Manuscript submitted to
AIMS' Journals
Volume **13**, Number **5**, XX **2020**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

THE HYPOELLIPTIC ROBIN PROBLEM FOR QUASILINEAR ELLIPTIC EQUATIONS

Dedicated to Professor Angelo Favini on the occasion of his 70th birthday

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(Communicated by the associate editor name)

Abstract. This paper is devoted to the study of a *hypoelliptic* Robin boundary value problem for quasilinear, second-order elliptic differential equations depending nonlinearly on the gradient. More precisely, we prove an existence and uniqueness theorem for the quasilinear hypoelliptic Robin problem in the framework of Hölder spaces under the *quadratic gradient growth* condition on the nonlinear term. The proof is based on the comparison principle for quasilinear problems and the Leray–Schauder fixed point theorem. Our result extends earlier theorems due to Nagumo, Akô and Schmitt to the hypoelliptic Robin case which includes as particular cases the Dirichlet, Neumann and regular Robin problems.

1. Introduction and Main Result.

Let Ω be a bounded domain of the Euclidean space $\mathbf{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$. its closure $\overline{\Omega} = \Omega \cup \partial\Omega$ is an N-dimensional, compact smooth manifold with boundary. We consider a second-order, *uniformly elliptic* differential operator

$$Au = -\sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + c(x)u$$

with real smooth coefficients on the closure $\overline{\Omega} = \Omega \cup \partial\Omega$ such that

(1) $a^{ij}(x) = a^{ji}(x)$ for $1 \leq i, j \leq N$, and there exists a constant $a_0 > 0$ such that

$$\sum_{i,j=1}^{N} a^{ij}(x)\, \eta_i \eta_j \geq a_0 |\eta|^2 \quad \text{for all } x \in \overline{\Omega} \text{ and } \eta \in \mathbf{R}^N.$$

(2) $c(x) \geq 0$ in Ω and $c(x)$ does not vanish *identically* in Ω.

In this paper we study the following *quasilinear* elliptic boundary value problem with non-homogeneous Robin condition: For a given function $\varphi(x')$ defined on $\partial\Omega$,

2010 *Mathematics Subject Classification.* Primary: 35J62; Secondary: 35H10, 35R25.

Key words and phrases. Quasilinear elliptic equation, hypoelliptic Robin problem, Nagumo condition, comparison principle, Leray–Schauder fixed point theorem.

find a function $u(x)$ in Ω such that

$$\begin{cases} -Au = \sum_{i,j=1}^{N} a^{ij}(x) \dfrac{\partial^2 u}{\partial x_i \partial x_j} - c(x)u = f(x,u,\nabla u) & \text{in } \Omega, \\ Bu(x') := a(x') \dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = \varphi(x') & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here:

(3) ∇u stands for the gradient of u

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_N} \right).$$

(4) $a(x')$ and $b(x')$ are real-valued, smooth functions on the boundary $\partial\Omega$.
(5) $\mathbf{n} = (n_1, n_2, \dots, n_N)$ is the unit outward normal to $\partial\Omega$.
(6) $\partial/\partial\boldsymbol{\nu}$ is the outward conormal derivative associated with the operator A (see Figure 1)

$$\frac{\partial}{\partial \boldsymbol{\nu}} = \sum_{i,j=1}^{N} a^{ij}(x')\, n_j \frac{\partial}{\partial x_i}.$$

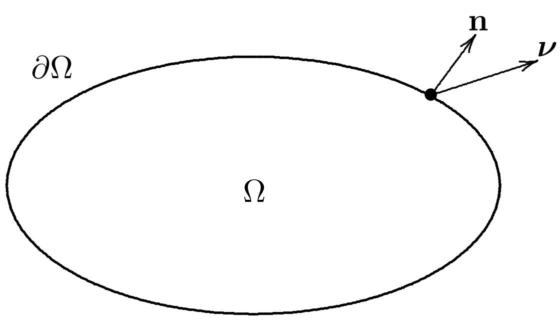

FIGURE 1. The unit outward normal $\mathbf{n}$ and the conormal $\boldsymbol{\nu}$ to $\partial\Omega$

Moreover, we impose the following two assumptions on the Robin boundary condition B:

(H.1) $a(x') \geq 0$ and $b(x') \geq 0$ on $\partial\Omega$.
(H.2) $a(x') + b(x') > 0$ on $\partial\Omega$.

It should be emphasized that the conditions (H.1) and (H.2) allow the problem (1) to include as particular cases the Dirichlet ($a(x') \equiv 0$), Neumann ($b(x') \equiv 0$) and regular Robin ($a(x') \equiv 1$) boundary conditions.

We give a simple but typical example of such functions $a(x')$ and $b(x')$ in the case where $N = 2$ ([22, Example 1.1]):

Example 1. Let $\Omega = \left\{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 < 1\right\}$ be the unit disk with the boundary $\partial\Omega = \left\{(x_1, x_2) \in \mathbf{R}^2 : x_1^2 + x_2^2 = 1\right\}$. For a local coordinate system $x_1 = \cos\theta$, $x_2 = \sin\theta$ with $\theta \in [0, 2\pi]$ on the unit circle $\partial\Omega$, we define a function $a(x_1, x_2)$

by the formula

$$a(x_1, x_2) = a(\cos\theta, \sin\theta) = \begin{cases} e^{\frac{2}{\pi}-\frac{1}{\theta}}\left(1-e^{\frac{2}{\pi}+\frac{1}{\theta-\frac{\pi}{2}}}\right) & \text{for } \theta \in \left[0, \frac{\pi}{2}\right], \\ 1 & \text{for } \theta \in \left[\frac{\pi}{2}, \pi\right], \\ e^{\frac{2}{\pi}+\frac{1}{\theta-\frac{3\pi}{2}}}\left(1-e^{\frac{2}{\pi}-\frac{1}{\theta-\pi}}\right) & \text{for } \theta \in \left[\pi, \frac{3\pi}{2}\right], \\ 0 & \text{for } \theta \in \left[\frac{3\pi}{2}, 2\pi\right], \end{cases}$$

and let

$$b(x_1, x_2) := 1 - a(x_1, x_2) \quad \text{on } \partial\Omega. \tag{2}$$

What is the important feature of the conditions (H.1) and (H.2) is that the so-called Lopatinski–Shapiro ellipticity condition is violated at the points where $a(x') = 0$ (see [22, Example 6.1]). More precisely, if we reduce the study of the problem (1) to that of a first order, pseudo-differential operator T on the boundary $\partial\Omega$, then the operator T is of the form

$$T = a(x')\sqrt{-\Delta'} + b(x')$$

where Δ' is the Laplace–Beltrami operator on $\partial\Omega$ (see [22, Chapter 7]). We can prove that if the conditions (H.1) and (H.2) are satisfied, then the operator T has a *parametrix* S in the Hörmander class $L^0_{1,1/2}(\partial\Omega)$ (see [22, Lemma 7.2]). Hence the operator T is *hypoelliptic* with loss of one derivative on $\partial\Omega$.

Remark 1. Amann–Crandall [4] studied the regular (non-degenerate) Robin case. More precisely, they assume that the boundary $\partial\Omega$ is the *disjoint* union of the two subsets $M = \{x \in \partial\Omega : a(x') = 0\}$ and $\partial\Omega \setminus M = \{x \in \partial\Omega : a(x') > 0\}$, each of which is an $(N-1)$-dimensional, compact smooth manifold. In this case, it is easy to see that the pseudo-differential operator $T = a(x')\sqrt{-\Delta'} + b(x')$ is *elliptic* of order *one* on $\partial\Omega \setminus M$ and of order *zero* on M, respectively.

The linear problem (1), that is,

$$f(x, z, p) = \sum_{i=1}^{N} b^i(x)\, p_i + c(x) z$$

is studied in great detail by Taira [18] and [22] in the frameworks of Hölder and Sobolev spaces. In the case where the function f is nonlinear in u but independent of ∇u, that is,

$$f(x, z, p) = f(x, z),$$

there is a similar result due to Taira [20] where a global static bifurcation theory is elaborated. We remark that Taira [19] studies the homogeneous problem (1) ($\varphi \equiv 0$) for linear elliptic operators of divergence form by using the super-subsolution method ([8, Section 6.3]).

On the other hand, the problem (1) with

$$f(x, z, p) = f(x)\, z^{(N+2)/(N-2)}, \quad N \geq 3,$$

is related to the so-called Yamabe problem which is a basic problem in Riemannian geometry (see [12], [14], [7], [17]).

In this paper the nonlinear term $f(x, z, p)$ of the problem (1) will be subject to the following three conditions:

(i) **Regularity conditions**:

$$f(x,z,p) \in C^{\alpha}(\overline{\Omega} \times \mathbf{R} \times \mathbf{R}^N) \text{ for } 0 < \alpha < 1. \tag{3a}$$

$$f(x,z,p) \text{ is continuously differentiable with respect to } z \text{ and } p. \tag{3b}$$

(ii) **Monotonicity condition**: There exists a constant $f_0 > 0$ such that

$$\frac{\partial f}{\partial z}(x,z,p) \geq f_0 \quad \text{for all } (x,z,p) \in \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^N. \tag{4}$$

(iii) **Quadratic gradient growth condition**: There exists a positive and non-decreasing function $f_1(t)$ such that

$$|f(x,z,p)| \leq f_1(|z|) \left(1 + |p|^2\right) \quad \text{for all } (x,z,p) \in \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^N. \tag{5}$$

By the quadratic gradient growth condition (5), we find that the nonlinear term $f(x,z,p)$ satisfies the so-called *Nagumo condition* (see [16, condition (2.6)]):

$$\int_0^{\infty} \frac{s}{1+s^2}\, ds = \frac{1}{2} \left[\ln\left(1+s^2\right)\right]_{s=0}^{s=\infty} = +\infty.$$

The main purpose of this paper is to extend Taira [19] to the non-homogeneous problem (1) allowing *quadratic nonlinearity* in f with respect to the gradient ∇u of the unknown function u. We derive an existence and uniqueness result for the problem (1) in the framework of Hölder spaces.

This paper is an expanded and revised version of the previous work Taira–Palagachev–Popivanov [23].

Following Taira [18], we introduce a variant of Hölder space

$$C_*^{1+\alpha}(\partial\Omega) = \left\{\varphi = a(x')\varphi_1 + b(x')\varphi_2 : \varphi_1 \in C^{1+\alpha}(\partial\Omega),\ \varphi_2 \in C^{2+\alpha}(\partial\Omega)\right\},$$

equipped with the norm

$$\|\varphi\|_{C_*^{1+\alpha}(\partial\Omega)} = \inf\left\{\|\varphi_1\|_{C^{1+\alpha}(\partial\Omega)} + \|\varphi_2\|_{C^{2+\alpha}(\partial\Omega)} : \varphi = a(x')\varphi_1 + b(x')\varphi_2\right\}.$$

Then it is easy to verify (see the proof of [22, Lemma 6.8]) that the function space $C_*^{1+\alpha}(\partial\Omega)$ is a Banach space with respect to the norm $\|\cdot\|_{C_*^{1+\alpha}(\partial\Omega)}$.

We remark that the space $C_*^{1+\alpha}(\partial\Omega)$ is an "interpolation space" between the Hölder spaces $C^{2+\alpha}(\partial\Omega)$ and $C^{1+\alpha}(\partial\Omega)$. More precisely, we have the assertions

$$\begin{cases} C_*^{1+\alpha}(\partial\Omega) = C^{2+\alpha}(\partial\Omega) & \text{if } a(x') \equiv 0 \text{ on } \partial\Omega \text{ (the Dirichlet case)}, \\ C_*^{1+\alpha}(\partial\Omega) = C^{1+\alpha}(\partial\Omega) & \text{if } b(x') \equiv 0 \text{ on } \partial\Omega \text{ (the Neumann case)}, \\ C_*^{1+\alpha}(\partial\Omega) = C^{1+\alpha}(\partial\Omega) & \text{if } a(x') > 0 \text{ on } \partial\Omega \text{ (the regular Robin case)}, \end{cases}$$

and, for general $a(x')$, we have the continuous injections

$$C^{2+\alpha}(\partial\Omega) \subset C_*^{1+\alpha}(\partial\Omega) \subset C^{1+\alpha}(\partial\Omega).$$

Now we are in a position to state our main result:

Theorem 1.1. *In addition to the conditions (H.1) and (H.2), we assume that the regularity conditions* (3)*, the monotonicity condition* (4) *and the quadratic gradient growth condition* (5) *are satisfied. Then the quasilinear problem* (1) *admits a unique classical solution $u \in C^{2+\alpha}(\overline{\Omega})$ for any $\varphi \in C_*^{1+\alpha}(\partial\Omega)$.*

It should be emphasized that Theorem 1.1 is a generalization of Nagumo [13, Theorem B], Akô [2, Main Theorem], Ladyzhenskaya–Ural'tseva [11, Chapter 6, Theorem 3.6] and Schmitt [16, Theorems 4.1 and 5.1] to the *hypoelliptic* Robin case which includes as particular cases the Dirichlet, Neumann and regular Robin problems.

For Theorem 1.1, we give a simple example of the function $f(x,z,p)$:

Example 2. $f(x,z,p) = z - |p|^2$. In this case we may take $f_0 = 1$ and $f_1(t) = 1+t$.

A typical example of our quasilinear problem (1) is given by the following:

$$\begin{cases} \Delta u = u - |\nabla u|^2 & \text{in } \Omega, \\ a(x')\dfrac{\partial u}{\partial \mathbf{n}} + (1 - a(x'))\, u = \varphi(x') & \text{on } \partial\Omega, \end{cases}$$

where (see Example 1)

$$0 \leq a(x') \leq 1 \quad \text{on } \partial\Omega.$$

In this case we may take

$$c(x) = \frac{1}{2},$$
$$f(x,z,p) = \frac{1}{2}\, z - |p|^2.$$

The rest of this paper is organized as follows. Section 2 is devoted to the precise definitions of Hölder spaces $C^{k+\alpha}(\Omega)$, $C^{k+\alpha}(\overline{\Omega})$ and L^p Sobolev spaces $W^{k,p}(\Omega)$. In Section 3 we establish *a priori* estimates of solutions $u \in C^{2+\alpha}(\overline{\Omega})$ of the *non-homogeneous* quasilinear problem (1) (Theorem 3.4). The deriving of the desired *a priori* estimate (15) is a two-step process consisting of successive bounds on the Hölder norms $\|u\|_{C(\overline{\Omega})}$ and $\|\nabla u\|_{C^\alpha(\overline{\Omega})}$ in the following way:

(1) The estimate of the uniform norm $\|u\|_{C(\overline{\Omega})}$: This follows by using the Bony maximum principle in the framework of L^p Sobolev spaces (Lemma 3.2).
(2) The *a priori* bound on the Hölder norm $\|\nabla u\|_{C^\alpha(\overline{\Omega})}$: First, we reduce it to an estimate for the Sobolev norm $\|\nabla u\|_{W^{1,p}(\Omega)}$ with $p = N/(1-\alpha)$, and then apply a $W^{2,p}(\Omega)$-*a priori* bound for the solutions of the *homogeneous* problem (1) ($\varphi \equiv 0$) proved by Taira [19, Proposition 2.3] (Theorem 3.3). In this procedure, A very important role is played by the monotonicity condition (4) and the quadratic gradient growth condition (5), as well as by the isomorphic properties in Hölder spaces and L^p Sobolev spaces of the linear operators appearing in the problem (1) ([18, Theorem 1.1]).

Section 4 is devoted to the proof of Theorem 1.1. This is carried out by making use of a version of the Leray–Schauder fixed point theorem due to Schaefer (Theorem 4.1) which reduces the solvability of the problem (1) to the establishment of a uniform *a priori* estimate in the Hölder space $C^{1+\alpha}(\overline{\Omega})$ for all solutions of a family of nonlinear problems related to the problem (1) (see the estimate (21)).

In Appendix we formulate various maximum principles due to Bony [5] for second-order, elliptic differential operators with discontinuous coefficients such as the weak and strong maximum principles (Theorems A.1 and A.3) and the Hopf boundary point lemma (Lemma A.2) in the framework of L^p Sobolev spaces.

2. **Function spaces.**

This preparatory section is devoted to the precise definitions of Hölder and Sobolev spaces of L^p type (see Gilbarg–Trudinger [10]).

Let $0 < \alpha < 1$. A function u defined on Ω is said to be *uniformly Hölder continuous* with exponent α in Ω if the quantity

$$[u]_{\alpha;\Omega} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha}$$

is finite. We say that u is *locally Hölder continuous* with exponent α in Ω if it is uniformly Hölder continuous with exponent α on compact subsets of Ω.

If $0 < \alpha < 1$, we define the Hölder space $C^{\alpha}(\Omega)$ as follows:

$$\begin{aligned} C^{\alpha}(\Omega) = &\text{ the space of functions in } C(\Omega) \text{ which are locally Hölder} \\ &\text{ continuous with exponent } \alpha \text{ on } \Omega. \end{aligned}$$

If k is a positive integer and $0 < \alpha < 1$, we define the Hölder space $C^{k+\alpha}(\Omega)$ as follows:

$$\begin{aligned} C^{k+\alpha}(\Omega) = &\text{ the space of functions in } C^k(\Omega) \text{ all of whose } k\text{-th order} \\ &\text{ derivatives are locally Hölder continuous with exponent } \alpha \\ &\text{ on } \Omega. \end{aligned}$$

We introduce various seminorms and norms on the spaces $C^k(\Omega)$ and $C^{k+\alpha}(\Omega)$ as follows:

$$\begin{aligned} [u]_{k,0;\Omega} &= \left|D^k u\right|_{0;\Omega} = \sup_{x\in\Omega}\sup_{|\beta|=k}\left|D^{\beta}u(x)\right|, \\ [u]_{k,\alpha;\Omega} &= \left[D^k u\right]_{\alpha;\Omega} = \sup_{|\beta|=k}\left[D^{\beta}u\right]_{\alpha;\Omega}. \end{aligned}$$

Furthermore, we let

$$\begin{aligned} C^{\alpha}(\overline{\Omega}) = &\text{ the space of functions in } C(\overline{\Omega}) \text{ which are Hölder} \\ &\text{ continuous with exponent } \alpha \text{ on } \overline{\Omega}, \end{aligned}$$

and

$$\begin{aligned} C^{k+\alpha}(\overline{\Omega}) = &\text{ the space of functions in } C^k(\overline{\Omega}) \text{ all of whose } k\text{-th order} \\ &\text{ derivatives are Hölder continuous with exponent } \alpha \\ &\text{ on } \overline{\Omega}. \end{aligned}$$

Similarly, we define the associated norms on the spaces $C^k(\overline{\Omega})$ and $C^{k+\alpha}(\overline{\Omega})$ as follows:

$$\begin{aligned} \|u\|_{C^k(\Omega)} &= |u|_{k;\Omega} = \sum_{j=0}^{k}\left|D^j u\right|_{0;\Omega}, \\ \|u\|_{C^{k+\alpha}(\Omega)} &= |u|_{k,\alpha;\Omega} = |u|_{k;\Omega} + \left[D^k u\right]_{\alpha;\Omega}. \end{aligned}$$

The usual Sobolev space $W^{k,p}(\Omega)$ for $k \in \mathbf{N}$ and $1 < p < \infty$ is defined as follows:

$$\begin{aligned} W^{k,p}(\Omega) = &\text{ the space of functions } u \in L^p(\Omega) \text{ whose derivatives } D^{\alpha}u, \\ &\ |\alpha| \leq k, \text{ in the sense of distributions are in } L^p(\Omega), \end{aligned}$$

and its norm $\|\cdot\|_{W^{k,p}(\Omega)}$ is given by the formula

$$\|u\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq k}\int_{\Omega}|D^{\alpha}u(x)|^p\,dx\right)^{1/p}.$$

3. ***A priori* estimates for the quasilinear problem (1).** In the proof of Theorem 1.1 we make use of a version of the Leray–Schauder fixed point theorem due to Schaefer (Theorem 4.1). For this purpose, we need to establish an *a priori* estimate for the $C^{1+\alpha}(\overline{\Omega})$-norm of each solution $u \in C^{2+\alpha}(\overline{\Omega})$ of the *non-homogeneous* quasilinear problem (1).

We start with the following *comparison principle* for quasilinear problems ([4, Lemma 2]):

Lemma 3.1. *Assume that the condition (H.1) and (H.2) are satisfied and further that $f(x,z,p)$ is* strictly increasing *in z for each $(x,p) \in \Omega \times \mathbf{R}^N$ and is differentiable with respect to p for each $(x,z) \in \Omega \times \mathbf{R}$. Let u, $v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy the conditions*

$$\begin{aligned} &\sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} - c(x)u - f(x,u,\nabla u) \\ \geq &\sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} - c(x)v - f(x,v,\nabla v) \quad \text{in } \Omega \end{aligned} \tag{6}$$

and

$$a(x') \frac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u \leq a(x') \frac{\partial v}{\partial \boldsymbol{\nu}} + b(x')v \quad \text{on } \partial\Omega. \tag{7}$$

Then it follows that $u(x) \leq v(x)$ on $\overline{\Omega}$.

Proof. Our proof is based on a reduction to absurdity. We let

$$w(x) := u(x) - v(x),$$

and assume, to the contrary, that the set

$$\Omega^+ = \{x \in \Omega : u(x) > v(x)\} = \{x \in \Omega : w(x) > 0\}$$

is non-empty (see Figure 2).

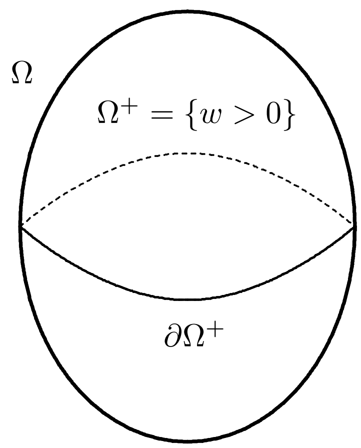

FIGURE 2. The open subset Ω^+ with boundary $\partial\Omega^+$

Since $f(x,z,p)$ is *strictly increasing* with respect to z, it follows from the inequality (6) that

$$
\begin{aligned}
&\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j} - c(x)w + f\left(x, u(x), \nabla v(x)\right) - f\left(x, u(x), \nabla u(x)\right) \qquad (8)\\
&= \left(\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} - c(x)u - \sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 v}{\partial x_i \partial x_j} + c(x)v \right)\\
&\quad + f\left(x, u(x), \nabla v(x)\right) - f\left(x, u(x), \nabla u(x)\right)\\
&\geq \left(f\left(x, u(x), \nabla u(x)\right) - f\left(x, v(x), \nabla v(x)\right)\right)\\
&\quad + f\left(x, u(x), \nabla v(x)\right) - f\left(x, u(x), \nabla u(x)\right)\\
&= f\left(x, u(x), \nabla v(x)\right) - f\left(x, v(x), \nabla v(x)\right)\\
&> 0 \quad \text{in } \Omega^+.
\end{aligned}
$$

However, we can rewrite the term

$$
f\left(x, u(x), \nabla v(x)\right) - f\left(x, u(x), \nabla u(x)\right)
$$

in the form

$$
\begin{aligned}
&f\left(x, u(x), \nabla v(x)\right) - f\left(x, u(x), \nabla u(x)\right)\\
&= -\int_0^1 \frac{d}{dt}\left(f\left(x, u(x), t\,\nabla w(x) + \nabla v(x)\right)\right) dt\\
&= -\sum_{i=1}^{N} \int_0^1 \frac{\partial f}{\partial p_i}\left(x, u(x), t\,\nabla w(x) + \nabla v(x)\right) \frac{\partial w}{\partial x_i}\, dt\\
&= \sum_{i=1}^{N} b^i(x)\, \frac{\partial w}{\partial x_i},
\end{aligned}
$$

where

$$
b^i(x) := -\int_0^1 \frac{\partial f}{\partial p_i}(x, u(x), t\nabla w(x) + \nabla v(x))\, dt, \quad 1 \leq i \leq N.
$$

Hence we have, by the inequality (8),

$$
\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x)\, \frac{\partial w}{\partial x_i} - c(x)w > 0 \quad \text{in } \Omega^+. \qquad (9)
$$

Now we take a point x_0 of the closure $\overline{\Omega}$ such that

$$
w(x_0) = \max_{x \in \overline{\Omega}} w(x) > 0.
$$

(1) First, we consider the case where $x_0 \in \Omega$: We remark that

$$
x_0 \in \Omega^+.
$$

Then it follows from an application of the strong maximum principle (see Theorem A.3) that

$$
w(x) \equiv w(x_0) \quad \text{in } \Omega^+.
$$

Hence we have the inequality

$$\begin{aligned}&\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 w}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} b^i(x)\frac{\partial w}{\partial x_i}(x) - c(x)w(x)\\ &= -c(x)w(x_0)\\ &\le 0 \quad \text{in } \Omega^+.\end{aligned}$$

This contradicts the inequality (9).

(2) Secondly, we consider the case where $x_0 \in \partial\Omega$: We remark that

$$x_0 \in \partial\Omega \cap \partial\Omega^+.$$

Then it follows from an application of the Hopf boundary point lemma (see Lemma A.2) that

$$\frac{\partial w}{\partial \boldsymbol{\nu}}(x_0) > 0.$$

Hence we have, by conditions (H.1) and (H.2),

$$Bw(x_0) = a(x_0)\frac{\partial w}{\partial \boldsymbol{\nu}}(x_0) + b(x_0)w(x_0) > 0.$$

However, it follows from the inequality (7) that

$$Bw(x_0) = Bu(x_0) - Bv(x_0) \le 0.$$

This is a contradiction.

Summing up, we have proved that the set Ω^+ is empty.

The proof of Lemma 3.1 is complete. □

3.1. ***A priori* estimate for the uniform norm $\|u\|_{C(\overline{\Omega})}$.** As a first step in obtaining an *a priori* estimate for the *non-homogeneous* problem (1), we consider the homogeneous case. Namely, let $u \in C^{2+\alpha}(\overline{\Omega})$ be a solution of the problem

$$\begin{cases} \sum_{i,j=1}^{N} a^{ij}(x)\dfrac{\partial^2 u}{\partial x_i \partial x_j} - c(x)u = f(x,u,\nabla u) & \text{in } \Omega, \\ a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Then we have the following *a priori* bound on the uniform norm $\|u\|_{C(\overline{\Omega})}$:

Lemma 3.2. *Assume that the conditions (H.1), (H.2), the regularity conditions (3) and the monotonicity condition (4) are satisfied. If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution of the homogeneous problem (8), then we have the* a priori *estimate*

$$\|u\|_{C(\overline{\Omega})} = \max_{x \in \overline{\Omega}} |u(x)| \le \frac{\max_{x \in \overline{\Omega}} |f(x,0,0)|}{f_0}. \tag{11}$$

Proof. The proof of Lemma 3.2 is divided into two steps.

Step (1): First, by letting

$$K := \frac{\max_{x \in \overline{\Omega}} |f(x,0,0)|}{f_0},$$

we obtain from the monotonicity condition (4) that

$$\begin{aligned}
&\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 K}{\partial x_i \partial x_j} - c(x)K - f(x,K,\nabla K)\\
&= -f(x,K,0) - c(x)K\\
&= -K\int_0^1 \frac{\partial f}{\partial z}(x,tK,0)\,dt - f(x,0,0) - c(x)K\\
&\leq -K\,f_0 - f(x,0,0) - c(x)K = -\left(\max_{x\in\overline{\Omega}}|f(x,0,0)| + f(x,0,0)\right) - c(x)K\\
&\leq 0 \quad \text{for each } x \in \Omega.
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
&\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} - c(x)u - f(x,u,\nabla u)\\
&= 0\\
&\geq \sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 K}{\partial x_i \partial x_j} - c(x)K - f(x,K,\nabla K) \qquad \text{in } \Omega.
\end{aligned}$$

On the other hand, we have the inequality

$$a(x')\frac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 \leq b(x')\,K = a(x')\frac{\partial K}{\partial \boldsymbol{\nu}} + b(x')\,K \quad \text{on } \partial\Omega.$$

Therefore, it follows from an application of Lemma 3.1 that

$$u(x) \leq K \quad \text{for all } x \in \overline{\Omega}.$$

Step (2): Secondly, if we let

$$\widetilde{f}(x,z,p) := -f(x,-z,-p) \quad \text{for all } (x,z,p) \in \overline{\Omega}\times\mathbf{R}\times\mathbf{R}^N,$$

then it follows that the function

$$v(x) := -u(x)$$

is a solution of the nonlinear problem

$$\begin{cases}
\sum_{i,j=1}^{N} a_{ij}(x)\dfrac{\partial^2 v}{\partial x_i \partial x_j} - c(x)v = \widetilde{f}(x,v,\nabla v) & \text{in } \Omega,\\
a(x')\dfrac{\partial v}{\partial \boldsymbol{\nu}} + b(x')v = 0 & \text{on } \partial\Omega.
\end{cases}$$

However, the nonlinear term $\widetilde{f}(x,z,p)$ satisfies the monotonicity condition (4):

$$\frac{\partial \widetilde{f}}{\partial z}(x,z,p) = \frac{\partial f}{\partial z}(x,-z,-p) \geq f_0 \quad \text{for all } (x,z,p) \in \overline{\Omega}\times\mathbf{R}\times\mathbf{R}^N.$$

Hence, by arguing just as in Step (1) with $u(x)$ and $f(x,z,p)$ replaced by $v(x)$ and $\widetilde{f}(x,z,p)$, respectively, we obtain that

$$-u(x) = v(x) \leq K \quad \text{for all } x \in \overline{\Omega},$$

since we have the formula

$$\frac{\max_{x\in\overline{\Omega}}|\widetilde{f}(x,0,0)|}{f_0} = \frac{\max_{x\in\overline{\Omega}}|f(x,0,0)|}{f_0} = K.$$

The proof of Lemma 3.2 is complete. $\square$

3.2. ***A priori* estimate for the Hölder norm $\|u\|_{C^{1+\alpha}(\overline{\Omega})}$.** In the following the letter C stands for a generic positive constant depending only on known quantities but not on u, which may vary from a line into another.

We start with an *a priori* bound on the Hölder norm $\|u\|_{C^{1+\alpha}(\overline{\Omega})}$ for the homogeneous problem (10):

Theorem 3.3. *In addition to the conditions (H.1) and (H.2), we assume that the regularity conditions* (3)*, the monotonicity condition* (4) *and the quadratic gradient growth condition* (5) *are satisfied. Then there exists a positive constant C, independent of u, such that*

$$\|u\|_{C^{1+\alpha}(\overline{\Omega})} \leq C \tag{12}$$

for every solution $u \in C^{2+\alpha}(\overline{\Omega})$ of the homogeneous *problem*

$$\begin{cases} \sum_{i,j=1}^{N} a^{ij}(x) \dfrac{\partial^2 u}{\partial x_i \partial x_j} - c(x)u = f(x, u, \nabla u) & \text{in } \Omega, \\ a(x') \dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Proof. First, it follows from an application of the Morrey lemma (see [1, Theorem 4.12, Part II], [10, Chapter 12, Lemma 12.2]) that the imbedding

$$W^{2,p}(\Omega) \subset C^{1+\alpha}(\overline{\Omega}), \quad \alpha = 1 - \frac{N}{p}$$

holds true for $p > N$. Hence we have, with some constant $C > 0$,

$$\|u\|_{C^1(\overline{\Omega})} \leq C \|u\|_{W^{2,p}(\Omega)}, \tag{13a}$$

$$[\nabla u]_{\alpha;\overline{\Omega}} \leq C \|u\|_{W^{2,p}(\Omega)}. \tag{13b}$$

Namely, the *a priori* bound (12) on the Hölder norm $\|u\|_{C^{1+\alpha}(\overline{\Omega})}$ can be reduced to a uniform estimate (with respect to u) of the Sobolev norm $\|u\|_{W^{2,p}(\Omega)}$ for every solution u of the *homogeneous* problem (10).

However, since the quadratic gradient growth condition (5) is satisfied, we can apply [19, Proposition 2.3] to find a non-negative and increasing function $\gamma(t)$, depending only on known quantities, such that

$$\|u\|_{W^{2,p}(\Omega)} \leq \gamma\left(\|u\|_{C(\overline{\Omega})}\right) \tag{14}$$

for every solution $u \in W^{2,p}(\Omega)$ of the homogeneous problem (10).

Indeed, the proof of [19, Proposition 2.3] remains valid for our operator

$$Au = -\sum_{i,j=1}^{N} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + c(x)u.$$

We remark that the proof of [19, Proposition 2.3] is based on methods developed by Tomi [24] and Amann [3].

Therefore, the desired *a priori* bound (12) follows by combining the estimates (13), (14) and the *a priori* bound (11) on the uniform norm $\|u\|_{C(\overline{\Omega})}$.

The proof of Theorem 3.3 is complete. $\square$

The purpose of this subsection is to generalize Theorem 3.3 to the non-homogeneous boundary condition case:

Theorem 3.4. *In addition to the conditions (H.1) and (H.2), we assume that the regularity conditions* (3)*, the monotonicity condition* (4) *and the quadratic gradient growth condition* (5) *are satisfied. Then there exists a positive constant C, independent of u, such that*

$$\|u\|_{C^{1+\alpha}(\overline{\Omega})} \leq C \tag{15}$$

for every solution $u \in C^{2+\alpha}(\overline{\Omega})$ of the non-homogeneous *problem* (1) *with $\varphi \in C_*^{1+\alpha}(\partial\Omega)$:*

$$\begin{cases} \sum_{i,j=1}^N a^{ij}(x)\dfrac{\partial^2 u}{\partial x_i \partial x_j} - c(x)u = f(x, u, \nabla u) & \text{in } \Omega, \\ a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here the constant C depends on the norm $\|\varphi\|_{C_*^{1+\alpha}(\partial\Omega)}$.

Proof. To deal with the non-homogeneous problem (1), we remark that [18, Theorem 1.1] implies the existence of a unique solution $v \in C^{2+\alpha}(\overline{\Omega})$ of the linear problem

$$\begin{cases} \sum_{i,j=1}^N a^{ij}(x)\dfrac{\partial^2 v}{\partial x_i \partial x_j} - c(x)v = 0 & \text{in } \Omega, \\ a(x')\dfrac{\partial v}{\partial \boldsymbol{\nu}} + b(x')v = \varphi & \text{on } \partial\Omega, \end{cases}$$

with the estimate

$$\|v\|_{C^{2+\alpha}(\overline{\Omega})} \leq C_1 \|\varphi\|_{C_*^{1+\alpha}(\partial\Omega)}. \tag{16}$$

Now we introduce a new nonlinear term

$$\overline{f}(x, z, p) := f\left(x, z + v(x), p + \nabla v(x)\right) \quad \text{for all } (x, z, p) \in \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^N.$$

If u is a solution of the problem (1), then the function

$$w(x) := u(x) - v(x)$$

solves the *homogeneous* nonlinear problem

$$\begin{cases} \sum_{i,j=1}^N a^{ij}(x)\dfrac{\partial^2 w}{\partial x_i \partial x_j} - c(x)w = \overline{f}(x, w, \nabla w) & \text{in } \Omega, \\ a(x')\dfrac{\partial w}{\partial \boldsymbol{\nu}} + b(x')w = 0 & \text{on } \partial\Omega. \end{cases} \tag{17}$$

It is easy to see that the nonlinear term $\overline{f}(x, z, p)$ satisfies the monotonicity condition (4)

$$\begin{aligned} \frac{\partial \overline{f}}{\partial z}(x, z, p) &= \frac{\partial f}{\partial z}\left(x, z + v(x), p + \nabla v(x)\right) \\ &\geq f_0 \quad \text{for all } (x, z, p) \in \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^N. \end{aligned}$$

Moreover, we have the inequality

$$\begin{aligned} &\left|\overline{f}(x, z, p)\right| \\ &= \left|f\left(x, z + v(x), p + \nabla v(x)\right)\right| \leq f_1(|z + v(x)|)\left(1 + |p + \nabla v(x)|^2\right) \\ &\leq f_1\left(|z| + |v(x)|\right) 2\left(1 + |p|^2 + |\nabla v(x)|^2\right) \quad \text{for all } (x, z, p) \in \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^N. \end{aligned}$$

However, by the estimate (16) it follows that

$$\max_{x \in \overline{\Omega}} |v(x)| + \max_{x \in \overline{\Omega}} |\nabla v(x)| \leq C_1 \|\varphi\|_{C_*^{1+\alpha}(\partial\Omega)},$$

so that

$$\begin{aligned}
&|v(x)| \leq C_\varphi, \\
&|z + v(x)| \leq |z| + C_\varphi, \\
&|p|^2 + |\nabla v(x)|^2 \leq |p|^2 + C_\varphi^2,
\end{aligned}$$

where

$$C_\varphi := C_1 \|\varphi\|_{C_*^{1+\alpha}(\partial\Omega)}.$$

Hence we find that the nonlinear term $\overline{f}(x, z, p)$ satisfies the quadratic gradient growth condition (5) with a new function $\overline{f}_1(t)$:

$$|\overline{f}(x, z, p)| \leq \overline{f}_1(|z|) \left(1 + |p|^2\right) \quad \text{for all } (x, z, p) \in \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^N.$$

By applying Theorem 3.3 to the homogeneous problem (17), we obtain from the estimate (12) that

$$\|w\|_{C^{1+\alpha}(\overline{\Omega})} \leq C_2, \tag{18}$$

with some constant $C_2 > 0$.

Therefore, the desired estimate (15) follows by combining the estimates (16) and (18) with

$$C := C_2 + C_\varphi = C_2 + C_1 \|\varphi\|_{C_*^{1+\alpha}(\partial\Omega)}.$$

Indeed, it suffices to note that

$$\begin{aligned}
\|u\|_{C^{1+\alpha}(\overline{\Omega})} = \|v + w\|_{C^{1+\alpha}(\overline{\Omega})} &\leq \|v\|_{C^{1+\alpha}(\overline{\Omega})} + \|w\|_{C^{1+\alpha}(\overline{\Omega})} \\
&\leq C_1 \|\varphi\|_{C_*^{1+\alpha}(\partial\Omega)} + C_2.
\end{aligned}$$

The proof of Theorem 3.4 is complete. □

4. **Proof of Theorem 1.1.** This section is devoted to the proof of Theorem 1.1.

(I) The *uniqueness* result follows immediately from the comparison principle (Lemma 3.1).

(II) To derive the *existence* result, we shall make use of the following version of the Leray–Schauder fixed point theorem due to Schaefer (see [6, Theorem 3.4.8], [8, Example 5.8.4], [9, Theorem 5.4.14]; [10, Theorem 11.3], [15, Theorem 6.3.1]):

Theorem 4.1 (Schaefer)**.** *Let $f(x, t)$ be a one-parameter family of* compact *operators defined on a Banach space X for $t \in [0, 1]$, with $f(x, t)$ uniformly continuous in t for fixed $x \in X$. Furthermore, assume that* every *solution of the equation $x = f(x, t)$ for each $t \in [0, 1]$ is contained in the fixed open ball $\Sigma = \{x \in X : \|x\| < M\}$. Then, assuming $f(\cdot, 0) \equiv 0$, the operator $f(\cdot, 1)$ has a fixed point $x \in \Sigma$.*

The proof of the existence part is divided into four steps.

Step 1: Let $\varphi \in C_*^{1+\alpha}(\partial\Omega)$. For any given function $v \in C^{1+\alpha}(\overline{\Omega})$, we consider the linear problem

$$\begin{cases} \sum_{i,j=1}^N a^{ij}(x) \dfrac{\partial^2 u}{\partial x_i \partial x_j} - c(x) u = f(x, v, \nabla v) & \text{in } \Omega, \\ a(x') \dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x') u = \varphi(x') & \text{on } \partial\Omega. \end{cases} \tag{19}$$

In view of the *regularity conditions* (3), it follows that

$$f(x, v, \nabla v) \in C^\alpha(\overline{\Omega}).$$

Hence [18, Theorem 1.1] asserts that there exists a unique solution $u \in C^{2+\alpha}(\overline{\Omega})$ of the linear problem (19). In this way, we can define a nonlinear operator $\mathcal{H}$ by the formula

$$\begin{aligned}\mathcal{H} : C^{1+\alpha}(\overline{\Omega}) &\longrightarrow C^{2+\alpha}(\overline{\Omega}) \\ v &\longmapsto u.\end{aligned}$$

Then it follows from [18, Theorem 1.1] that $\mathcal{H}$ is a continuous operator. Indeed, it suffices to note that the mapping

$$\left(\sum_{i,j=1}^{N} a^{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} - c(x)u,\ a(x') \frac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u \right) : C^{2+\alpha}(\overline{\Omega}) \\ \longrightarrow C^{\alpha}(\overline{\Omega}) \oplus C^{1+\alpha}_{*}(\partial\Omega)$$

is an algebraic and topological *isomorphism* for $\alpha \in (0,1)$. This implies the continuity of $\mathcal{H}$ considered as an operator from $C^{1+\alpha}(\overline{\Omega})$ into $C^{2+\alpha}(\overline{\Omega})$. Furthermore, since the space $C^{2+\alpha}(\overline{\Omega})$ is compactly imbedded into the space $C^{1+\alpha}(\overline{\Omega})$ (see [10, Lemma 6.36]), we derive immediately also the *compactness* of the mapping

$$\mathcal{H} \colon C^{1+\alpha}(\overline{\Omega}) \longrightarrow C^{1+\alpha}(\overline{\Omega}).$$

The situation can be visualized as follows:

$$\mathcal{H} \colon C^{1+\alpha}(\overline{\Omega}) \longrightarrow C^{2+\alpha}(\overline{\Omega}) \underset{\text{compactly}}{\hookrightarrow\hookrightarrow} C^{1+\alpha}(\overline{\Omega}).$$

Step 2: Now, for each $\rho \in [0,1]$ we consider the equation

$$u = \rho\, \mathcal{H} u \quad \text{in } C^{1+\alpha}(\overline{\Omega}),$$

that is, the non-homogeneous problem

$$\begin{cases} \sum_{i,j=1}^{N} a^{ij}(x) \dfrac{\partial^2 u}{\partial x_i \partial x_j} - c(x)u = \rho\, f\,(x, u, \nabla u) & \text{in } \Omega, \\ a(x') \dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = \rho\, \varphi(x') & \text{on } \partial\Omega. \end{cases} \tag{20}$$

We shall prove the following *uniform* a priori *estimate* for every solution $u = u_\rho$ of the non-homogeneous problem (20)

$$\|u_\rho\|_{C^{1+\alpha}(\overline{\Omega})} \leq C', \tag{21}$$

with a constant $C' > 0$ independent of ρ and u.

Substep 2.1: If $v \in C^{2+\alpha}(\overline{\Omega})$ is a unique solution of the linear problem

$$\begin{cases} \sum_{i,j=1}^{N} a^{ij}(x) \dfrac{\partial^2 v}{\partial x_i \partial x_j} - c(x)v = 0 & \text{in } \Omega, \\ a(x') \dfrac{\partial v}{\partial \boldsymbol{\nu}} + b(x')v = \varphi & \text{on } \partial\Omega, \end{cases}$$

we let

$$v_\rho(x) := \rho\, v(x) \quad \text{for all } 0 \leq \rho \leq 1.$$

Then it follows that v_ρ is the unique solution of the linear problem

$$\begin{cases} \sum_{i,j=1}^{N} a^{ij}(x) \dfrac{\partial^2 v_\rho}{\partial x_i \partial x_j} - c(x)v_\rho = 0 & \text{in } \Omega, \\ a(x') \dfrac{\partial v_\rho}{\partial \boldsymbol{\nu}} + b(x')v_\rho = \rho\, \varphi & \text{on } \partial\Omega, \end{cases} \tag{22}$$

with the estimate

$$\begin{aligned} \|v_\rho\|_{C^{2+\alpha}(\overline{\Omega})} &= \rho \|v\|_{C^{2+\alpha}(\overline{\Omega})} \leq \rho C_1 \|\varphi\|_{C_*^{1+\alpha}(\partial\Omega)} \\ &\leq C_1 \|\varphi\|_{C_*^{1+\alpha}(\partial\Omega)} \quad \text{for all } 0 \leq \rho \leq 1. \end{aligned} \tag{23}$$

Here it should be noticed (see the estimate (15)) that the constant C_1 depends on the norm $\|\varphi\|_{C_*^{1+\alpha}(\partial\Omega)}$.

Substep 2.2: For every solution $u = u_\rho$ of the non-homogeneous problem (20), we let

$$w_\rho(x) := u_\rho(x) - v_\rho(x) = u_\rho(x) - \rho\, v(x) \quad \text{for all } 0 \leq \rho \leq 1.$$

Then it follows from the problems (20) and (22) that the function w_ρ is a unique solution of the *homogeneous* nonlinear problem

$$\begin{cases} \sum_{i,j=1}^N a^{ij}(x) \dfrac{\partial^2 w_\rho}{\partial x_i \partial x_j} - c(x) w_\rho = \rho\, f\left(x, u_\rho, \nabla u_\rho\right) - \rho\, v(x) & \text{in } \Omega, \\ a(x') \dfrac{\partial w_\rho}{\partial \boldsymbol{\nu}} + b(x') w_\rho = 0 & \text{on } \partial\Omega. \end{cases} \tag{24}$$

We remark that $w_0 = 0$ for $\rho = 0$ as it follows from the uniqueness result in [18, Theorem 1.1].

Therefore, if we introduce a new nonlinear term

$$\overline{f}_\rho(x,z,p) := \rho\, f\left(x, z + v_\rho(x), p + \nabla v_\rho(x)\right) \quad \text{for all } (x,z,p) \in \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^N,$$

then the non-homogeneous nonlinear problem (24) can be expressed as follows:

$$\begin{cases} \sum_{i,j=1}^N a^{ij}(x) \dfrac{\partial^2 w_\rho}{\partial x_i \partial x_j} - c(x) w_\rho = \overline{f}_\rho\left(x, w_\rho, \nabla w_\rho\right) & \text{in } \Omega, \\ a(x') \dfrac{\partial w_\rho}{\partial \boldsymbol{\nu}} + b(x') w_\rho = 0 & \text{on } \partial\Omega. \end{cases} \tag{25}$$

We verify that the nonlinear term $\overline{f}_\rho(x,z,p)$ satisfies the *monotonicity condition* (4). Indeed, it follows that

$$\begin{aligned} &\frac{\partial \overline{f}_\rho}{\partial z}(x,z,p) \\ &= \rho \frac{\partial f}{\partial z}\left(x, z + v_\rho(x), p + \nabla v_\rho(x)\right) \geq \rho\, f_0 \quad \text{for all } (x,z,p) \in \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^N. \end{aligned}$$

On the other hand, it follows from the estimate (23) that

$$\max_{x \in \overline{\Omega}} |v_\rho(x)| + \max_{x \in \overline{\Omega}} |\nabla v_\rho(x)| \leq C_1 \|\varphi\|_{C_*^{1+\alpha}(\partial\Omega)} \quad \text{for all } 0 \leq \rho \leq 1,$$

so that

$$\begin{aligned} |v_\rho(x)| &= \rho\, |v(x)| \leq C_\varphi, \\ |z + v_\rho(x)| &\leq |z| + C_\varphi, \\ |p|^2 + |\nabla v_\rho(x)|^2 &\leq |p|^2 + C_\varphi^2. \end{aligned}$$

Hence we have the inequality

$$\begin{aligned} &\left|\overline{f}_\rho(x,0,0)\right| \\ &= \rho \left|f\left(x, v_\rho(x), \nabla v_\rho(x)\right)\right| \\ &\leq \rho \max\left\{|f(x,z,p)| : x \in \overline{\Omega},\ |z| \leq C_\varphi,\ |p| \leq C_\varphi\right\} \\ &= \rho\, L_\varphi \quad \text{for all } x \in \overline{\Omega} \text{ and } 0 < \rho \leq 1, \end{aligned}$$

where

$$L_\varphi := \max\left\{|f(x,z,p)| : x \in \overline{\Omega},\ |z| \leq C_\varphi,\ |p| \leq C_\varphi\right\}.$$

This proves that

$$\frac{\max_{x\in\overline{\Omega}} |\overline{f}_\rho(x,0,0)|}{\rho\, f_0} \leq \frac{\rho\, L_\varphi}{\rho\, f_0} = \frac{L_\varphi}{f_0} \quad \text{for all } 0 < \rho \leq 1.$$

By applying Lemma 3.2 and Theorem 3.3 to the homogeneous problem (25), we obtain the *uniform estimate*

$$\|w_\rho\|_{C^{1+\alpha}(\overline{\Omega})} \leq C_3 \quad \text{for all } 0 \leq \rho \leq 1, \tag{26}$$

with some constant $C_3 > 0$ independent of ρ. We recall that $w_0 = 0$ for $\rho = 0$.

Substep 2.3: Moreover, we have the inequality

$$\begin{aligned}
&\left|\overline{f}_\rho(x,z,p)\right| = \left|\rho\, f\left(x, z + v_\rho(x), p + \nabla v_\rho(x)\right)\right| \\
&\leq \rho\, f_1(|z + v_\rho(x)|)\left(1 + |p + \nabla v_\rho(x)|^2\right) \\
&\leq 2\,\rho\, f_1\left(|z| + |v_\rho(x)|\right)\left(1 + |p|^2 + |\nabla v_\rho(x)|^2\right) \\
&\leq 2\, f_1\left(|z| + C_\varphi\right)\left(1 + |p|^2 + C_\varphi^2\right) \\
&\qquad \text{for all } (x,z,p) \in \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^N.
\end{aligned}$$

Hence we find that the nonlinear term $\overline{f}_\rho(x,z,p)$ satisfies the *quadratic gradient growth condition* (5) with a new function $\overline{f}_1(t)$:

$$|\overline{f}_\rho(x,z,p)| \leq \overline{f}_1(|z|)\left(1 + |p|^2\right) \quad \text{for all } (x,z,p) \in \overline{\Omega} \times \mathbf{R} \times \mathbf{R}^N.$$

Therefore, the desired estimate (21) follows by combining the estimates (23) and (26) with

$$C' := C_3 + C_\varphi = C_3 + C_1\,\|\varphi\|_{C_*^{1+\alpha}(\partial\Omega)}.$$

Indeed, it suffices to note that

$$\begin{aligned}
\|u_\rho\|_{C^{1+\alpha}(\overline{\Omega})} &= \|v_\rho + w_\rho\|_{C^{1+\alpha}(\overline{\Omega})} \leq \|v_\rho\|_{C^{1+\alpha}(\overline{\Omega})} + \|w_\rho\|_{C^{1+\alpha}(\overline{\Omega})} \\
&\leq C_1\,\|\varphi\|_{C_*^{1+\alpha}(\partial\Omega)} + C_3 \quad \text{for all } 0 \leq \rho \leq 1.
\end{aligned}$$

We remark that $u_0 = v_0 + w_0 = 0$ for $\rho = 0$.

Step 3: By using Schaefer's theorem (Theorem 4.1), we find that the properties of the operator $\mathcal{H}$ and the estimate (21) imply the existence of a fixed point $u \in C^{1+\alpha}(\overline{\Omega})$ of the operator $\mathcal{H}$. Namely, the function u satisfies the non-homogeneous problem (20) for $\rho = 1$:

$$\begin{cases}
\sum_{i,j=1}^N a^{ij}(x)\dfrac{\partial^2 u}{\partial x_i \partial x_j} - c(x)u = f\left(x, u, \nabla u\right) & \text{in } \Omega, \\
a(x')\dfrac{\partial u}{\partial \boldsymbol{\nu}} + b(x')u = \varphi(x') & \text{on } \partial\Omega.
\end{cases}$$

In this way, the fixed point u becomes a solution of the original nonlinear problem (1).

Step 4: Finally, the smoothing properties of $\mathcal{H}$ yield that

$$u = \mathcal{H}u \in C^{2+\alpha}(\overline{\Omega}).$$

Now the proof of Theorem 1.1 is complete. □

5. **Appendix: The maximum principle in Sobolev spaces.** In this appendix we formulate various maximum principles for second-order, elliptic differential operators with discontinuous coefficients such as the weak and strong maximum principles (Theorems A.1 and A.3) and the boundary point lemma (Lemma A.2) in the framework of L^p Sobolev spaces. The results here are adapted from Bony [5], Troianiello [25, Chapter 3] and also Taira [21, Chapter 8].

Let Ω be a bounded domain in Euclidean space $\mathbf{R}^N$, $N \geq 3$, with boundary $\partial\Omega$ of class $C^{1,1}$. We consider a second-order, *uniformly elliptic* differential operator A with real *discontinuous* coefficients of the form

$$Au := -\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x)\frac{\partial u}{\partial x_i} + c(x)u.$$

More precisely, we assume that the coefficients $a^{ij}(x)$, $b^i(x)$ and $c(x)$ of the differential operator A satisfy the following three conditions:

(1) $a^{ij}(x) \in L^\infty(\Omega)$, $a^{ij}(x) = a^{ji}(x)$ for almost all $x \in \Omega$ and there exist a constant $\lambda > 0$ such that

$$\frac{1}{\lambda}|\xi|^2 \leq \sum_{i,j=1}^{N} a^{ij}(x)\xi_i\xi_j \leq \lambda|\xi|^2 \quad \text{for almost all } x \in \Omega \text{ and all } \xi \in \mathbf{R}^N.$$

(2) $b^i(x) \in L^\infty(\Omega)$ for all $1 \leq i \leq N$.

(3) $c(x) \in L^\infty(\Omega)$ and $c(x) \geq 0$ for almost all $x \in \Omega$.

First, we state a variant of the *weak maximum principle* in the framework of L^p Sobolev spaces, due to Bony [5] ([25, Chapter 3, Lemma 3.25]):

Theorem A.1 (the weak maximum principle)**.** *If a function u in $W^{2,p}(\Omega)$, with $N < p < \infty$, satisfies the condition*

$$Au(x) \leq 0 \quad \text{for almost all } x \in \Omega,$$

then we have the inequality

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u^+,$$

where

$$u^+(x) = \max\{u(x), 0\} \quad \text{for } x \in \overline{\Omega}.$$

Secondly, the Hopf *boundary point lemma* reads as follows ([25, Chapter 3, Lemma 3.26]):

Lemma A.2 (Hopf)**.** *Assume that a function $u \in W^{2,p}(\Omega)$, $N < p < \infty$, satisfies the condition*

$$Au(x) \leq 0 \quad \text{for almost all } x \in \Omega.$$

If $u(x)$ attains a non-negative, strict local maximum at a point x_0' of $\partial\Omega$, then we have the inequality

$$\frac{\partial u}{\partial \boldsymbol{\nu}}(x_0') > 0$$

(see Figure 1).

Finally, we can obtain the following *strong maximum principle* for the operator A ([5, Théorème 2], [25, Chapter 3, Theorem 3.27]):

Theorem A.3 (the strong maximum principle). *Assume that a function $u \in W^{2,p}(\Omega)$, $N < p < \infty$, satisfies the condition*

$$Au(x) \leq 0 \quad \text{for almost all } x \in \Omega.$$

If $u(x)$ attains a non-negative maximum at an interior point x_0 *of Ω, then it is a (non-negative) constant function.*

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Received xxxx 20xx; revised xxxx 20xx.

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