0, \{N\}) \in \Omega^*_\Delta \setminus_{\min} \Omega^*_\Delta$ that is stochastically stable. Let $\bar{h} > 0$ be such that $s_{\min} + \bar{h}\Delta = s_{\max}$. Define for $h \in \{0, 1, \ldots, \bar{h}\}$,

$$U_h = \left\{ (s, \{N\}) \in \Omega^*_\Delta \,\middle|\, s_{(1)} = s_{\min} + h\Delta \right\}, \qquad U_{\le h} = \bigcup_{h' \le h} U_{h'}.$$

Then, U_h is the set of strict-core allocations with the richest player claiming $s_{\min} + h\Delta$, and $U_{\le h}$ is the set of strict-core allocations where the richest player claims at most $s_{\min} + h\Delta$.

Let h be such that $\omega_0 \in U_h$. Note that $h \ge 1$ because $\omega_0 \notin_{\min} \Omega^*_\Delta$. Construct a path $\{\omega_0, \ldots, \omega_L\}$ that satisfies the following two conditions: (i) $\omega_i = (s_i, \{N\}) \in U_h$, for all $i \in \{0, \ldots, L-1\}$ and $\omega_L = (s_L, \{N\}) \in U_{\le h-1}$; (ii) $c_{\omega_i, \omega_{i+1}} = R(\omega_i)$ and the number of the richest players in ω_{i+1} is less than that in ω_i, for all $i \in \{0, \ldots, L-2\}$. Lemma 5.2 implies that such a path exists. Add the constructed path to $T(\omega_0)$ and remove the existing edges emanating from $\omega_1, \ldots, \omega_L$. Let $T(\omega_L)$ denote the resulting set of edges, which must be an ω_L-tree. Observe that

$$W(\omega_0) = W(T(\omega_0)) > W(T(\omega_0)) + R(\omega_0) - R(\omega_L) \ge W(T(\omega_L)) \ge W(\omega_L).$$

This contradicts that ω_0 is stochastically stable. Because the choice of ω_0 is arbitrary, this proves the "only if" part of the claim.

Next, we prove the "if" part (i.e., all allocations in ${}_{\min}\Omega^*_\Delta$ are stochastically stable). If ${}_{\min}\Omega^*_\Delta$ has only one state, then the "only if" part implies that the state is uniquely stochastically stable. Suppose that $|{}_{\min}\Omega^*_\Delta| > 1$.[16] The "only if" part implies that there exists some $\omega^1 \in {}_{\min}\Omega^*_\Delta$ that is stochastically stable. Let $T(\omega^1)$ denote a minimum-cost tree of ω^1. Fix $\omega^K \in {}_{\min}\Omega^*_\Delta$, such that $\omega^K \neq \omega_1$.

Consider the following operation, starting with $k = 1$. This constructs an ω^K-tree with a weakly smaller stochastic potential than that of $T(\omega^1)$. In what follows, let $\omega^k = (s^k, \cdot)$.

(i) If s^k violates Condition (7), then Lemma 5.2 (iii) implies that there exists a sequence of transitions $d(\omega^k, \omega^K) = \{(\omega^k, \omega^{k+1}), \ldots, (\omega^{K-1}, \omega^K)\}$, such that $W(d(\omega^k, \omega^K)) = u(s^k_{(1)}) - u(s^k_{(1)} - \Delta)$. Construct a new tree $T(\omega^K)$ by adding edges of $d(\omega^k, \omega^K)$ to $T(\omega^k)$ and removing edges from $T(\omega^k)$ that emanate from $\omega^{k+1}, \ldots, \omega^K$. Stop the operation.

(ii) If s^k satisfies Condition (7), then let $I^k_{2\Delta} = \{i \in N : s^k_{(1)} - s^k_i \geq 2\Delta\}$, where $s^k_{(1)}$ is the claim of the richest player in s^k. Lemma 3.4 implies that $I^k_{2\Delta} \neq \varnothing$. Let j be the richest player in s^k (i.e., $s^k_j = s^k_{(1)}$). Choose $h \in I^k_{2\Delta}$. Let s^{k+1} be such that

$$s^{k+1}_j = s^k_j - \Delta, \qquad s^{k+1}_h = s^k_h + \Delta, \qquad s^{k+1}_i = s^k_i \quad \forall i \notin \{j, h\}.$$

Because s^k satisfies Condition (7), $s^{k+1} \in \mathscr{C}_\Delta$. Note that $\omega^{k+1} = (s^{k+1}, \{N\}) \in {}_{\min}\Omega^*_\Delta$. Construct a new tree $T(\omega^{k+1})$ by adding edge (ω^k, ω^{k+1}) to $T(\omega^k)$ and removing the edge from $T(\omega^k)$ that emanates from ω^{k+1}. The resulting set $T(\omega^{k+1})$ must be an ω^{k+1}-tree.

Stop if $s^{k+1} = s^K$. Otherwise, increment k by 1 (i.e., $k = k + 1$), and repeat the above operation.

Observe that the sum of the wealth differences between the richest player and the players in $I^k_{2\Delta}$, $\sum_{i \in I^k_{2\Delta}} s^k_{(1)} - s^k_i$, is strictly decreasing over k in operation (ii). The sum does not reach zero. If it is zero, then the allocation must be egalitarian, but the egalitarian allocation is not in the core for $|{}_{\min}\Omega^*_\Delta| > 1$. Thus, the process following the operation will reach some k such that either s^k violates Condition (7) or $s^{k+1} = s^K$ within a finite number of steps. When the operation stops, the set of edges $T(\omega^K)$ must be an ω^K-tree. Observe that

$$\begin{aligned} W(\omega^K) \leq W(T(\omega^K)) &\leq W(T(\omega^1)) + \sum_{k=1}^{K-1} R(\omega^k) - \sum_{k=2}^{K} R(\omega^k) \\ &= W(T(\omega^1)) = W(\omega^1). \end{aligned}$$

[16] The egalitarian allocation must not be in $\mathscr{C}_\Delta$ if $|{}_{\min}\Omega^*_\Delta| > 1$. If it is, then it must be the unique element of ${}_{\min}\Omega^*_\Delta$.

This implies that ω^K must be stochastically stable. Because the choice of ω^K is arbitrary, any strict-core allocation in ${}_{\min}\Omega^*_\Delta$ is stochastically stable. $\square$

Proof of Lemma 5.6. Claim (i):
We prove this lemma by way of contradiction. Suppose that $\omega = (s, \mathcal{M}^*) \in \Omega^*_\Delta$, such that $\sum_{J \in \mathcal{M}^*} v(J) < V_{\max}$ (i.e., $\mathcal{M}^* \notin \mathcal{M}_{\max}$). Let $\hat{\mathcal{M}}$ be such that $\sum_{\hat{J} \in \hat{\mathcal{M}}} v(\hat{J}) = V_{\max}$. Observe that

$$\sum_{\hat{J} \in \hat{\mathcal{M}}} v(\hat{J}) > \sum_{J \in \mathcal{M}^*} \sum_{i \in J} s_i = \sum_{\hat{J} \in \hat{\mathcal{M}}} \sum_{i \in \hat{J}} s_i.$$

Then, there exists some $\hat{J} \in \hat{\mathcal{M}}$ such that $v(\hat{J}) > \sum_{i \in \hat{J}} s_i$. This contradicts that s is a strict-core allocation.

Claim (ii):
The proof is by way of contradiction. Suppose that $|\mathcal{M}_{\max}| \geq 2$. Choose $(s', \mathcal{M}') \in \Omega^*_\Delta$. The proof of Claim (i) implies that $\mathcal{M}' \in \mathcal{M}_{\max}$. Choose $\mathcal{M} \in \mathcal{M}_{\max} \setminus \{\mathcal{M}'\}$ and a claim profile s, such that $\sum_{i \in J} s_i = v(J)$ for all $J \in \mathcal{M}$. Observe that

$$\begin{aligned}
& \sum_{J \in \mathcal{M}} \sum_{i \in J} s_i = \sum_{J' \in \mathcal{M}'} \sum_{i \in J'} s'_i \\
\Leftrightarrow & \sum_{J \in \mathcal{M} \cap \mathcal{M}'} \sum_{i \in J} s_i + \sum_{J \in \mathcal{M} \setminus \mathcal{M}'} \sum_{i \in J} s_i = \sum_{J \in \mathcal{M}' \cap \mathcal{M}} \sum_{i \in J} s'_i + \sum_{J' \in \mathcal{M}' \setminus \mathcal{M}} \sum_{i \in J'} s'_i \\
\Leftrightarrow & \sum_{J \in \mathcal{M} \setminus \mathcal{M}'} \sum_{i \in J} s_i = \sum_{J' \in \mathcal{M}' \setminus \mathcal{M}} \sum_{i \in J'} s'_i.
\end{aligned} \tag{11}$$

There must exist $J \in \mathcal{M} \setminus \mathcal{M}'$, such that $v(J) \geq \sum_{i \in J} s'_i$. Otherwise, $v(J) < \sum_{i \in J} s'_i$ for all $J \in \mathcal{M} \setminus \mathcal{M}'$, which contradicts the last equality in Equation (11). Then, J weakly blocks s', which contradicts that s' is a strict-core allocation. $\square$

Proof of Lemma 5.7. We group non-equilibrium states, or $\Omega \setminus \Omega^*_\Delta$, into four cases. For Case I, we show that the process will reach some strict-core allocation. For Cases II and III, we show that the process eventually falls into Case I. For Case IV, the process will reach some state of the other cases.
Case I:
Suppose that the current state is $(s, \mathcal{M}^*)$, which satisfies the following conditions:

$$\begin{array}{ll}
\text{(i)} \;\; s \notin \mathscr{C}_\Delta, & \text{(ii)} \;\; \sum_{i \in J} s_i = v(J) \;\; \forall J \in \mathcal{M}^*, \\
\text{(iii)} \;\; \mathcal{M}^* \in \mathcal{M}_{\max}, & \text{(iv)} \;\; s_i \geq v(\{i\}) \;\; \forall i \in N.
\end{array} \tag{12}$$

Let $(s^*, \mathcal{M}^*) \in \Omega^*_\Delta$. Because $s \notin \mathscr{C}_\Delta$, there exists J, such that $v(J) \geq \sum_{i \in J} s_i$. Let J form a coalition with s' in which $s'_i \geq s_i \geq v(\{i\})$ for all $i \in J$. Let $\{M_1, \ldots, M_k\} \subseteq \mathcal{M}^*$ be a set of teams that are dissolved as a result of J being formed. Then, there must exist some $M_j \in \{M_1, \ldots, M_k\}$, such that $\sum_{i \in M_j \cap J} s_i^* > \sum_{i \in M_j \cap J} s_i'$.[17] Note that $M_j \setminus J \neq \varnothing$.[18] Let M_j form a coalition with s'' in which $s_i'' = v(\{i\})$ for $i \in M_j \setminus J$, and $s_i'' = s_i'$ for $i \in M_j \cap J$.[19] Then, let $i \in M_j \setminus J$ form a singleton team. Because $s_i'' = v(\{i\})$ for $i \in M_j \setminus J$, the transition cost is zero. The resulting state is such that all players of $\{M_1, \ldots, M_k\}$ form singleton teams. Let M_x form a coalition with $s^*_{M_x}$ for all $M_x \in \{M_1, \ldots, M_k\}$ in the subsequent periods. The resulting state must be either some $(s^*, \mathcal{M}^*) \in \Omega^*_\Delta$ or $(s, \mathcal{M}^*)$, which satisfies condition (12). Note that all dissolved teams in the above operations form again and that there are more teams whose claims are consistent with s^* after the operations. Repeat Case I until the process reaches some $(s^*, \mathcal{M}^*) \in \Omega^*_\Delta$.

Case II:
Suppose that the current state is $(s, \mathcal{M}^*)$, which satisfies (i),(iii), and (iv) and violates (ii) of Equation (12); that is, there exists $J^* \in \mathcal{M}^*$, such that $\sum_{i \in J^*} s_i < v(J^*)$. Then, let J^* form a coalition with s' in which $s_i' \geq s_i$, for all $i \in J^*$ and $\sum_{i \in J^*} s_i = v(J^*)$; that is, J^* redistributes its surplus in an efficient way.

Repeat the above operation until there is no team J such that $\sum_{i \in J} s_i < v(J)$. The resulting state will either be some $(s^*, \mathcal{M}^*) \in \Omega^*_\Delta$ or be $(s, \mathcal{M}^*)$, satisfying the conditions of (12) in Case I.

Case III:
Suppose that the current state is $(s, \mathcal{M}) \in \Omega$, which satisfies (iv) and violates (iii) of Equation (12) (i.e., $\sum_{J \in \mathcal{M}} v(J) < V_{\max}$). Let $(s^*, \mathcal{M}^*) \in \Omega^*_\Delta$. Observe that

$$\sum_{J^* \in \mathcal{M}^*} v(J^*) > \sum_{J \in \mathcal{M}} \sum_{i \in J} s_i = \sum_{J^* \in \mathcal{M}^*} \sum_{i \in J^*} s_i.$$

Then, there exists J^*, such that $v(J^*) > \sum_{i \in J^*} s_i$. Choose some s'_{J^*}, such that $\sum_{i \in J^*} s_i' = v(J^*)$ and $s_i' \geq s_i$ for all $i \in J^*$.[20] Let J^* form a coalition with s'_{J^*}. Let $\mathcal{M}'$ denote the resulting set of teams. Observe that if $\mathcal{M}' \neq \mathcal{M}^*$, then the resulting state must fall into Case III. Any member i of J^* will earn $s_i' \geq s_i$. Any member i of a dissolved team will earn $v(\{i\})$.

Repeat the above operation until the state becomes some $(s, \mathcal{M}^*)$. The resulting state will either be some $(s^*, \mathcal{M}^*) \in \Omega^*_\Delta$ or be $(s, \mathcal{M}^*)$, which satisfies the conditions of (12) in Case I.

Case IV:
Suppose that the current state is $(s, \mathcal{M})$, which violates (iv) (i.e, $s_i < v(\{i\})$ for some $i \in N$). For each i, such that $s_i < v(\{i\})$, let i form a singleton team and earn $v(\{i\})$. The resulting state will

[17] If such M_j does not exist, then J weakly blocks s^*, which contradicts that s^* is a strict-core allocation.

[18] If $M_j \setminus J = \varnothing$, then $M_j \subseteq J$, which implies that $(M_j \cap J) \in \mathcal{M}^*$. From condition (ii), we have that $\sum_{i \in M_j \cap J} s_i = v(M_j \cap J) = \sum_{i \in M_j \cap J} s_i^* > \sum_{i \in M_j \cap J} s_i'$, which implies that some member of $(M_j \cap J)$ will be strictly worse off by deviating from s. This contradicts that J weakly blocks s.

[19] Every player i of $M_j \setminus J$ forms a singleton team and earns his/her reservation surplus, $v(\{i\})$. They accept s'' with positive probability. In addition, note that s'' is feasible because M_j is chosen such that $\sum_{i \in M_j \cap J} s_i^* > \sum_{i \in M_j \cap J} s_i'$.

[20] s_i' is not necessarily equal to s_i^*; that is, s_i' may or may not be part of a strict-core allocation.

fall either in Case I or in Case III. □

Proof of Theorem 5.8. We prove that the first two claims of Lemma 5.2 still hold for $(s, \mathcal{M}) \in \Omega^*_\Delta \setminus {}_{\min}\Omega^*_\Delta$. Then, observing that $R(s, \{\mathcal{M}\}) = u(s_{\min}) - u(s_{\min} - \Delta)$ for $(s, \mathcal{M}) \in {}_{\min}\Omega^*_\Delta$, the subsequent proof is the same as the "only if" part of the proof of Theorem 5.3. Thus, we omit the subsequent proof.

Suppose that the current state is $(s, \mathcal{M}^*) \in \Omega^*_\Delta \setminus {}_{\min}\Omega^*_\Delta$. Recall that $I_\$(s) \in \{i : s_i = s_{(1)}\}$. Let $J_\$ \in \mathcal{M}^*$ denote a team that includes $i_\$ \in I_\$(s)$. Choose $i_\$ \in I_\$(s)$ and $h \in J_\$$, such that $s_h \leq s_{i_\$} - 2\Delta$. Such $i_\$$ and h exist. Otherwise, any transfer will result in at least one player receiving at least $s_{(1)}$. Then, $s_{(1)} = s_{\min}$ and $(s, \mathcal{M}^*) \in {}_{\min}\Omega^*_\Delta$ must hold, which is a contradiction. Let s' be such that $s'_{i_\$} = s_{(1)} - \Delta$, $s'_h = s_h + \Delta$ for $h \in J_\$$, and $s'_i = s_i$ otherwise. Observe that s' satisfies Inequality (2), and that the escape cost from $(s, \mathcal{M}^*)$ to $(s', \mathcal{M}^*)$ is given similarly to that of Equation (8). That is, we show that $R(s, \{\mathcal{M}^*\}) = u(s_{(1)}) - u(s_{(1)} - \Delta)$. The resulting allocation s' is either one in which the richest player claims $s_{(1)} - \Delta$ or one in which the next richest players claim $s_{(1)}$. This proves the first two claims of Lemma 5.2 for $(s, \mathcal{M}) \in \Omega^*_\Delta \setminus {}_{\min}\Omega^*_\Delta$. □

Proofs for Section 6

Proof of Lemma 6.2. Let $(s^{sw}, \{N\}) \in \Omega^*_\Delta$ maximize the social welfare. Assume that $(s^{sw}, \{N\}) \notin {}_{\min}\Omega^{**}_\Delta$. Let $(s^*, \{N\}) \in {}_{\min}\Omega^{**}_\Delta$. Furthermore, let

$$x \in \operatorname*{argmin}_{i \in N} u_i(s^{sw}_i) - u_i(s^{sw}_i - \Delta), \qquad y \in \operatorname*{argmin}_{i \in N} u_i(s^*_i) - u_i(s^*_i - \Delta).$$

Here, x is a player with the lowest marginal utility in s^{sw}, and y is such a player in s^*. Note that $s^{sw}_x > s^*_x$, which is implied by

$$u_x(s^{sw}_x) - u_x(s^{sw}_x - \Delta) < u_y(s^*_y) - u_y(s^*_y - \Delta) \leq u_i(s^*_i) - u_i(s^*_i - \Delta) \quad \forall i \in N.$$

Note too that $s^*_x \geq v(\{x\}) + \Delta$ implies that $s^{sw}_x \geq v(\{x\}) + 2\Delta$.

Suppose that $\sum_{i \in J} s^{sw}_i \geq v(J) + 2\Delta$, for all $J \subset N$ with $x \in J$. Because $s^{sw}_x > s^*_x$, there exists some $j \neq x$, such that $s^{sw}_j < s^*_j$. Observe that

$$u_x(s^{sw}_x) - u_x(s^{sw}_x - \Delta) < u_j(s^*_j) - u_j(s^*_j - \Delta) \leq u_j(s^{sw}_j + \Delta) - u_j(s^{sw}_j).$$

A transfer of Δ from x to j in s^{sw} results in an allocation that is in the strict core and increases the social welfare. This contradicts that s^{sw} maximizes the social welfare. This completes the proof for $n = 2$. This is because $J = \{x\}$ is the only subset for $n = 2$ that satisfies $J \subset N$, with $x \in J$, and $s^{sw}_x \geq v(\{x\}) + 2\Delta$.

Suppose that $\sum_{i \in J^{sw}} s^{sw}_i = v(J^{sw}) + \Delta$ for some $J^{sw} \subset N$, with $x \in J^{sw}$. Note that $|J^{sw}| = 2$. Note too that such J^{sw} must be unique. If J^{sw} is not unique, any transfer from x to another player will result in an allocation that is not in the strict core. This means that s^*, where $s_x^* < s_x^{sw}$, cannot be in the strict core, which is a contradiction.

Now, observe that $u_x(s_x^{sw}) - u_x(s_x^{sw} - \Delta) \geq u_i(s_i^{sw} + \Delta) - u_i(s_i^{sw})$ for all $i \in J^{sw} \setminus \{x\}$. Otherwise, a transfer from x to some $i \in J^{sw}$ should increase the social welfare. Because s^* is a strict-core allocation, $\sum_{i \in J^{sw}} s_i^* \geq v(J^{sw}) + \Delta$. Then, $s_x^{sw} > s_x^*$ implies that there exists $z \in J^{sw}$, who must be better off in s^* than s^{sw} (i.e., $s_z^* > s_z^{sw}$). However, this implies that

$$u_z(s_z^*) - u_z(s_z^* - \Delta) \leq u_z(s_z^{sw} + \Delta) - u_z(s_z^{sw}) \leq u_x(s_x^{sw}) - u_x(s_x^{sw} - \Delta) < u_y(s_y^*) - u_y(s_y^* - \Delta),$$

which contradicts that player y has the lowest marginal utility in s^*. □

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