monoidal category

Theorem

words of fixed length n

$\Rightarrow G_n = G_{n,b}$ basic graph

$\rho b_0 = 0 \quad \rho(-) = 0$

rank

~~$\rho(w)$~~

$\rho(v \square w) = \rho(v) + \rho(w) + \text{length}(w) - 1$

$\rho w = 0 \Rightarrow$ all pairs of parentheses in w start at the front

$\boxed{G_n : \text{commute}}$

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & V_1 \arrow[d] & V_2 \arrow[l] \arrow[r] \arrow[d] & V_3 \arrow[d] & W \arrow[l] \arrow[d] \\
W^{(n)} = & (w^{\cdot}) = & W^{(n)} = & W^{(n)} = & W^{(n)}
\end{tikzcd}$$

any two directed paths (all are canonical path λ^{-1}) from V_i to $W^{(n)}$

by induction the rank of $V_A = V$

Suppose it true for all V of smaller rank,

consider two different directed paths.

starting at V with directed basic arrows ρ and γ

Both ρ and γ decrease rank

$$\begin{tikzcd}
 & V = u \square w \arrow[dl, "\rho"'] \arrow[dr, "\gamma"] & \\
V' \arrow[r] \arrow[d] & Z \arrow[d] & V'' \arrow[l] \\
W^{(n)} = & W^{(n)} = & W^{(n)}
\end{tikzcd}$$

by a case subdivision

(1) $\rho = \gamma \Rightarrow Z = V' = V''$

(2) $\rho \neq \gamma \quad V = u \square w$

$\rho = \beta' \square 1_w$: ρ acts inside the first factor

$\rho = 1_u \square \beta''$: β acts inside the second factor

$\rho = \alpha_{u,s,t}$ where $V = u \square w = u \square (s \square t)$

compare ρ and γ

both act inside the same factor u

$\Rightarrow$ we can use induction on the length n

ρ acts inside u and γ

$$\begin{tikzcd}
 & u \square w \arrow[dl, "\rho' \square 1"'] \arrow[dr, "1 \square \gamma'"] & \\
u' \square w \arrow[dr, "1 \square \gamma'"'] & & u \square w' \arrow[dl, "\rho' \square 1"] \\
 & u' \square w' &
\end{tikzcd}$$

$\square$ bifunctor

残っている場合

one of ρ or γ, say ρ, is $\rho = \alpha = \alpha_{u,s,t}$

since $\gamma \neq \rho$, γ must act inside u or inside w

If γ acts inside u, we use a diamond from $u \square (s \square t)$ to $(u' \square s) \square t$, which commutes because α is natural

If γ is inside $w = s \square t$ and actually inside s or inside t, naturality of α gives a similar diamond

There remains only the case where γ is inside $s \square t$ but not inside s or t.

Then γ must be an instance of α, t must be a product $t = p \square q$

and our diamond must then start with

$$\begin{tikzcd}
 & \gamma = u \square w = u \square (s \square (p \square q)) \arrow[dl, "\rho = \alpha"'] \arrow[dr, "1 \square \alpha = \gamma"] & \\
(u \square s) \square (p \square q) & & u \square ((s \square p) \square q)
\end{tikzcd}$$

pentagon

G_n : commutative in B as far as associativity is concerned

$\lambda_a : e \square a \cong a \quad \rho_a : a \square e \cong a$

λ と ρ が入ってくる場合

G_n' : vertices all words of length n including words involving e

edges all basic arrows.

G_n' : infinite

contains G_n

G_n' is commutative

For any word W $\quad W \xrightarrow{\text{path}} W^{(n)}$

composite arrow is equal to that for a different path which first removes all e's, then applies α

If e is removed by $\lambda : e \square b \cong b$

after some application of α,

then that e can be removed before

either by naturality of α or diagram ($e \square a$; λ, ρ)

$$\begin{tikzcd}
a \square (e \square c) \arrow[r, "\alpha"] \arrow[d, "1 \square \lambda"'] & (a \square e) \square c \arrow[d, "\rho \square 1"] \\
a \square c \arrow[r, phantom, "="] & a \square c
\end{tikzcd}$$

$$\begin{tikzcd}
e \square (b \square c) \arrow[r, "\alpha"] \arrow[d, "\lambda"'] & (e \square b) \square c \arrow[d, "\lambda \square 1"] \\
b \square c & b \square c
\end{tikzcd}$$

\begin{tikzcd}
a\square(b\square e) \arrow[r, "\alpha"] \arrow[d, "1\square\rho"'] & (a\square b)\square e \arrow[d, "\rho"] \\
a\square b \arrow[r, phantom, "="] & a\square b
\end{tikzcd}

$\lambda_e = \rho_e : e\square e \to e$

this reduced path composite equal to that for a canonical path in which all e's are removed in some specified order (say starting with the left-most occurrence of e)

$G'_n \Rightarrow G'_n$

monoidal category

(例) $\langle Set, \times, 1\rangle$

$\langle Top, \times, *\rangle$

$\langle Ab, \otimes, \mathbb{Z}\rangle$

$\langle Cat, \times, \mathbb{1}\rangle$

Category $\mathbb{C}$

objects a, b class

$\mathbb{C}(a,b)$

合成

$\mathbb{C}(b,c)\times\mathbb{C}(a,b) \to \mathbb{C}(a,c)$

$* \xrightarrow{id} \mathbb{C}(a,a)$

\begin{tikzcd}
\mathbb{C}(c,d)\times\mathbb{C}(b,c)\times\mathbb{C}(a,b) \arrow[r] \arrow[d] & \mathbb{C}(c,d)\times\mathbb{C}(a,c) \arrow[d] \\
\mathbb{C}(b,d)\times\mathbb{C}(a,b) \arrow[r] & \mathbb{C}(a,d)
\end{tikzcd}

\begin{tikzcd}
*\times\mathbb{C}(a,b) \arrow[r, "id\times 1"] \arrow[dr, "="'] & \mathbb{C}(b,b)\times\mathbb{C}(a,b) \arrow[d] \\
& \mathbb{C}(a,b)
\end{tikzcd}

$\langle Set, \times, *\rangle$

$\langle V, \times, *\rangle$ monoidal

enriched category theory

enriched over $\underbrace{\langle V, \times, *\rangle}_{\text{monoidal category}}$