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A strong maximum principle for globally hypoelliptic operators

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Dedicated to Professor Izumi Kubo on the occasion of his 80th birthday

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Abstract In this paper we study the strong maximum principle for *globally hypoelliptic* differential operators of second-order, and reveal the underlying analytical mechanism of propagation of maximums in terms of the Lie algebra generated by diffusion vector fields and the Fichera function. Our formulation of the strong maximum principle is *coordinate-free*. The results here may be applied to questions of uniqueness for degenerate elliptic boundary value problems on a manifold. Furthermore, the mechanism of propagation of maximums plays an important role in the interpretation and study of Markov processes from the viewpoint of functional analysis.

Keywords Globally hypoelliptic operator · strong maximum principle · Lie algebra · Fichera function

Mathematics Subject Classification (2010) 35J70 · 35H10 · 58G20 · 35K65

1 Introduction

The purpose of this paper is to study an intimate connection between Markov processes and partial differential equations. This will play an important role in the study of Markov processes via the theory of partial differential equations (see [31] and [32]). The present paper is an expanded and revised version of the previous works [28] through [31]. inspired by a probabilistic characterization of the strong maximum principle due to Stroock–Varadhan [27].

We begin with the following elementary result:

> *Let I be an open interval of* $\mathbf{R}$. *If* $u \in C^2(I)$, $\frac{d^2u}{dx^2}(x) \geq 0$ *in* I *and if* u *takes its maximum at a point of* I, *then* u *is a constant.*

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This result can be extended to the n-dimensional case, with the operator d^2/dx^2 replaced by the usual Laplacian

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Let Ω be a connected open subset of $\mathbf{R}^n$. If $u \in C^2(\Omega)$, $\Delta u \geq 0$ in Ω and u takes its maximum at a point of Ω, then u is a constant. (1.1)

Result (1.1) is well known by the name of the *strong maximum principle* for the Laplacian.

Let L be a second-order, differential operator with real coefficients, defined in a *connected* open subset Ω of Euclidean space $\mathbf{R}^n$, such that

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u. \tag{1.2}$$

Here:

(1) The $a^{ij}(x)$ are the components of a C^∞ symmetric contravariant tensor of type $\binom{2}{0}$ on Ω and satisfy the condition

$$\sum_{i,j=1}^{n} a^{ij}(x)\, \xi_i \xi_j \geq 0 \quad \text{for all } x \in \Omega \text{ and all } \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbf{R}^n.$$

(2) $b^i \in C^\infty(\Omega)$ for $1 \leq i \leq n$.
(3) $c \in C^\infty(\Omega)$ with $L1(x) = c(x) \leq 0$ in Ω.

A *strong maximum principle* for the differential operator L is a statement of the form:

If $u \in C^2(\Omega)$, $Lu \geq 0$ in Ω and if u takes its greatest non-negative value M at a point x of Ω, then it follows that $u \equiv M$ in Ω. (SMP)

By using a modification of the technique originally introduced by E. Hopf [12] for elliptic operators and later adapted by L. Nirenberg [19] and A. Friedman [8] for parabolic ones, it can be proved that the maximum is propagated along a finite chain of trajectories of the diffusion vector fields

$$\pm X_j = \pm \sum_{k=1}^{n} a^{jk}(x) \frac{\partial}{\partial x_k}, \quad 1 \leq j \leq n$$

(see part (i) of Lemma 4.1 below).

On the other hand, Chow's theorem [5, Satz C] states that if the Lie algebra

$$\mathcal{L}\,(X_1, X_2, \ldots, X_n)$$

over $\mathbf{R}$ generated by the family $\{X_j\}_{j=1}^n$ of vector fields has rank n at a point x of Ω, then there exists a neighborhood $V(x)$ of x such that *any point* of $V(x)$ can be joined to x by a finite chain of trajectories of the vector fields $\{\pm X_j\}_{j=1}^n$. A *coordinate-free* definition of the Lie algebra $\mathcal{L}\,(X_1, X_2, \ldots, X_n)$ is given by the formulas (2.4) and (2.5) in the next section.

In this way, we arrive at the following well-known result (see [2, Théorème 3.1], [21, Theorem 3.1.11]):

Theorem 1.1 *If the Lie algebra $\mathcal{L}(X_1, X_2, \ldots, X_n)$ has rank n at* every point *of Ω, then the strong maximum principle (SMP) holds true for the operator L.*

We give a typical example for Theorem 1.1 in the plane $\mathbf{R}^2$ ($n = 2$):

Example 1.1 Let $\Omega = \{(x_1, x_2) : -1 < x_1 < 1,\ -1 < x_2 < 1\}$ (the rectangle). If we let

$$L_1 = \frac{\partial^2}{\partial x_1^2} + x_1^2 \frac{\partial^2}{\partial x_2^2},$$

then we have the formulas (see Remark (2.1))

$$X_1 = \frac{\partial}{\partial x_1},$$
$$X_2 = x_1^2 \frac{\partial}{\partial x_2}.$$

It is easy to see that the vector fields

$$X_1 = \frac{\partial}{\partial x_1},$$
$$[X_1, [X_1, X_2]] = 2\frac{\partial}{\partial x_2}$$

span all vector fields at every point of Ω. Namely, we have the assertion

$$\operatorname{rank} \mathcal{L}(X_1, X_2) = 2 \quad \text{at every point of } \Omega.$$

The purpose of this paper is to study the case where the rank of Lie algebra $\mathcal{L}(X_1, X_2, \ldots, X_n)$ is less than n in some subset $\mathcal{M}$ of Ω under the rank condition (H) which is formulated in Section 2. In Section 3, we impose a geometrical assumption on the exceptional set $\mathcal{M}$ (assumption (3.1)) in the rank condition (H). Then, following Fichera [7] we introduce the Fichera function $b(x)$ by formula (3.7) (Lemmas 3.1 and 3.2). In Section 4 we give sufficient conditions for the strong maximum principle (SMP) to hold true for the operator L in terms of the Lie algebra $\mathcal{L}(X_1, X_2, \ldots, X_n)$ and the Fichera function $b(x)$ (Theorem 4.1). In Section 5, by making use of the *probabilistic representation formula* (5.6) for the Dirichlet problem we give (partially) necessary conditions for the strong maximum principle (SMP) to hold true for the operator L (Theorem 5.1). In Section 6 we formulate another maximum principle, called the *sharp maximum principle* (PMP), for the operator L in terms of *subunit vectors* whose notion is introduced by Fefferman–Phong [6] (condition (6.1)). We give four simple examples of the strong maximum principle (SMP) via the sharp maximum principle (PMP) (Examples 6.1 through 6.4).

2 Differential operators and Lie algebras

In this section, we consider the case where Ω is an n-dimensional, C^∞ *manifold.* To state a hypothesis concerning the differential operator L *not assumed* to be written as sums of squares of vector fields, we introduce some notation and definitions.

2.1 The Lie algebra $\mathcal{L}(X_1, X_2, \ldots, X_n)$

We denote by $\Gamma(\Omega, T^*(\Omega))$ and by $\Gamma(\Omega, T(\Omega))$ the space of C^∞ covector fields on Ω and the space of C^∞ vector fields on Ω, respectively.

First, we consider the space $\Gamma(\Omega, T(\Omega) \otimes_S T(\Omega))$ of C^∞ symmetric contravariant tensor fields of type $\binom{2}{0}$ on Ω. If $x = (x_1, x_2, \ldots, x_n)$ is a local coordinate system in a chart on Ω, we define the symmetric tensor product $\otimes_S$ as follows (see [18, Lemma 2.2.13]):

$$\frac{\partial}{\partial x_i} \otimes_S \frac{\partial}{\partial x_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} \otimes \frac{\partial}{\partial x_i}\right) \quad \text{for } 1 \leq i, j \leq n. \tag{2.1}$$

For the leading coefficients $a^{ij}(x)$ of L, we let

$$X_j = \sum_{k=1}^{n} a^{jk}(x) \frac{\partial}{\partial x_k} \quad \text{for } 1 \leq j \leq n.$$

Since we have the assertion

$$\Phi = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial}{\partial x_i} \otimes_S \frac{\partial}{\partial x_j} \in \Gamma\left(\Omega, T(\Omega) \otimes_S T(\Omega)\right), \tag{2.2}$$

we can define a mapping

$$\Psi : \Gamma\left(\Omega, T^*(\Omega)\right) \longrightarrow \Gamma\left(\Omega, T(\Omega)\right)$$

by the formula

$$\Psi(\zeta) = \Phi(\zeta, \cdot) \quad \text{for every } \zeta \in \Gamma(\Omega, T^*(\Omega)). \tag{2.3}$$

In terms of local coordinates $x = (x_1, x_2, \ldots, x_n)$ in a chart on Ω, we have the formula

$$\Gamma\left(\Omega, T^*(\Omega)\right) \ni \zeta = \sum_{i=1}^{n} \zeta_i(x)\, dx_i \longmapsto \Psi(\zeta) = \sum_{i,j=1}^{n} a^{ij}(x)\, \zeta_i(x) \frac{\partial}{\partial x_j} \in \Gamma\left(\Omega, T(\Omega)\right).$$

The proof of well-definedness of formulas (2.2) and (2.3) will be given in the next Subsection 2.2 (Lemma 2.1), due to its length.

In this way, we can associate the family $\{X_j\}_{j=1}^n$ with a family $\mathcal{Y}$ of C^∞ vector fields on Ω defined as follows:

$$\mathcal{Y} = \text{ the image of } \Psi = \left\{\Psi(\zeta) : \zeta \in \Gamma\left(\Omega, T^*(\Omega)\right)\right\}. \tag{2.4}$$

In this paper, the Lie algebra $\mathcal{L}(X_1, X_2, \ldots, X_n)$ is meant as the Lie algebra $\mathcal{L}(\mathcal{Y})$ over $\mathbf{R}$ generated by the family $\mathcal{Y}$:

$$\mathcal{L}(X_1, X_2, \ldots, X_n) = \mathcal{L}(\mathcal{Y}). \tag{2.5}$$

It should be emphasized that this definition of the Lie algebra $\mathcal{L}(X_1, X_2, \ldots, X_n)$ is *coordinate-free*.

Remark 2.1 If the differential operator L is of the form

$$L = \sum_{j=1}^{r} Y_j^2 + Z,$$

where Y_1, Y_2, ..., Y_r and Z are smooth vector fields on Ω, then the Lie algebra $\mathcal{L}(X_1, X_2, \dots, X_n)$ may be replaced by the Lie algebra $\mathcal{L}(Y_1, Y_2, \dots, Y_r)$ generated by the family $\{Y_j\}_{j=1}^{r}$, just as in Hörmander [14] and Bony [2] (see [31, the proof of Theorem 7.2.4]).

In this paper, we let

$$\mathcal{M} = \{x \in \Omega : \text{the rank of } \mathcal{L}(X_1, X_2, \dots, X_n) \text{ at } x \text{ is less than } n\},$$

and impose the following *rank condition* on the Lie algebra $\mathcal{L}(X_1, X_2, \dots, X_n)$ defined by formulas (2.4) and (2.5):

(H) The exceptional set $\mathcal{M}$ is a closed *hypersurface* in Ω such that

$$\operatorname{rank} \mathcal{L}(X_1, X_2, \dots, X_n) = n \quad \text{outside } \mathcal{M}.$$

2.2 The well-definedness of formulas (2.2) and (2.3)

Now we are in a position to prove the well-definedness of the mapping Ψ defined by formulas (2.2) and (2.3):

Lemma 2.1 *Let x_0 be an arbitrary point of Ω and consider, in a neighborhood U of x_0, a C^2-diffeomorphism*

$$y = F(x) = (F_1(x), F_2(x), \dots, F_n(x)) \quad \text{for } x \in U. \tag{2.6}$$

Assume that

$$\zeta = \sum_{i=1}^{n} \zeta_i(x)\, dx_i = \sum_{\ell=1}^{n} \eta_\ell(y)\, dy_\ell \in \Gamma\left(\Omega, T^*(\Omega)\right). \tag{2.7}$$

Then we have, by formulas (2.6) and (2.7),

$$\begin{aligned} \Psi(\zeta) &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a^{ij}(x)\, \zeta_i(x) \right) \frac{\partial}{\partial x_j} \qquad (2.8) \\ &= \sum_{m=1}^{n} \left(\sum_{\ell=1}^{n} \widetilde{a}^{\ell m}(y)\, \eta_\ell(y) \right) \frac{\partial}{\partial y_m} \in \Gamma\left(\Omega, T(\Omega)\right), \end{aligned}$$

where the $\widetilde{a}^{\ell m}(y)$ are the components of a C^∞ symmetric contravariant tensor of type $\binom{2}{0}$ defined by the formulas

$$\begin{aligned} \widetilde{a}^{\ell m}(y) &= \sum_{i,j=1}^{n} a^{ij}\left(F^{-1}(y)\right) \frac{\partial y_\ell}{\partial x_i} \frac{\partial y_m}{\partial x_j} \qquad (2.9) \\ &= \sum_{i,j=1}^{n} a^{ij}\left(F^{-1}(y)\right) \frac{\partial F_\ell}{\partial x_i} \frac{\partial F_m}{\partial x_j} \quad \text{for } 1 \le \ell, m \le n. \end{aligned}$$

Proof First, since we have, by formula (2.6),

$$\frac{\partial}{\partial x_i} = \sum_{\ell=1}^{n} \frac{\partial y_\ell}{\partial x_i} \frac{\partial}{\partial y_\ell} = \sum_{\ell=1}^{n} \frac{\partial F_\ell}{\partial x_i} \frac{\partial}{\partial y_\ell} \quad \text{for } 1 \leq i \leq n. \tag{2.10a}$$

$$\frac{\partial}{\partial x_j} = \sum_{m=1}^{n} \frac{\partial y_m}{\partial x_j} \frac{\partial}{\partial y_m} = \sum_{m=1}^{n} \frac{\partial F_m}{\partial x_j} \frac{\partial}{\partial y_m} \quad \text{for } 1 \leq j \leq n, \tag{2.10b}$$

we obtain from formula (2.1) that

$$\begin{aligned}
&\frac{\partial}{\partial x_i} \otimes_S \frac{\partial}{\partial x_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j} + \frac{\partial}{\partial x_j} \otimes \frac{\partial}{\partial x_i}\right) \\
&= \frac{1}{2}\left(\sum_{\ell=1}^{n} \frac{\partial y_\ell}{\partial x_i} \frac{\partial}{\partial y_\ell} \otimes \sum_{m=1}^{n} \frac{\partial y_m}{\partial x_j} \frac{\partial}{\partial y_m} + \sum_{m=1}^{n} \frac{\partial y_m}{\partial x_j} \frac{\partial}{\partial y_m} \otimes \sum_{\ell=1}^{n} \frac{\partial y_\ell}{\partial x_i} \frac{\partial}{\partial y_\ell}\right) \\
&= \sum_{\ell,m=1}^{n} \frac{\partial y_\ell}{\partial x_i} \frac{\partial y_m}{\partial x_j} \cdot \frac{1}{2}\left(\frac{\partial}{\partial y_\ell} \otimes \frac{\partial}{\partial y_m} + \frac{\partial}{\partial y_m} \otimes \frac{\partial}{\partial y_\ell}\right) \\
&= \sum_{\ell,m=1}^{n} \frac{\partial y_\ell}{\partial x_i} \frac{\partial y_m}{\partial x_j} \left(\frac{\partial}{\partial y_\ell} \otimes_S \frac{\partial}{\partial y_m}\right).
\end{aligned}$$

Hence we have the formula

$$\begin{aligned}
\Phi &= \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial}{\partial x_i} \otimes_S \frac{\partial}{\partial x_j} \\
&= \sum_{i,j=1}^{n} a^{ij}(x) \left(\sum_{\ell,m=1}^{n} \frac{\partial y_\ell}{\partial x_i} \frac{\partial y_m}{\partial x_j} \left(\frac{\partial}{\partial y_\ell} \otimes_S \frac{\partial}{\partial y_m}\right)\right) \\
&= \sum_{\ell,m=1}^{n} \left(\sum_{i,j=1}^{n} a^{ij}\left(F^{-1}(y)\right) \frac{\partial F_\ell}{\partial x_i} \frac{\partial F_m}{\partial x_j}\right) \frac{\partial}{\partial y_\ell} \otimes_S \frac{\partial}{\partial y_m} \\
&= \sum_{\ell,m=1}^{n} \widetilde{a}^{\ell m}(y) \frac{\partial}{\partial y_\ell} \otimes_S \frac{\partial}{\partial y_m}.
\end{aligned}$$

This proves the desired assertion (2.2).

Secondly, by formula (2.7) it follows that

$$\zeta = \sum_{\ell=1}^{n} \eta_\ell(y)\, dy_\ell = \sum_{\ell=1}^{n} \eta_\ell(y) \left(\sum_{i=1}^{n} \frac{\partial y_\ell}{\partial x_i}\, dx_i\right) = \sum_{i=1}^{n} \left(\sum_{\ell=1}^{n} \frac{\partial y_\ell}{\partial x_i}\, \eta_\ell(y)\right) dx_i,$$

so that

$$\zeta_i(x) = \sum_{\ell=1}^{n} \frac{\partial y_\ell}{\partial x_i}\, \eta_\ell(y) = \sum_{\ell=1}^{n} \frac{\partial F_\ell}{\partial x_i}\, \eta_\ell(y) \quad \text{for } 1 \leq i \leq n. \tag{2.11}$$

Therefore, by using formulas (2.10b) and (2.11) we obtain that

$$\sum_{i,j=1}^{n} a^{ij}(x)\, \zeta_i(x) \frac{\partial}{\partial x_j} = \sum_{i,j=1}^{n} a^{ij}(x) \left(\sum_{\ell=1}^{n} \frac{\partial y_\ell}{\partial x_i}\, \eta_\ell(y)\right) \left(\sum_{m=1}^{n} \frac{\partial y_m}{\partial x_j} \frac{\partial}{\partial y_m}\right)$$

$$= \sum_{\ell,m=1}^{n} \left(\sum_{i,j=1}^{n} a^{ij} \left(F^{-1}(y) \right) \frac{\partial y_\ell}{\partial x_i} \frac{\partial y_m}{\partial x_j} \right) \eta_\ell(y) \frac{\partial}{\partial y_m}$$
$$= \sum_{\ell,m=1}^{n} \widetilde{a}^{\ell m}(y)\, \eta_\ell(y) \frac{\partial}{\partial y_m}.$$

This proves the desired assertion (2.8).

The proof of Lemma 2.1 is complete. □

3 The Fichera function

In this section, we impose the following geometrical assumption on the exceptional set $\mathcal{M}$ in the rank condition (H) (see Figure 3.1):

The set $\mathcal{M}$ is a closed *hypersurface* defined locally by the equation $\mathcal{M} = \{x \in \Omega : \varphi(x) = 0\}$ with $\operatorname{grad} \varphi \neq 0$. (3.1)

Here

$$\boldsymbol{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) = \operatorname{grad} \varphi$$

is the *normal* to the hypersurface $\mathcal{M}$.

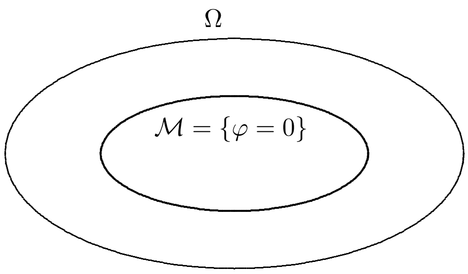


Fig. 3.1 The closed hypersurface $\mathcal{M}$ in Ω

First, we introduce two disjoint subsets Σ_3 and Σ^0 of $\mathcal{M}$ respectively by the formulas

$$\Sigma_3 = \left\{ x \in \mathcal{M} : \sum_{i,j=1}^{n} a^{ij}(x)\, \nu_i \nu_j > 0 \right\},$$
$$\Sigma^0 = \left\{ x \in \mathcal{M} : \sum_{i,j=1}^{n} a^{ij}(x)\, \nu_i \nu_j = 0 \right\} = \mathcal{M} \setminus \Sigma_3.$$

The next lemma justifies the definition of the sets Σ_3 and Σ^0 (cf. [21, Theorem 1.1.1]):

Lemma 3.1 *The sets Σ_3 and Σ^0 are invariant under C^2-diffeomorphisms preserving normal vectors to the set $\mathcal{M}$.*

Proof Let x_0 be an arbitrary point of $\mathcal{M}$ and consider, in a neighborhood U of x_0, a C^2-diffeomorphism

$$y = F(x) = (F_1(x), F_2(x), \dots, F_n(x)), \quad x \in U, \tag{3.2}$$

which preserves the normal vector $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_n)$ to $\mathcal{M}$. Then we have, by assumption (3.1),

$$\mathcal{M} \cap U = \{y \in U : \Phi(y) = 0\}, \quad \Phi = \varphi \circ F^{-1}, \tag{3.3}$$

and

$$\nu_i = \frac{\partial \varphi}{\partial x_i} = \sum_{\ell=1}^{n} \frac{\partial \Phi}{\partial y_\ell} \frac{\partial y_\ell}{\partial x_i} = \sum_{\ell=1}^{n} \frac{\partial \Phi}{\partial y_\ell} \frac{\partial F_\ell}{\partial x_i} \quad \text{for } 1 \leq i \leq n. \tag{3.4}$$

Furthermore, we can rewrite the differential operator L in the form

$$\begin{aligned} L &= \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x_i} + c(x) \\ &= \sum_{\ell,m=1}^{n} \left(\sum_{i,j=1}^{n} a^{ij}\left(F^{-1}(y)\right) \frac{\partial F_\ell}{\partial x_i} \frac{\partial F_m}{\partial x_j} \right) \frac{\partial^2}{\partial y_\ell \partial y_m} \\ &\quad + \sum_{\ell=1}^{n} \left(\sum_{i=1}^{n} b^i\left(F^{-1}(y)\right) \frac{\partial F_\ell}{\partial x_i} + \sum_{i,j=1}^{n} a^{ij}\left(F^{-1}(y)\right) \frac{\partial^2 F_\ell}{\partial x_i \partial x_j} \right) \frac{\partial}{\partial y_\ell} \\ &\quad + c\left(F^{-1}(y)\right) \\ &:= \sum_{\ell,m=1}^{n} \widetilde{a}^{\ell m}(y) \frac{\partial^2}{\partial y_\ell \partial y_m} + \sum_{\ell=1}^{n} \widetilde{b}^{\ell}(y) \frac{\partial}{\partial y_\ell} + \widetilde{c}(y). \end{aligned} \tag{3.5}$$

However, we have, by formulas (3.4) and (2.9),

$$\begin{aligned} \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} &= \sum_{i,j=1}^{n} a^{ij}(x) \left(\sum_{\ell,m=1}^{n} \frac{\partial \Phi}{\partial y_\ell} \frac{\partial y_\ell}{\partial x_i} \frac{\partial \Phi}{\partial y_m} \frac{\partial y_m}{\partial x_j} \right) \\ &= \sum_{\ell,m=1}^{n} \left(\sum_{i,j=1}^{n} a^{ij}\left(F^{-1}(y)\right) \frac{\partial y_\ell}{\partial x_i} \frac{\partial y_m}{\partial x_j} \right) \frac{\partial \Phi}{\partial y_\ell} \frac{\partial \Phi}{\partial y_m} \\ &= \sum_{\ell,m=1}^{n} \widetilde{a}^{\ell m}(y) \frac{\partial \Phi}{\partial y_\ell} \frac{\partial \Phi}{\partial y_m}. \end{aligned} \tag{3.6}$$

This proves the invariance of the sets Σ_3 and Σ^0, since the diffeomorphism F preserves normal vectors and so $\operatorname{grad} \Phi$ has the same direction as the inward normal $\boldsymbol{\nu}$. Indeed, we obtain from formula (3.6) that

$$\sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} > 0 \iff \sum_{\ell,m=1}^{n} \widetilde{a}^{\ell m}(y) \frac{\partial \Phi}{\partial y_\ell} \frac{\partial \Phi}{\partial y_m} > 0,$$

$$\sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} = 0 \iff \sum_{\ell,m=1}^{n} \widetilde{a}^{\ell m}(y) \frac{\partial \Phi}{\partial y_\ell} \frac{\partial \Phi}{\partial y_m} = 0.$$

The proof of Lemma 3.1 is complete. □

Following Fichera [7], we introduce a function $b(x)$, called the *Fichera function*, defined by the formula

$$b(x) = \sum_{i=1}^{n} \left(b^i(x) - \sum_{j=1}^{n} \frac{\partial a^{ij}}{\partial x_j}(x) \right) \nu_i = \sum_{i=1}^{n} \left(b^i(x) - \sum_{j=1}^{n} \frac{\partial a^{ij}}{\partial x_j}(x) \right) \frac{\partial \varphi}{\partial x_i}. \tag{3.7}$$

The next lemma justifies the definition (3.7) of the Fichera function $b(x)$ on the set Σ^0 (cf. [21, Lemma 1.1.1]):

Lemma 3.2 *The Fichera function $b(x)$ defined on the set Σ^0 is invariant under C^2-diffeomorphisms preserving normal vectors to the set $\mathcal{M}$.*

Proof Let x_0 be an arbitrary point of $\mathcal{M}$ and consider, in a neighborhood U of x_0, a C^2-diffeomorphism

$$y = F(x) = (F_1(x), F_2(x), \dots, F_n(x)), \tag{3.2}$$

as in the proof of Lemma 3.1.

Then we have, by formulas (3.5) and (3.4),

$$\begin{aligned}
\widetilde{b}(y) &= \sum_{\ell=1}^{n} \left(\widetilde{b}^\ell(y) - \sum_{m=1}^{n} \frac{\partial \widetilde{a}^{\ell m}}{\partial y_m}(y) \right) \frac{\partial \Phi}{\partial y_\ell} \\
&= \sum_{\ell=1}^{n} \left(\sum_{i=1}^{n} b^i(x) \frac{\partial F_\ell}{\partial x_i} + \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 F_\ell}{\partial x_i \partial x_j} \right) \frac{\partial \Phi}{\partial y_\ell} - \sum_{\ell=1}^{n} \sum_{m=1}^{n} \frac{\partial \widetilde{a}^{\ell m}}{\partial y_m}(y) \frac{\partial \Phi}{\partial y_\ell} \\
&= \sum_{i=1}^{n} b^i(x) \frac{\partial \varphi}{\partial x_i} + \sum_{\ell=1}^{n} \left(\sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 F_\ell}{\partial x_i \partial x_j} \right) \frac{\partial \Phi}{\partial y_\ell} - \sum_{\ell=1}^{n} \sum_{m=1}^{n} \frac{\partial \widetilde{a}^{\ell m}}{\partial y_m}(y) \frac{\partial \Phi}{\partial y_\ell}.
\end{aligned} \tag{3.8}$$

Moreover, we can calculate the last term in formula (3.8) as follows:

$$\begin{aligned}
&\sum_{\ell=1}^{n} \sum_{m=1}^{n} \frac{\partial \widetilde{a}^{\ell m}}{\partial y_m}(y) \frac{\partial \Phi}{\partial y_\ell} \\
&= \sum_{\ell=1}^{n} \sum_{m=1}^{n} \frac{\partial}{\partial y_m} \left(\sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial F_\ell}{\partial x_i} \frac{\partial F_m}{\partial x_j} \right) \frac{\partial \Phi}{\partial y_\ell} \\
&= \sum_{m=1}^{n} \sum_{i,j=1}^{n} \frac{\partial a^{ij}}{\partial y_m} \frac{\partial F_m}{\partial x_j} \frac{\partial \varphi}{\partial x_i} + \sum_{\ell=1}^{n} \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 F_\ell}{\partial x_i \partial x_j} \frac{\partial \Phi}{\partial y_\ell} \\
&\quad + \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial \varphi}{\partial x_i} \sum_{m=1}^{n} \frac{\partial^2 F_m}{\partial y_m \partial x_j},
\end{aligned} \tag{3.9}$$

since we have, by formulas (2.10b),

$$\sum_{m=1}^{n} \frac{\partial^2 F_\ell}{\partial y_m \partial x_i} \frac{\partial F_m}{\partial x_j} = \sum_{m=1}^{n} \frac{\partial F_m}{\partial x_j} \frac{\partial}{\partial y_m} \left(\frac{\partial F_\ell}{\partial x_i} \right) = \frac{\partial^2 F_\ell}{\partial x_i \partial x_j}.$$

Therefore, by combining formulas (3.8) and (3.9) we obtain that

$$\widetilde{b}(y) = \sum_{\ell=1}^{n} \left(\widetilde{b}^\ell(y) - \sum_{m=1}^{n} \frac{\partial \widetilde{a}^{\ell m}}{\partial y_m}(y) \right) \frac{\partial \Phi}{\partial y_\ell} \tag{3.10}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} b^i(x)\frac{\partial \varphi}{\partial x_i} + \sum_{\ell=1}^{n}\left(\sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial^2 F_\ell}{\partial x_i \partial x_j}\right)\frac{\partial \Phi}{\partial y_\ell} - \sum_{\ell=1}^{n}\sum_{m=1}^{n}\frac{\partial \widetilde{a}^{\ell m}}{\partial y_m}(y)\frac{\partial \Phi}{\partial y_\ell} \\
&= \sum_{i=1}^{n} b^i(x)\frac{\partial \varphi}{\partial x_i} - \sum_{i,j=1}^{n}\left(\sum_{m=1}^{n}\frac{\partial a^{ij}}{\partial y_m}\frac{\partial F_m}{\partial x_j}\right)\frac{\partial \varphi}{\partial x_i} - \sum_{i,j=1}^{n} a^{ij}(x)\frac{\partial \varphi}{\partial x_i}\sum_{m=1}^{n}\frac{\partial^2 F_m}{\partial y_m \partial x_j} \\
&= \sum_{i=1}^{n} b^i(x)\frac{\partial \varphi}{\partial x_i} - \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{\partial a^{ij}}{\partial x_j}\right)\frac{\partial \varphi}{\partial x_i} - \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a^{ij}(x)\frac{\partial \varphi}{\partial x_i}\right)\sum_{m=1}^{n}\frac{\partial^2 F_m}{\partial y_m \partial x_j} \\
&= \sum_{i=1}^{n}\left(b^i(x) - \sum_{j=1}^{n}\frac{\partial a^{ij}}{\partial x_j}\right)\frac{\partial \varphi}{\partial x_i} - \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a^{ij}(x)\frac{\partial \varphi}{\partial x_i}\right)\sum_{m=1}^{n}\frac{\partial^2 F_m}{\partial y_m \partial x_j} \\
&= b(x) - \sum_{j=1}^{n}\left(\sum_{i=1}^{n} a^{ij}(x)\frac{\partial \varphi}{\partial x_i}\right)\sum_{m=1}^{n}\frac{\partial^2 F_m}{\partial y_m \partial x_j}.
\end{aligned}
$$

On the other hand, by applying Oleĭnik–Radkevič [21, Corollary 1 to Lemma 1.7.1] we have the inequality

$$
\left|\sum_{i=1}^{n} a^{ij}(x)\frac{\partial \varphi}{\partial x_i}\right|^2 \leq 2\, a^{jj}(x) \sum_{i,k=1}^{n} a^{ik}(x)\frac{\partial \varphi}{\partial x_i}\frac{\partial \varphi}{\partial x_k} \quad \text{for } 1 \leq j \leq n.
$$

Hence, on the set

$$
\varSigma^0 = \left\{ x \in \mathcal{M} : \sum_{i,k=1}^{n} a^{ik}(x)\, \nu_i\, \nu_k = 0 \right\},
$$

it follows that

$$
\sum_{i=1}^{n} a^{ij}(x)\frac{\partial \varphi}{\partial x_i} = 0 \quad \text{for } 1 \leq j \leq n.
$$

Namely, we find that the last term in the right-hand side of formula (3.10) vanishes on the set $\varSigma^0$:

$$
\sum_{j=1}^{n}\left(\sum_{i=1}^{n} a^{ij}(x)\frac{\partial \varphi}{\partial x_i}\right)\sum_{m=1}^{n}\frac{\partial^2 F_m}{\partial y_m \partial x_j} = 0 \quad \text{on } \varSigma^0. \tag{3.11}
$$

In this way, we obtain from assertion (3.11) and formula (3.10) that

$$
b(x) = \widetilde{b}(y) \quad \text{on } \varSigma^0.
$$

This proves the invariance of the Fichera function $b(x)$ on the set $\varSigma^0$.

The proof of Lemma 3.2 is complete. □

4 Sufficient conditions

Our first main result is stated as follows (cf. [21, Theorem 3.1.13]):

Theorem 4.1 (Sufficiency) *Assume that the exceptional set $\mathcal{M}$ is a closed, connected* hypersurface *such that the complement $\Omega \setminus \mathcal{M}$ has two connected components. Furthermore, we make the following assumption:*

(A) The function

$$\sum_{i,j=1}^{n} a^{ij}(x')\,\nu_i\,\nu_j + \left|b(x')\right|$$

does not vanish identically on the hypersurface $\mathcal{M}$ and, for each point x' of the set

$$\mathcal{N} = \left\{ x' \in \mathcal{M} : \sum_{i,j=1}^{n} a^{ij}(x')\,\nu_i\,\nu_j + \left|b(x')\right| = 0 \right\},$$

there exists a point x'_0 of $\mathcal{M} \setminus \mathcal{N}$ such that x' and x'_0 are connected by a finite chain of trajectories of the vector fields $\{\pm X_j\}_{j=1}^{n}$ (see Figure 4.1).

Then the strong maximum principle (SMP) holds true for the operator L if and only if one of the following two conditions (B) and (C) is satisfied (see Figure 4.2):

(B) The function

$$\sum_{i,j=1}^{n} a^{ij}(x')\,\nu_i\,\nu_j$$

does not vanish identically on the hypersurface $\mathcal{M}$.

(C) If the function

$$\sum_{i,j=1}^{n} a^{ij}(x')\,\nu_i\,\nu_j$$

vanishes identically on the hypersurface $\mathcal{M}$, then there exist at least *two points x'_+ and x'_- of $\mathcal{M}$ such that*

$$b\left(x'_\pm\right) = \sum_{i=1}^{n} \left(b^i\left(x'_\pm\right) - \sum_{j=1}^{n} \frac{\partial a^{ij}}{\partial x_j}\left(x'_\pm\right) \right) \nu_i \gtrless 0.$$

Remark 4.1 The assumption (A) implies that the maximum starting at $\mathcal{N}$ (where the maximum may stay) *goes out of* $\mathcal{N}$. On the other hand, the maximum can traverse $\mathcal{M}$ from *every side* of $\mathcal{M}$ if and only if one of the conditions (B) and (C) is satisfied.

We give two simple examples for Theorem 4.1 in the plane $\mathbf{R}^2$ ($n = 2$):

Example 4.1 Let $\Omega = \{(x_1, x_2) : -1 < x_1 < 1,\ -1 < x_2 < 1\}$ (the rectangle). We let

$$L_2 = \frac{\partial^2}{\partial x_1^2} + \exp\left[-\frac{1}{x_1^2}\right] \frac{\partial^2}{\partial x_2^2}.$$

Then we have the assertions

$$\mathcal{M} = \{(0, x_2) : -1 < x_2 < 1\} = \{0\} \times (-1, 1),$$

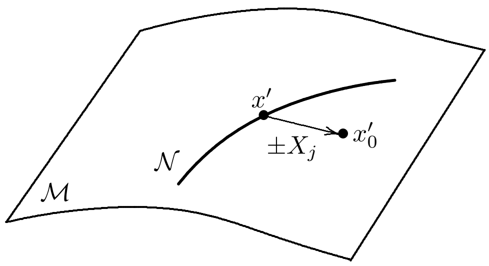


Fig. 4.1 The condition (A)

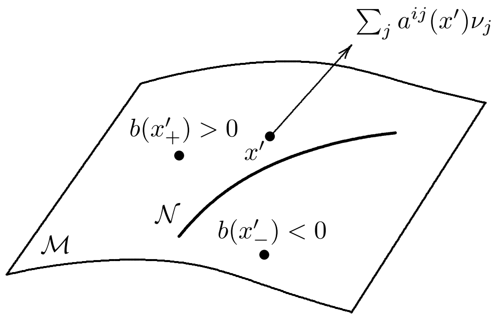


Fig. 4.2 The conditions (B) and (C)

$$\mathcal{N} = \emptyset.$$

It is easy to see that the conditions (A) and (B) hold true. Indeed, it suffices to note that

$$\begin{cases} \sum_{i,j=1}^{2} a^{ij}\,(x_1,x_2)\,\nu_i\,\nu_j \equiv 1 & \text{on } \mathcal{M}, \\ b\,(x_1,x_2) \equiv 0 & \text{on } \mathcal{M}. \end{cases}$$

Example 4.2 Let $\Omega = \{(x_1,x_2) : -1 < x_1 < 1,\ -1 < x_2 < 1\}$ (the rectangle). We let

$$L_3 = \exp\left[-\frac{1}{x_1^2}\right]\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + x_2\frac{\partial}{\partial x_1}.$$

Then we have the assertions

$$\mathcal{M} = \{(0,x_2) : -1 < x_2 < 1\} = \{0\} \times (-1,1),$$
$$\mathcal{N} = \{(0,0)\}.$$

It is easy to see that the conditions (A) and (C) hold true. Indeed, it suffices to note that

$$\begin{cases} \sum_{i,j=1}^{2} a^{ij}\,(x_1,x_2)\,\nu_i\,\nu_j \equiv 0 & \text{on } \mathcal{M}, \\ b\,(x_1,x_2) = x_2 & \text{on } \mathcal{M}. \end{cases}$$

4.1 Proof of Theorem 4.1

Assume that

$$\begin{cases} u \in C^2(\Omega), \\ Lu \geq 0 \quad \text{in } \Omega, \\ M = \sup_{x \in \Omega} u(x) \geq 0, \end{cases}$$

and let

$$F = \{x \in \Omega : u(x) = M\}.$$

Our proof is essentially based on the following lemma (cf. [1], [2], [11], [21], [24]):

Lemma 4.1 *(i) Let $x(t)$ be a finite chain of trajectories of the diffusion vector fields*

$$\pm X_j = \pm \sum_{k=1}^{n} a^{jk}(x) \frac{\partial}{\partial x_k} \quad \textit{for } 1 \leq j \leq n.$$

If we have the assertion

$$x(t_0) \in F \quad \textit{for some } t_0,$$

then it follows that

$$x(t) \in F \quad \textit{for all } t \geq t_0.$$

(ii) Let x_0 be a point of $\mathcal{M}$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x_0)\, \nu_i \nu_j = 0 \quad \textit{and } L\varphi(x_0) > 0.$$

If $x_0 \in F$, then the connected component

$$\{x \in \Omega : \varphi(x) > 0\}$$

of $\Omega \setminus \mathcal{M}$ is contained in the set F

Similarly, let x_0 be a point of $\mathcal{M}$ such that

$$\sum_{i,j=1}^{n} a^{ij}(x_0)\, \nu_i \nu_j = 0 \quad \textit{and } L\varphi(x_0) < 0.$$

If $x_0 \in F$, then the connected component

$$\{x \in \Omega : \varphi(x) < 0\}$$

of $\Omega \setminus \mathcal{M}$ is contained in the set F.

Proof (i) In order to prove part (i), it suffices to note that [2, Proposition 3.1] remains valid for the operator L of the form (1.2), as is proved by Oleĭnik–Radkevič [21, Theorem 3.1.6]. More precisely, the reader might refer to Redheffer [24, Lemma 7] and Amano [1, Lemma 3].

(ii) Without loss of generality, we may choose a local coordinate system

$$y = (y', y_n) = (y_1, \ldots, y_{n-1}, y_n)$$

in a neighborhood U of x_0 such that (see Figure 4.3)

(1) $x_0 =$ the origin, $\varphi(y) = y_n$ and $\boldsymbol{\nu} = (0, \ldots, 0, 1)$;
(2) U is a unit ball about the origin;
(3) L is of the form

$$L = \sum_{i,j=1}^{n} \alpha^{ij}(y) \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^{n} \beta^i(y) \frac{\partial}{\partial y_i} + c(y).$$

Here we remark that

$$\alpha^{nn}(0) = 0,$$

since we have the condition

$$\sum_{i,j=1}^{n} a^{ij}(x_0)\, \nu_i\, \nu_j = 0.$$

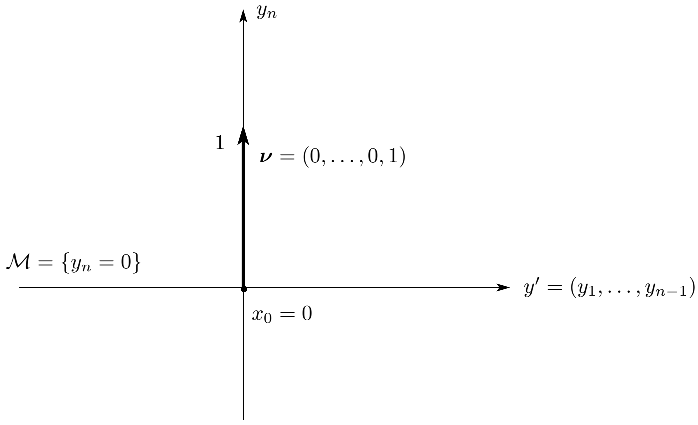


Fig. 4.3 The local coordinate system $y = (y', y_n) = (y_1, \ldots, y_{n-1}, y_n)$ near $x_0 = 0$

By replacing y_n by $-y_n$ if necessary, it suffices to consider the case where

$$L\varphi(x_0) = \beta^n(0) > 0. \tag{4.1}$$

Our proof of part (ii) is based on a reduction to absurdity. We assume, to the contrary, that the assertion does not hold true. Then it follows from part (i) and Chow's theorem [5, Satz C] that there is *no* point of F in the upper half-space $\{y_n > 0\}$. We let

$$h(y) = \varepsilon \left(y_1^2 + y_2^2 + \ldots + y_{n-1}^2 \right) + (y_n - 1)^2,$$

where $0 < \varepsilon < 1$ will be chosen later on. We remark that the ellipsoid

$$\left\{ y \in \mathbf{R}^n : h(y) \leq 1 \right\} = \{ h(y) \leq 1 \}$$

touches the set F only at the origin.

Consider the function

$$v(y) = \exp\,[1 - h(y)] - 1.$$

Since $\alpha^{nn}(0) = 0$, we have the formula

$$Lv(0) = -2\varepsilon \sum_{i=1}^{n-1} \alpha^{ii}(0) + 2\beta^n(0).$$

By condition (4.1), we can choose ε so small that

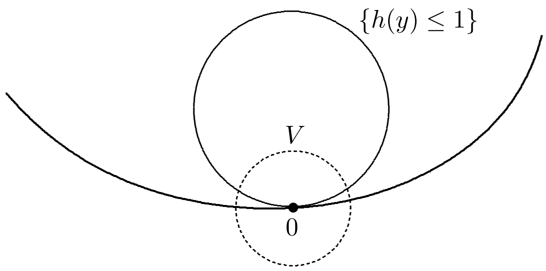
$$Lv > 0 \quad \text{in some neighborhood } V \text{ of the origin.} \tag{4.2}$$

If λ is a positive constant, we let

$$w(y) := u(y) + \lambda\, v(y).$$

Since $Lu \geq 0$ in Ω, it follows from assertion (4.2) that

$$Lw > 0 \quad \text{in } V. \tag{4.3}$$



$F = \{x \in \Omega : u(x) = M\}$

Fig. 4.4 The closed set F and the neighborhood V of $x_0 = 0$

We remark that the distance between the set $\partial V \cap F$ and the set

$$\{v(y) \geq 0\} = \{\exp\,[1 - h(y)] \geq 1\} = \{h(y) \leq 1\}$$

is positive, where ∂V is the boundary of V (see Figure 4.4). Hence we can choose the constant λ so small that

$$w(y) < M \quad \text{on } \partial V.$$

However, since $w(0) = u(0) = M$, it follows that the function $w(y)$ takes its greatest non-negative value *inside* V. Therefore, we have the inequality

$$\begin{aligned} Lw(0) &= \sum_{i,j=1}^{n} \alpha^{ij}(0) \frac{\partial^2 w}{\partial x_i \partial x_j}(0) + \sum_{i=1}^{n} \beta^i(0) \frac{\partial w}{\partial x_i}(0) + c(0)w(0) \\ &= \sum_{i,j=1}^{n} \alpha^{ij}(0) \frac{\partial^2 w}{\partial x_i \partial x_j}(0) + c(0)w(0) \leq \sum_{i,j=1}^{n} \alpha^{ij}(0) \frac{\partial^2 w}{\partial x_i \partial x_j}(0) \\ &\leq 0. \end{aligned}$$

This contradicts inequality (4.3).

The proof of lemma 4.1 is complete. $\square$

4.2 End of Proof of Theorem 4.1

The proof of Theorem 4.1 is divided into two steps.

Step 1: First, we have the following formula for the Fichera function $b(x)$ on the set $\mathcal{M}$ (cf. [27, the proof of Lemma 7.1]):

Claim 4.1 *If the function*

$$\sum_{i,j=1}^{n} a^{ij}(x)\,\nu_i\,\nu_j$$

vanishes identically on $\mathcal{M}$*, then we have the formula*

$$L\varphi(x) = b(x) = \sum_{i=1}^{n} \left(b^i(x) - \sum_{j=1}^{n} \frac{\partial a^{ij}}{\partial x_j}(x) \right) \nu_i \quad \textit{on } \mathcal{M}. \tag{4.4}$$

Proof Given each point $x_0 \in \mathcal{M}$, we choose a local coordinate system

$$x = (x_1, \ldots, x_{n-1}, x_n)$$

in such a way that

$$\begin{aligned} &x_0 = 0, \\ &\varphi(x) = x_n \quad \text{near the point } x_0, \\ &\boldsymbol{\nu} = (0, \ldots, 0, 1). \end{aligned}$$

In terms of this local coordinate system, it suffices to show that

$$\sum_{j=1}^{n} \frac{\partial a^{nj}}{\partial x_j}(0) = 0. \tag{4.5}$$

Indeed, we then have the formula

$$\begin{aligned} L\varphi(x_0) = L\varphi(0) &= \sum_{i=1}^{n} a^{ij}(0) \frac{\partial^2 \varphi}{\partial x_i \partial x_j}(0) + \sum_{k=1}^{n} b^k(0) \frac{\partial \varphi}{\partial x_k}(0) = b^n(0) \\ &= b^n(0) - \sum_{j=1}^{n} \frac{\partial a^{nj}}{\partial x_j}(0) = b(0) \\ &= b\left(x_0\right). \end{aligned}$$

This proves the desired assertion (4.4).

First, since $0 \in \mathcal{M}$, it follows that

$$a^{nn}\left(x', 0\right) = 0 \quad \text{for all } x' = (x_1, \ldots, x_{n-1}) \text{ near } 0 \in \mathbf{R}^{n-1}. \tag{4.6}$$

Hence, by applying Oleĭnik–Radkevič [21, Corollary 1 to Lemma 1.7.7] we obtain that

$$\begin{aligned} &\left| a^{nj}\left(x', 0\right) \right|^2 \\ &\leq 2\, a^{nn}\left(x', 0\right)\, a^{jj}\left(x', 0\right) \quad \text{for all } x' \text{ near the origin in } \mathbf{R}^{n-1} \text{ if } 1 \leq j \leq n-1, \end{aligned}$$

so that

$$a^{nj}\left(x', 0\right) = 0 \quad \text{for all } x' \text{ near the origin in } \mathbf{R}^{n-1} \text{ if } 1 \leq j \leq n-1.$$

This proves that

$$\sum_{j=1}^{n-1} \frac{\partial a^{nj}}{\partial x_j}(0) = 0. \tag{4.7}$$

On the other hand, since we have the assertion

$$a^{nn}\left(x', x_n\right) \geq 0 \quad \text{for all } x = \left(x', x_n\right) \text{ near } 0 \in \mathbf{R}^n,$$

it follows from assertion (4.6) that

$$\frac{\partial a^{nn}}{\partial x_n}(0) = 0. \tag{4.8}$$

Therefore, the desired assertion (4.5) follows immediately from assertions (4.7) and (4.8).

The proof of Claim 4.1 is complete. □

Step 2: By virtue of formula (4.4), it follows from part (i) of Lemma 4.1 that the assumption (A) implies that the maximum starting at $\mathcal{N}$ goes out of $\mathcal{N}$. Furthermore, we obtain from Lemma 4.1 that if x_0 is a point of $\mathcal{M}$ such that either

$$\sum_{i,j=1}^{n} a^{ij}\left(x_0\right) \nu_i \nu_j > 0$$

or

$$\sum_{i,j=1}^{n} a^{ij}\left(x_0\right) \nu_i \nu_j = 0 \quad \text{and} \quad L\varphi(x_0) \neq 0,$$

then the maximum starting at x_0 either goes into both sides of $\mathcal{M}$ or goes into one side of $\mathcal{M}$, respectively.

Summing up, we obtain that the maximum starting at $\mathcal{M}$ *goes out of* $\mathcal{M}$.

On the other hand, it follows from part (ii) of Lemma 4.1 that the condition (C) implies that the maximum can traverse $\mathcal{M}$ from every side of $\mathcal{M}$.

Therefore, we can conclude that, under the assumption (A), the maximum at any point x_0 of Ω can be propagated into the *whole* Ω if and only if one of the conditions (B) and (C) holds true (see [27, Theorem 4.1]).

The proof of Theorem 4.1 is complete. □

5 Necessary conditions

The conditions (B) and (C) are *partially* necessary. In fact, we can prove the following second main result:

Theorem 5.1 (Necessity) *Assume that the exceptional set $\mathcal{M}$ is a closed connected hypersurface such that the complement $\Omega \setminus \mathcal{M}$ has* two *connected components and that its one component G is* relatively compact *in Ω. Let*

$$\boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_n)$$

be the inward normal *to the domain G at the boundary portion $\mathcal{M}$. Assume that*

$$\sum_{i,j=1}^{n} a^{ij}(x')\,\nu_i\,\nu_j \equiv 0 \quad \text{on } \mathcal{M} \tag{5.1}$$

and that

$$b(x') = \sum_{i=1}^{n} \left(b^i(x') - \sum_{j=1}^{n} \frac{\partial a^{ij}}{\partial x_j}(x') \right) \nu_i \neq 0 \quad \text{on } \mathcal{M}. \tag{5.2}$$

Then the strong maximum principle (SMP) does not *hold true for the operator L. More precisely, we can construct a function $u^0 \in C^\infty(\Omega)$ such that*

$$\begin{cases} u^0 \leq 0 & \text{in } \Omega, \\ Lu^0 \geq 0 & \text{in } \Omega, \\ u^0 = 0 & \text{on } \Omega \setminus G, \\ u^0(x_0) < 0 & \text{for some point } x_0 \in G. \end{cases} \tag{5.3}$$

Remark 5.1 Some remarks are in order.

1° Condition (A) holds true, but conditions (B) and (C) do not hold true. Indeed, it suffices to note that

$$\sum_{i,j=1}^{n} a^{ij}(x')\,\nu_i\,\nu_j + b(x') = b(x') \neq 0 \quad \text{on } \mathcal{M}.$$

2° The assertions (5.3) imply that the maximum zero *cannot* go into G from the outside $\Omega \setminus G$.

Theorem 5.1 will be proved in the next subsection, due to its length.
We give two simple examples for Theorem 5.1 in the plane $\mathbf{R}^2$ $(n = 2)$:

Example 5.1 Let $\Omega = \{(x_1, x_2) : -1 < x_1 < 1,\ -1 < x_2 < 1\}$ (the rectangle). We let

$$L_\pm = \exp\left[-\frac{1}{x_1^2}\right] \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \pm \frac{\partial}{\partial x_1}.$$

Then we have the assertions

$$\begin{aligned} &\mathcal{M} = \{(0, x_2) : -1 < x_2 < 1\} = \{0\} \times (-1, 1), \\ &\mathcal{N} = \emptyset, \\ &G_+ = \{(x_1, x_2) : -1 < x_1 < 0,\ -1 < x_2 < 1\} = (-1, 0) \times (-1, 1), \\ &G_- = \{(x_1, x_2) : 0 < x_1 < 1,\ -1 < x_2 < 1\} = (0, 1) \times (-1, 1). \end{aligned}$$

It is easy to see that the conditions (5.1) and (5.2) hold true for the operators $L_\pm$. Indeed, it suffices to note that

$$\sum_{i,j=1}^{2} a^{ij}(x_1, x_2)\,\nu_i\,\nu_j + b(x_1, x_2) = \pm 1 \neq 0 \quad \text{on } \mathcal{M}.$$

5.1 Proof of Theorem 5.1

We only consider the case where

$$b(x') = \sum_{i=1}^{n} \left(b^i(x') - \sum_{j=1}^{n} \frac{\partial a^{ij}}{\partial x_j}(x') \right) \nu_i < 0 \quad \text{on } \mathcal{M}, \tag{5.4}$$

since the case where $b(x') > 0$ on $\mathcal{M}$ can be treated in a similar way. The proof of Theorem 5.1 is divided into three steps.

Step 1: First, we choose a function $\psi \in C_0^\infty(\Omega)$ such that

$$\begin{cases} \psi \leq 0 & \text{in } G, \\ \psi(x_0) = -1 & \text{for some point } x_0 \in G. \end{cases} \tag{5.5}$$

Then it follows from an application of Stroock–Varadhan [27, Corollary 8.2] that there exists a unique solution $u \in L^\infty(G)$ of the Dirichlet problem

$$\begin{cases} Lu = -\psi & \text{in } G, \\ u = 0 & \text{on } \partial G, \end{cases}$$

and further that it is given by the following *probabilistic representation formula*:

$$u(x) = E_x \left[\int_0^{\tau'} \psi(\xi(t)) \exp\left(\int_0^t c(\xi(s))\, ds \right) dt \right]. \tag{5.6}$$

Here:

(1) $\xi(t) = \left(\xi^1(t), \xi^2(t), \dots, \xi^n(t)\right)$ is a *diffusion process* governed by the system of stochastic differential equations

$$d\xi^i(t) = \sqrt{2} \sum_{j=1}^{n} \sigma^{ij}\left(\xi(t)\right) \circ dB^j(t) + \left(b^i\left(\xi(t)\right) - \sum_{j=1}^{n} \frac{\partial a^{ij}}{\partial x_j}\left(\xi(t)\right) \right) dt,$$

where $\sigma = \left(\sigma^{ij}\right)$ is a unique square root of the non-negative definite matrix $\left(a^{ij}\right)$, $B(t) = \left(B^1(t), B^2(t), \dots, B^n(t)\right)$ is an n-dimensional standard Brownian motion and the stochastic integral is defined in the Fisk–Stratonovich sense (cf. [25]).

(2) τ' is the *exit time* from the compact set $\overline{G} = G \cup \mathcal{M}$ defined by the formula

$$\tau' = \inf\left\{t \geq 0 : \xi(t) \notin \overline{G}\right\}.$$

Indeed, since the Lie algebra $\mathcal{L}\left(X_1, X_2, \dots, X_n\right)$ has rank n at each point of G and since we have, by condition (5.2),

$$b(x') = \sum_{i=1}^{n} \left(b^i(x') - \sum_{j=1}^{n} \frac{\partial a^{ij}}{\partial x_j}(x') \right) \nu_i < 0 \quad \text{on } \mathcal{M},$$

it is easily seen from Chow's theorem [5, Satz C] and Stroock–Varadhan [27, Remark 5.2] that

$$\sup_{x \in \overline{G}} E_x\left[\tau'\right] < \infty.$$

It should be emphasized that the probabilistic formula (5.6) can be traced back to the pioneering work of Kakutani [17] on harmonic functions.

Therefore, our desired assertion follows from an application of [27, Corollary 8.2] with

$$\begin{aligned} \Sigma_2 &:= \left\{ x' \in \partial G : \sum_{i,j=1}^{n} a^{ij}(x')\, \nu_i\, \nu_j = 0,\ b(x') < 0 \right\} = \mathcal{M}, \\ \Sigma_3 &:= \partial G \setminus \mathcal{M}. \end{aligned}$$

Step 2: Secondly, since the function $\psi(x)$ satisfies the conditions (5.5), we obtain from the probabilistic representation formula (5.6) that

$$\begin{cases} u^0 \leq 0 & \text{in } G, \\ u^0(x_0) < 0 & \text{for some point } x_0 \in G. \end{cases} \tag{5.7}$$

Now we let

$$u^0(x) = \begin{cases} u(x) & \text{for } x \in G, \\ 0 & \text{for } x \in \Omega \setminus G, \end{cases}$$

and

$$(Lu)^0(x) = \begin{cases} Lu(x) & \text{for } x \in G, \\ 0 & \text{for } x \in \Omega \setminus G. \end{cases}$$

By virtue of condition (5.1), in a tubular neighborhood V of $\mathcal{M}$ we can express the operator L in the form

$$L = a(x) \frac{\partial^2}{\partial \nu^2} + \ell_1(x, D') \frac{\partial}{\partial \nu} + \ell_2(x, D').$$

Here

(1) $\partial/\partial\nu$ denotes the *inward normal* differentiation to the domain G at the boundary portion $\mathcal{M}$.
(2) $\ell_j(x, D')$, $j = 1, 2$, is a differential operator of order j acting along the parallel surfaces of $\mathcal{M}$ on V.
(3) $a(x) = \frac{\partial a}{\partial \nu}(x) = 0$ on $\mathcal{M}$, since $a(x) \geq 0$ on V.

Therefore, by using the well-known *jump formula* (see [4, Chapitre 1, formule (1.4.1)]; [13, formula (2.2.1)]) we obtain that

$$L\left(u^0\right) = (Lu)^0 + a(x) \frac{\partial}{\partial \nu} \left(u\, \delta_{\mathcal{M}}\right) + \ell_1(x, D') \left(u\, \delta_{\mathcal{M}}\right), \tag{5.8}$$

where $\delta_{\mathcal{M}}$ is the surface measure on the hypersurface $\mathcal{M}$ defined by the formula

$$\langle \delta_{\mathcal{M}}, \theta \rangle = \int_{\mathcal{M}} \theta|_{\mathcal{M}}\, d\sigma \quad \text{for all } \theta \in C_0^\infty(V),$$

and $d\sigma$ is the surface element on $\mathcal{M}$.

However, since $a(x) = \frac{\partial a}{\partial \nu}(x) = 0$ and $u(x) = 0$ on $\mathcal{M}$, it follows from formula (5.8) and assertion (5.5) that

$$L\left(u^0\right) = (Lu)^0 = (-\psi)^0 = -\psi \geq 0 \quad \text{in } \Omega. \tag{5.9}$$

Step 3: Finally, by assertions (5.7) and (5.9) it remains to show that $u^0 \in C^\infty(\Omega)$.

To do so, we recall that a differential operator A with C^∞ coefficients in Ω is said to be *hypoelliptic* if it satisfies the condition

$$\operatorname{sing\,supp} u = \operatorname{sing\,supp} Au \quad \text{for every distribution } u \in \mathcal{D}'(\Omega).$$

It is easy to see that this condition is equivalent to the following: For any open subset Ω_1 of Ω, we have the assertion

$$u \in \mathcal{D}'(\Omega),\ Au \in C^\infty(\Omega_1) \implies u \in C^\infty(\Omega_1).$$

For example, it is known (see [4], [16], [21]) that all elliptic operators are hypoelliptic. Moreover, we say that A is *globally hypoelliptic* if it satisfies the weaker condition

$$u \in \mathcal{D}'(\Omega),\ Au \in C^\infty(\Omega) \implies u \in C^\infty(\Omega).$$

It should be noticed that these two notions may be transferred to manifolds.

Since we have, by assumption (3.1), conditions (5.1) and (5.4),

$$\begin{aligned} &\mathcal{M} = \{x \in \Omega : \varphi(x) = 0\}, \\ &\boldsymbol{\nu} = \operatorname{grad} \varphi \neq 0, \\ &\sum_{i,j=1}^{n} a^{ij}(x)\, \nu_i\, \nu_j \equiv 0 \quad \text{on } \mathcal{M}, \\ &b(x') = L\varphi(x') < 0 \quad \text{on } \mathcal{M}, \end{aligned}$$

it follows from an application of Oleĭnik–Radkevič [21, Theorem 2.6.3] (see also [23, Theorem 7] and [15, Theorem 5.9]) that the operator L is globally hypoelliptic in Ω.

Hence, we obtain from equation (5.9) that

$$u^0 \in C^\infty(\Omega),$$

since $\psi \in C_0^\infty(\Omega)$.

Summing up, we have constructed a function $u^0 \in C^\infty(\Omega)$ which satisfies the desired conditions (5.3).

Now the proof of Theorem 5.1 is complete. □

6 Propagation of maximums in terms of subunit vectors

In this section, we formulate another maximum principle for the differential operator

$$Lu = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b^i(x) \frac{\partial u}{\partial x_i} + c(x)u \tag{1.2}$$

in terms of *subunit vectors* introduced by Fefferman–Phong [6]. This section is adapted from [31, Chapter 7].

Now we consider the following problem:

Problem 6.1 Let x be a point of Ω. Determine the largest connected, relatively closed subset $D(x)$ of Ω, containing x, such that

$$\begin{cases} \text{If } u \in C^2(\Omega),\ Lu \geq 0 \text{ in } \Omega \text{ and if } u \text{ takes its greatest non-negative} \\ \text{value } M \text{ at } x, \text{ then } u \equiv M \text{ throughout } D(x). \end{cases}$$

The set $D(x)$ is called the *propagation set* of x in Ω.

We shall give a coordinate-free description of the set $D(x)$ in terms of subunit vectors. A tangent vector

$$X = \sum_{j=1}^{n} \gamma^j \frac{\partial}{\partial x_j} \in T_x(\Omega)$$

at $x \in \Omega$ is said to be *subunit* for the differential operator L if it satisfies the condition

$$\left(\sum_{j=1}^{n} \gamma^j \eta_j \right)^2 \leq \sum_{i,j=1}^{n} a^{ij}(x) \eta_i \eta_j \quad \text{for all } \eta = \textstyle\sum_{j=1}^{n} \eta_j dx_j \in T_x^*(\Omega), \tag{6.1}$$

Since this notion is *coordinate-free*, we rotate the coordinate axes so that the symmetric matrix $\left(a^{ij}\right)$ is diagonalized at x:

$$\left(a^{ij}(x)\right) = \left(\lambda_i \delta_{ij}\right), \quad \lambda_1 > 0, \ldots, \lambda_r > 0, \quad \lambda_{r+1} = \ldots = \lambda_n = 0,$$

where $r = \operatorname{rank}\left(a^{ij}(x)\right)$. Then it is easy to see that the tangent vector X is subunit for L if and only if it is contained in the following ellipsoid of dimension r:

$$\frac{(\gamma^1)^2}{\lambda_1} + \ldots + \frac{(\gamma^r)^2}{\lambda_r} \leq 1, \quad \gamma^{r+1} = \ldots = \gamma^n = 0. \tag{6.2}$$

A *subunit trajectory* is a Lipschitz path $\gamma : [t_1, t_2] \to \Omega$ such that the tangent vector

$$\dot{\gamma}(t) = \frac{d}{dt}\left(\gamma(t)\right)$$

is subunit for L at $\gamma(t)$ for almost every t. We remark that if $\dot{\gamma}(t)$ is subunit for L, so is $-\dot{\gamma}(t)$; hence subunit trajectories are not oriented.

We let

$$X_0(x) = \sum_{i=1}^{n} \left(b^i(x) - \sum_{j=1}^{n} \frac{\partial a^{ij}}{\partial x_j}(x) \right) \frac{\partial}{\partial x_i}.$$

The vector field $X_0(x)$ is called the *drift vector field* in probability theory, while it is the so-called *subprincipal part* of the operator L in terms of the theory of partial differential equations (see [16, formula (1.8.32)]).

A *drift trajectory* is a curve $\theta : [t_1, t_2] \to \Omega$ such that

$$\dot{\theta}(t) = X_0\left(\theta(t)\right) \quad \text{on } [t_1, t_2],$$

and this curve is oriented in the direction of increasing t.

Our main result reads as follows (see [31, Theorem 7.2.1]):

The propagation set $D(x)$ of x in Ω contains the closure $D'(x)$ in Ω of (PMP) *all points $y \in \Omega$ which can be joined to x by a finite number of subunit and drift trajectories.*

This result (PMP) is called the *sharp maximum principle* for the differential operator L (cf. [11], [24]). The sharp maximum principle says that if the matrix $\left(a^{ij}(x)\right)$ is non-degenerate at a point x, that is, if $r = \operatorname{rank}\left(a^{ij}(x)\right) = n$, then the maximum propagates in an open neighborhood of x; but if the matrix $\left(a^{ij}(x)\right)$ is degenerate at x, then the maximum propagates only in a "thin" ellipsoid of dimension r (cf. formula (6.2)) and in the direction of X_0. Now we see the reason why the strong maximum principle (1.1) holds true for the Laplacian Δ.

We give four simple examples of the strong maximum principle (SMP) via the sharp maximum principle (PMP) in the case where Ω is the square $(-1,1)\times(-1,1)$ in the plane $\mathbf{R}^2$:

Example 6.1 $L_1 = \partial^2/\partial x_1^2 + x_1^2\partial^2/\partial x_2^2$. The subunit vector fields for L_1 are generated by the following:

$$\left(\frac{\partial}{\partial x_1}, x_1\frac{\partial}{\partial x_2}\right).$$

Hence we have the assertion

$$\text{The set } D'\left((x_1,x_2)\right) \text{ is equal to } \Omega \text{ for every } (x_1,x_2)\in\Omega. \tag{6.3}$$

Namely, the strong maximum principle (SMP) holds true for the operator L_1, as is shown in Example 1.1.

Example 6.2 $L_4 = x_1^2\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$. The subunit vector fields for L_4 are generated by the following:

$$\left(x_1\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right). \tag{6.4}$$

Thus we have the assertion

$$D'\left((x_1,x_2)\right) = \begin{cases} [0,1)\times(-1,1) & \text{if } x_1 > 0, \\ \{0\}\times(-1,1) & \text{if } x_1 = 0, \\ (-1,0]\times(-1,1) & \text{if } x_1 < 0. \end{cases}$$

It can be shown that the strong maximum principle (SMP) does not hold true for the operator L_4, just as in Example 5.1.

Example 6.3 $L_5 = x_1^2\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + x_2\partial/\partial x_1$. The subunit vector fields for L_5 are generated by formula (6.4), and the drift vector field is equal to the following:

$$X_0\left(x_1,x_2\right) = \left(x_2 - 2x_1\right)\frac{\partial}{\partial x_1}.$$

Thus, by virtue of the drift vector field we have assertion (6.3), and so the strong maximum principle (SMP) holds true for the operator L_5. We remark that the assumptions (A) and (C) hold true for the operator L_5, just as in Example 4.2.

Example 6.4 $L_6 = x_1^2 \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial/\partial x_1$. The subunit vector fields for L_6 are generated by the vector fields (6.4), and the drift vector field is equal to the following:

$$X_0\,(x_1, x_2) = (1 - x_1)\,\frac{\partial}{\partial x_1}.$$

Hence we have the assertion

$$D'\,((x_1, x_2)) = \begin{cases} \Omega & \text{if } x_1 < 0, \\ [0,1) \times (-1,1) & \text{if } x_1 \geq 0. \end{cases}$$

It can be shown that the strong maximum principle (SMP) does not hold true for the operator L_6 in some weak sense, just as in Example 5.1.

7 Concluding remarks

Some remarks are in order.

(1) The maximum principles in this paper are adapted from Amano [1], Bony [2], Gilbarg–Trudinger [10], Hill [11], Oleĭnik [20], Oleĭnik–Radkevič [21] and Redheffer [24]. For a general study of maximum principles, the reader might refer to Bony–Courrège–Priouret [3], Protter–Weinberger [22] and also [32, Chapter 8].
(2) Our formulation of the strong maximum principle is coordinate-free. The results here may be applied to questions of uniqueness for degenerate elliptic boundary value problems on a manifold.
(3) The underlying analytical mechanism of propagation of maximums plays an important role in the probabilistic interpretation and study of Markov processes from the viewpoint of functional analysis (see Bony–Courrège–Priouret [3, Chapitre I], [29] and [32, Part III]).
(4) The propagation set $D(x)$ coincides with the *support* of the Markov process corresponding to the operator L, which is the closure of the collection of all possible trajectories of a Markovian particle, starting at x, with generator L (see Stroock–Varadhan [26, Theorem 5.2]).
(5) It seems quite likely that there is an intimate connection between propagation of maximums and propagation of singularities for degenerate elliptic differential operators of second-order (see Fujiwara–Omori [9], Hörmander [14], Yoshino [33] and [30]).

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